Statistics of Extremes in Eigenvalue-Counting Staircases

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We consider the number $\mathcal{N}_{\theta_A}(\theta)$ of eigenvalues $e^{i\theta_j}$ of a random unitary matrix, drawn from $\mathrm{CUE}_\beta(N)$, in the interval $\theta_j \in [\theta_A, \theta]$. The deviations from its mean, $\mathcal{N}_{\theta_A}(\theta) - \mathbb{E}[\mathcal{N}_{\theta_A}(\theta)]$, form a random process as function of θ. We study the maximum of this process, by exploiting the mapping onto the statistical mechanics of log-correlated random landscapes. By using an extended Fisher-Hartwig conjecture supplemented with the freezing duality conjecture for log-correlated fields, we obtain the cumulants of the distribution of that maximum for any $\beta > 0$. It exhibits combined features of standard counting statistics of fermions (free for $\beta = 2$ and with Sutherland-type interaction for $\beta \neq 2$) in an interval and extremal statistics of the fractional Brownian motion with Hurst index $H = 0$. The $\beta = 2$ results are expected to apply to the statistics of zeroes of the Riemann Zeta function.

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Characterizing the full counting statistics of the fluctuations of the number $\mathcal{N}$ of $1d$ fermions in an interval is important in numerous physical contexts, both for ground state and dynamical properties. It appears, e.g., in shot noise [1], in fermion chains [2,3], in interacting Bose gases [4], in nonequilibrium Luttinger liquids [5], in trapped fermions [6–8], and for studying related observables, such as the entanglement entropy [9–11] or the statistics of local magnetization in quantum spin chains [12]. An equivalent problem can be formulated as counting eigenvalues of large random matrices (RM). As is well known since Dyson's work [13], such eigenvalues behave as classical particles with 1D Coulomb repulsion at inverse temperature $\beta > 0$. Namely, consider a unitary $N \times N$ matrix U and denote the corresponding unimodular eigenvalues as $z_j = e^{i\theta_j}$, $j = 1, \ldots, N$, with phases $\theta_i \in]-\pi, \pi]$. Then for any given $\beta > 0$ one can construct the so-called circular β ensemble $\mathrm{CUE}_\beta(N)$ in such a way that the expectation of a function depending only on the eigenvalues of U will be given by

$$\mathbb{E}(F) = c_N \prod_{j=1}^{N} \int_{-\pi}^{\pi} d\theta_i \prod_{1\le j<k\le N} |e^{i\theta_j} - e^{i\theta_k}|^\beta F, \qquad (1)$$

where $F \equiv F(\theta_1, \ldots, \theta_n)$. For $\beta = 2$ such matrices can be thought of as drawn uniformly according to the corresponding Haar's measure on $U(N)$, whereas for a generic $\beta > 0$ the explicit construction is more involved, see Ref. [14]. For any $\beta > 1$, the right-hand side (r.h.s.) of Eq. (1) equals the quantum expectation value of F in the ground state of N spinless fermions, of coordinates θ_i on the unit circle, described by the Sutherland Hamiltonian [15] $H = -\sum_i (\partial^2/\partial\theta_i^2) + \sum_{i<j}\{\beta(\beta-2)/8\sin^2[(\theta_i - \theta_j)/2]\}$. For $\beta = 2$, Eq (1) thus describes noninteracting fermions, while for $\beta \neq 2$ the fermions interact via an inverse square distance pairwise potential.

Let us now define the number of eigenvalues or fermions, $\mathcal{N}_{\theta_A}(\theta)$, in the interval $\theta_j \in [\theta_A, \theta]$ as

$$\mathcal{N}_{\theta_A}(\theta) = \sum_{j=1}^{N} [\chi(\theta - \theta_j) - \chi(\theta_A - \theta_j)],$$
$$\chi(u) = \begin{cases} 1, & u > 0 \\ 0, & u < 0 \end{cases}. \qquad (2)$$

As a function of θ this is a staircase with unit jumps upwards at random positions $\theta_j \in [\theta_A, \theta]$. The mean slope (i.e., the mean density of eigenvalues or fermions) being constant, the mean profile is $\mathbb{E}[\mathcal{N}_{\theta_A}(\theta)] = N(\theta - \theta_A)/2\pi$. In a given random matrix realization or sample one can define the deviation to the mean, $\delta\mathcal{N}_{\theta_A}(\theta) = \mathcal{N}_{\theta_A}(\theta) - \mathbb{E}[\mathcal{N}_{\theta_A}(\theta)]$, and study it as a random process as a function of θ, i.e., as a function of the length of the interval $\theta - \theta_A$, see Figs. 1 and 2. From the view of such a process, the standard results on fermion counting statistics [2], encoding the full distribution of $\delta\mathcal{N}_{\theta_A}(\theta)$ for a *fixed* value of θ, is a very local information. Such information is clearly insufficient for understanding various *non-local* properties of the process, such as characterizing maximal deviation of the staircase from its mean, i.e., $\max_{\theta\in[\theta_A,\theta_B]} |\mathcal{N}_{\theta_A}(\theta) - \mathbb{E}[\mathcal{N}_{\theta_A}(\theta)]|$. After normalization this is the Kolmogorov-Smirnov (KS) statistics, an

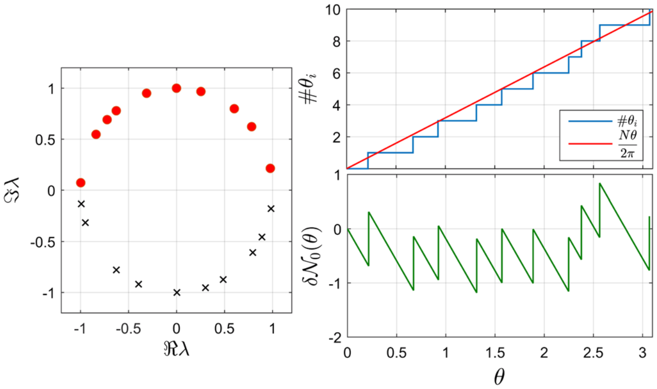

FIG. 1. Constructing an instance of $\delta N_0(\theta)$ for $\theta \in [0, \pi]$ for $\beta = 2$ and $N = 20$. Left: eigenvalues $\lambda = e^{i\theta_i}$. Right: counting staircase (top), with mean subtracted (bottom).

outstanding open problem for spectra of random matrices [16], [17].

In this Letter we study the value distribution separately for the maximum (and, equivalently, the minimum) of the centered process by explicitly calculating the cumulants of the probability density function (PDF) for the maximum value defined as

$$\delta\mathcal{N}_m = \max_{\theta\in[\theta_A,\theta_B]} \{\mathcal{N}_{\theta_A}(\theta) - \mathbb{E}[\mathcal{N}_{\theta_A}(\theta)]\} \tag{3}$$

on an interval $[\theta_A, \theta_B] \subset]-\pi, \pi]$, of a fixed length $\ell = \theta_B - \theta_A$. To derive the PDF of $\delta\mathcal{N}_m$ in the limit $N \gg 1$ we will show that for scales larger than $1/N$ the process $\delta\mathcal{N}_{\theta_A}(\theta)$ is very close to a special version of a 1D log-correlated Gaussian field, the so-called fractional Brownian motion with Hurst index $H = 0$, denoted as fBm0, defined in Ref. [18] and whose extrema where investigated recently [19,20]. However, it turns out that the relation to fBm0 alone is insufficient to fully determine the statistics of $\delta\mathcal{N}_m$. Namely, we will demonstrate that although the process $\delta\mathcal{N}_{\theta_A}(\theta)$ for large $N \gg 1$ is very close to the fBm0 at *different* points, the *non-Gaussian* features which characterize its *single-point* statistics show up in a nontrivial way in the PDF of its maximum $\delta\mathcal{N}_m$. These single-point features are inherited from the discrete nature of the number of fermions or eigenvalues as exemplified, e.g., in fermion counting statistics [2].

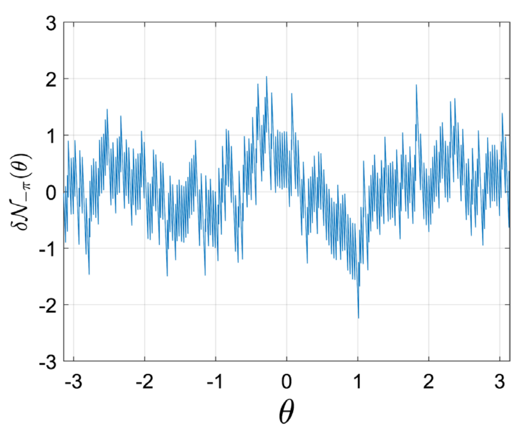

FIG. 2. A single realization of $\delta N_{-\pi}(\theta)$ for the full circle $\theta \in [-\pi, \pi]$ for $\beta = 2$ and $N = 200$.

We now describe our main findings by first assuming that the Dyson parameter is rational and can be represented as $\beta/2 = s/r$ where s and r are mutually prime, and relaxing this assumption later on. We find that, for any fixed interval, the mean value of the maximum $\delta\mathcal{N}_m$ defined in Eq. (3) exhibits, for $N \to \infty$, the universal behavior of the log-correlated fields [21–24]:

$$2\pi\sqrt{\frac{\beta}{2}}\mathbb{E}(\delta\mathcal{N}_m) \simeq 2\log N - \frac{3}{2}\log\log N + c_\ell^{(\beta)}, \tag{4}$$

where $c_\ell^{(\beta)} = O(1)$ is an unknown ℓ-dependent constant. The variance for the maximum $\delta\mathcal{N}_m$ exhibits to the leading order the extensive universal logarithmic growth typical for *pinned* log-correlated fields [19], on top of which we can evaluate the corrections of the order of unity:

$$\mathbb{E}^c(\delta\mathcal{N}_m^2) \simeq \frac{2}{\beta(2\pi)^2}[2\log N + \tilde{C}_2^{(\beta)} + C_2(\ell)]. \tag{5}$$

Finally, the higher cumulants converge to a finite limit as $N \to \infty$:

$$\mathbb{E}^c(\delta\mathcal{N}_m^k) \simeq \frac{2^{k/2}}{\beta^{k/2}(2\pi)^k}[\tilde{C}_k^{(\beta)} + C_k(\ell)], \tag{6}$$

where the constants $C_k(\ell) = O(1)$ depend on the length ℓ of the interval and will be given below in two limiting cases. The ℓ-independent constants $\tilde{C}_k^{(\beta)}$ for $k \geq 2$ are given by

$$\tilde{C}_k^{(\beta)} = \frac{d^k}{dt^k}\bigg|_{t=0} \log[A_\beta(t)A_\beta(-t)], \tag{7}$$

where

$$A_\beta(t) = r^{-t^2/2}\prod_{\nu=0}^{r-1}\prod_{p=0}^{s-1}\frac{G(1 - \frac{p}{s} + \frac{\nu + it\sqrt{\frac{2}{\beta}}}{r})}{G(1 - \frac{p}{s} + \frac{\nu}{r})}. \tag{8}$$

Here $G(z)$ denotes the standard Barnes function satisfying $G(z+1) = \Gamma(z)G(z)$, with $G(1) = 1$. Note that all the odd coefficients $\tilde{C}_{2k+1}^{(\beta)}$ vanish. Specifying for $\beta = 2$, one has $A_2(t) = G(1 + it)$, leading to $\tilde{C}_2^{(2)} = 2(1 + \gamma_E)$ and $\tilde{C}_4^{(2)} = -12\zeta(3)$. Notably, using (7), (8), we were able to obtain a formula for the $\tilde{C}_k^{(\beta)}$ as a single infinite series [25], which shows that they are smooth as a function of the

Dyson parameter β, thus relaxing the assumption of rationality. As discussed below, the factors $A_\beta(t)$, hence $\tilde{C}_k^{(\beta)}$, are intimately but nontrivially related to the cumulants of the number of fermions (free for $\beta = 2$ and with Sutherland-type interaction for $\beta \neq 2$) in a mesoscopic interval of the circle.

By contrast the factors $C_k(\ell)$ are β independent and originate from the problem of the maximum of a fBm0 on the interval $[\theta_A, \theta_B]$. For the ℓ-dependent constants we obtain an explicit formula in two cases:

(i) maximum over the full circle $\ell = 2\pi$.—In that case $[\theta_A, \theta_B] =]-\pi, \pi]$ and we find for any $k \geq 2$

$$C_k(2\pi) = (-1)^k \frac{d^k}{dt^k}\bigg|_{t=0} \log\left[\frac{\Gamma(1+t)^2 G(2-2t)}{G(2-t)^3 G(2+t)}\right], \quad (9)$$

which is related to the fBm0 bridge on $]-\pi, \pi]$ studied in Ref. [19];

(ii) *maximum over a mesoscopic interval $1/N \ll \ell \ll 1$.*—For $k \geq 2$ we obtain in this regime

$$C_k(\ell) \simeq 2\log \ell\, \delta_{k,2} \quad (10)$$

$$+(-1)^k \frac{d^k}{dt^k}\bigg|_{t=0} \left[\frac{2\Gamma(1+t)^2 G(2-2t)}{G(2+t)^2 G(2-t) G(4-t)}\right]. \quad (11)$$

This result is related to the fBm0 on an interval, with one pinned and one free end, studied in Ref. [19]. Note that the variance depends logarithmically on ℓ at small ℓ, whereas higher cumulants have limits as $\ell \to 0$. Note that the $l \to 0$ limit is expected to provide the $L \gg 1$ asymptotic for statistics of the maximum of $\mathcal{N}_{\theta_A}(\theta)$ in intervals of the order $2\pi L/N$, comparable with the mean eigenvalue spacing. The universal statistics of CUE$_\beta$ eigenvalues at such local scales is described by the so called *sine-β* process [26] and the associated counting function has been studied in [27].

Finally, addressing the question of the *location* of the maximum in Eq. (3), $\theta_m \in [\theta_A, \theta_B]$, let us define $y_m = (\theta_m - \theta_A)/\ell$. For the mesoscopic interval, we predict the PDF of y_m to be symmetric around $\frac{1}{2}$, with $\mathbb{E}(y_m^2) = 17/50$ and $\mathbb{E}(y_m^4) = 311/1470$, thus deviating from the uniform distribution. For the full circle we find a uniform distribution for θ_m [28]. However, joint moments for the position *and* value of the maximum show the effect of pinning at $\theta = \theta_A$ (see details in Ref. [25]).

To elucidate the relation to fBm0, let us recall that the process $\delta N_{\theta_A}(\theta)$ is exactly given by the difference [25]

$$\delta\mathcal{N}_{\theta_A}(\theta) = \frac{1}{\pi}\mathrm{Im}\log\xi_N(\theta) - \frac{1}{\pi}\mathrm{Im}\log\xi_N(\theta_A), \quad (12)$$

where $\xi_N(\theta) = \det(1 - e^{-i\theta}U)$ is the characteristic polynomial (*CP*). As shown in Ref. [29] for $\beta = 2$ (see Ref. [30] for general $\beta > 0$) the joint probability density of $\mathrm{Im}\log\xi_N(\theta)$ at two distinct points $\theta_1 \neq \theta_2$ converges as $N \to +\infty$ to that of a Gaussian process $W_\beta(\theta)$ of zero mean and covariance

$$\mathbb{E}[W_\beta(\theta_1)W_\beta(\theta_2)] = -\frac{1}{2\beta}\log\left[4\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)\right], \quad (13)$$

a particular instance of the 1D log-correlated Gaussian field. Since Eq. (12) implies that $\delta\mathcal{N}_{\theta_A}(\theta = \theta_A) = 0$ in any realization, the relevant object is the pinned log-correlated process closely related to fBm0. The log-correlated fields being highly singular always require a regularization to study their value distribution. The imaginary parts of the $\log\xi_N(\theta)$ for $N \gg 1$ provides such a natural regularization [30–33], being asymptotically a random process W, which shares the covariance (13) but with a finite variance $\mathbb{E}[W(\theta)^2] = \beta^{-1}\log N + O(1)$. Via Eq. (12) this provides the well-known asymptotic of the eigenvalues or fermions number variance: $\mathbb{E}[\delta\mathcal{N}^2(\theta)] \simeq (2/\beta\pi^2)\log N$. We shall see, however [25], that naively replacing the difference $\delta\mathcal{N}_{\theta_A}(\theta)$ with its Gaussian approximation $1/\pi[W_\beta(\theta) - W_\beta(\theta_A)]$ (related to the *bosonization* of the fermionic problem) is not sufficient for characterizing the maximum of the process.

Gaussian fields characterized by a logarithmic covariance appear in chaos and turbulence [34], branching random walks and polymers on trees [21,22], multifractal disordered systems [35,36], two-dimensional gravity [37,38]. Early works on their *extrema* revealed a connection to a remarkable freezing transition [21,22,35]. Through exact solutions, it led to predictions for the PDF of the maximum value of a log-correlated field on the circle and on the interval [39,40], involving the freezing duality conjecture (FDC) (see Ref. [20] for an extensive discussion). This led to further results in theoretical and mathematical physics [23,41–45] and probability [24,46–54]. While the log-correlated context of random *CP* attracted a lot of attention [30,48–50,55–62], none of these studies yet addressed the eigenvalue or zeros counting function in the intervals $\ell = O(1)$.

To study the maximum of the random field $\delta\mathcal{N}(\theta)$ we follow Refs. [20,39,40,55,56] and introduce a statistical mechanics problem of the partition sum:

$$Z_b = \frac{N}{2\pi}\int_{\theta_A}^{\theta_B} d\phi\, e^{2\pi b\sqrt{\beta/2}\,\delta\mathcal{N}_{\theta_A}(\phi)}, \quad (14)$$

The "inverse temperature" is equal to $-2\pi b\sqrt{\beta/2}$, and we choose $b > 0$ since we are studying here the maximum retrieved from the free energy $\mathcal{F}$ for $b \to +\infty$ as

$$\delta\mathcal{N}_m = \lim_{b\to+\infty}\mathcal{F}, \qquad \mathcal{F} = \frac{1}{2\pi b\sqrt{\beta/2}}\log Z_b. \quad (15)$$

To study the statistics of the associated free energy we start with considering the integer moments of Z_b given by

$$\mathbb{E}[Z_b^n] = \left(\frac{N}{2\pi}\right)^n \int_{\theta_A}^{\theta_B} e^{-b\sqrt{\beta/2}\sum_{a=1}^n N(\phi_a-\theta_A)} \prod_{a=1}^n d\phi_a \times \mathbb{E}\left[\prod_{j=1}^N e^{2\pi b\sqrt{\beta/2}\sum_{a=1}^n [\chi(\phi_a-\theta_j)-\chi(\theta_A-\theta_j)]}\right]. \quad (16)$$

The expectation value in Eq. (16) over the $\mathrm{CUE}_\beta(N)$ computed using Eq. (1) has the form $\mathbb{E}[\prod_{j=1}^N g(\theta_j)]$, where we defined

$$\log g(\theta) = 2\pi b\sqrt{\beta/2}\sum_{a=1}^n [\chi(\phi_a-\theta)-\chi(\theta_A-\theta)]. \quad (17)$$

This can be further rewritten for any $\phi_a, \theta, \theta_A \in]-\pi,\pi]$ with $\phi_a > \theta_A$ as

$$\log g(\theta) = b\sqrt{\beta/2}\left[\sum_{a=1}^n \phi_a - n\theta_A + n\arg e^{i(\theta_A-\theta+\pi)} - \sum_{a=1}^n \arg e^{i(\phi_a-\theta+\pi)}\right], \quad (18)$$

where we define the arg function as

$$\arg e^{i\phi} = \begin{cases} \phi & -\pi < \phi \le \pi \\ \phi - 2\pi & \pi < \phi \le 3\pi. \end{cases} \quad (19)$$

For $\beta = 2$, $\mathbb{E}[\prod_{j=1}^N g(\theta_j)] = \det_{1\le j,k\le N}[g_{j-k}]$ is a Toeplitz determinant, where $g_p = \int_{-\pi}^{\pi}(d\theta/2\pi)e^{-ip\theta}g(\theta)$ is the associated symbol, and $g(\theta)$ according to Eqs. (18)–(19) has jump singularities. The corresponding asymptotics as $N\to\infty$ is given by the famous Fisher-Hartwig (FH) formula [63] proved rigorously in Ref. [64]. A general rational β extension of the FH formula has been conjectured in Ref. [65]. Specifying the expressions in Ref. [65] to our case gives for $N\to+\infty$ and $nb^2 < 1$

$$\mathbb{E}[Z_b^n] \simeq \left(\frac{N}{2\pi}\right)^n N^{b^2(n+n^2)}|A_\beta(b)|^{2n}|A_\beta(bn)|^2 \times \int_{\theta_A}^{\theta_B} \prod_{1\le a<c\le n} |1-e^{i(\phi_a-\phi_c)}|^{-2b^2} \times \prod_{1\le a\le n} |1-e^{i(\phi_a-\theta_A)}|^{2nb^2} \prod_{a=1}^n d\phi_a, \quad (20)$$

where the function $A_\beta(b)$ is defined in Eq. (8). Had we used instead an approximation replacing the difference $\delta\mathcal{N}_{\theta_A}(\theta)$ in the large-N limit with the logarithmically correlated Gaussian process $W_\beta(\theta)$ defined via Eqs. (12)–(13), we would reproduce the Coulomb gas factors in Eq. (20) but miss the factors $A_\beta(b)$; see Ref. [25]. Hence, this product encapsulates the residual non-Gaussianity of the process.

Let us first discuss the simplest case $n = 1$ when Eq. (20) can be interpreted, via Eq. (14), as giving

$$\mathbb{E}(e^{2\pi b\delta\mathcal{N}_{\theta_A}(\theta)}) \simeq N^{2b^2}|A_\beta(b)|^4\left(4\sin^2\frac{\theta-\theta_A}{2}\right)^{b^2}. \quad (21)$$

This formula can be interpreted as the generating function for the full counting statistics for the number of Sutherland-model fermions in an interval of size $\theta-\theta_A$, which seems not to be addressed in the literature apart from the free-fermion case $\beta = 2$ [2,66] and $\beta = 4$ [67].

Further progress is possible in the two cases when the Coulomb integrals in Eq. (20) can be explicitly calculated.

(i) *Full circle* $\theta_A = -\pi$, $\theta_B = \pi$.—In that case the Coulomb integral is known as the Morris integral [68] leading to

$$\mathbb{E}[Z_b^n] \simeq \left(\frac{N}{2\pi}\right)^n N^{b^2(n+n^2)}|A_\beta(b)|^{2n}|A_\beta(bn)|^2 \times \mathbf{M}(n, a=-nb, b), \quad (22)$$

where $\mathbf{M}(n,a,b)$ is defined as Eq. (14) in Ref. [19]. This result is valid in the high temperature phase with $nb^2 < 1$. From this expression for integer moments there is a well-defined procedure to obtain the double-sided Laplace transform (DSLT) of the free energy first in the high temperature phase $b < 1$ via an analytic continuation. Defining $t = -bn$ we obtain

$$\mathbb{E}(e^{-2\pi\sqrt{\beta/2}(\mathcal{F}-\mathcal{F}_1)t}) \simeq N^{-tQ+t^2}A_\beta(t)A_\beta(-t) \times \Gamma(1+tb)\frac{G_b(Q-2t)G_b(Q)^3}{G_b(Q-t)^3G_b(Q+t)}, \quad (23)$$

where $\mathcal{F}_1$ is a constant [69] and $Q = b + 1/b$ and $G_b(x)$ is the generalized Barnes function, see Eq. (44) in Refs. [40] and [70]. We note that if we multiply both sides of the equation by $\Gamma(1+t/b)$, the r.h.s. is invariant by duality $b\to 1/b$, since formally $G_b(z) = G_{1/b}(z)$. According to the FDC [20,40] we obtain the DSLT in the low temperature phase $b > 1$. The result can be written as

$$\mathbb{E}(e^{-2\pi\sqrt{\beta/2}\mathcal{F}t})\Gamma\left(1+\frac{t}{b}\right) = \mathbb{E}(e^{-2\pi\sqrt{\beta/2}\delta\mathcal{N}_m t}), \quad (24)$$

where the r.h.s. is our main result; i.e., the DSLT of the PDF of $\delta\mathcal{N}_m$ for the full circle [71]

$$\mathbb{E}(e^{-2\pi\sqrt{\beta/2}\delta\mathcal{N}_m t}) \simeq N^{-2t+t^2}e^{ct}A_\beta(t)A_\beta(-t) \times \frac{\Gamma(1+t)^2G(2-2t)}{G(2-t)^3G(2+t)}, \quad (25)$$

which, according to Eq. (15), is the $b \to +\infty$ limit of the left-hand side of Eq. (24). Here $c = \frac{3}{2}\log\log(N) + c'$ and c' is a constant that we cannot determine by this method. Expansion of Eq. (25) around $t = 0$ leads to the large N asymptotics (4)–(6) for the cumulants, together with the predicted values for the coefficients $\tilde{C}_k^{(\beta)}$ in Eq. (7) and $C_k(2\pi)$ in Eq. (9). The $C_k(2\pi)$ equal, up to a factor $(-1)^k$, the cumulants C_k given in Ref. [19] for the fBm0 bridge, checked against numerics there for $k = 2$, 3, 4. These coefficients are studied in more details in Ref. [25].

(ii) Mesoscopic interval.—A similar calculation gives the maximum over a mesoscopic interval $1/N \ll \ell \ll 2\pi$. Relegating the details to Ref. [25] we simply quote our second main result, the DSLT of the PDF of $\delta\mathcal{N}_m$ for the small interval limit of small $\ell \ll 1$:

$$\mathbb{E}(e^{-2\pi\sqrt{\beta/2}\delta\mathcal{N}_m t}) \simeq (N\ell)^{-2t+t^2} e^{ct} A_\beta(t) A_\beta(-t),$$

$$\Gamma(1+t)^2 \frac{2G(2-2t)}{G(2+t)^2 G(2-t) G(4-t)}, \tag{26}$$

where $c = \frac{3}{2}\log\log N + c''$. Expansion around $t = 0$ leads to the same coefficients $\tilde{C}_k^{(\beta)}$, which are thus independent of ℓ [as can be seen already from Eq. (16)] and to the result for $C_k(\ell)$ in Eq. (10), again related to the ones for the fBm0 on an interval given and numerically checked in Ref. [19].

In conclusion, we obtained the cumulants of the maximum of the deviation of the counting function from its mean on an interval, for eigenvalues of random unitary matrices and for free and interacting fermions on the circle. They inherit features both from the fBm0 log-correlated field and from the fermionic full counting statistics. Finally, our result for the distribution of $\delta\mathcal{N}_m$ provides a first step to study the Kolmogorov-Smirnov statistics for the counting staircases, which would further require the joint PDF of the maximum and minimum (usually nontrivially correlated [72]).

The results for the mesoscopic interval are expected to be universal for a broader class of random matrix ensembles, as well as for fermions on a lattice in the dilute limit [73]. Finally, it is natural to conjecture that for $\beta = 2$ universality extends to describing the statistics of the counting staircases for the nontrivial zeroes t_n of the Riemann zeta-function $\zeta(1/2 + it)$ in mesoscopic intervals of the critical line $t \in \mathbb{R}$. Such zeroes are known to be extremely faithful to the random matrix statistics when analyzed in appropriate scales [74] underlying a fruitful line of applications of associated CP to understand ensuing features of $\zeta(1/2 + it)$ [29,30,75–78].

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