Statistical Properties of the Quantum Internet

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Steady technological advances are paving the way for the implementation of the quantum internet, a network of locations interconnected by quantum channels. Here we propose a model to simulate a quantum internet based on optical fibers and employ network-theory techniques to characterize the statistical properties of the photonic networks it generates. Our model predicts a continuous phase transition between a disconnected and a highly connected phase and that the typical photonic networks do not present the small world property. We compute the critical exponents characterizing the phase transition, provide quantitative estimates for the minimum density of nodes needed to have a fully connected network and for the average distance between nodes. Our results thus provide quantitative benchmarks for the development of a quantum internet.

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Network science is a multidisciplinary field that offers a common language to study statistical properties of a variety of systems such as social, biological, and economical networks [1]. On the basis of its success is the fact that systems are seeing simply as graphs, i.e., a set of nodes interacting via edges. In this approach it is not the particular working or behavior of the individual constituents that matters, but how connected they are. This viewpoint led to the discovery that systems that are very different in nature, such as the internet, scientific collaborations, or protein networks, are very similar from a network perspective. Furthermore, understanding the network connectivity allows to design better man-made networks, such as power grids, transport networks or company organization.

A new type of communication network, the quantum internet, is currently under development [2–4]. It consists of distant parties connected by quantum channels through which quantum bits can be exchanged. This new network will boost our capabilities of communication by allowing the execution of protocols which are more efficient than their classical counterpart, or that have no classical analog whatsoever. The main example of such advantage is the possibility of securing messages with quantum cryptography [5], currently one of the most advanced quantum technologies. Other anticipated applications are clock synchronization [6] and private quantum computation on a cloud [7,8]. From a fundamental perspective, quantum networks will also allow us to reach physical phenomena that have no classical analog. An example is the distribution of entanglement across the network, which will allow distant parties to perform quantum teleportation or to establish correlations with no classical explanation and defy our notions of causality [9]. Assuming an underlying quantum network, a number of protocols for entanglement distribution and communication in quantum networks have been proposed [10–16]. However, to the best of our knowledge, such underlying quantum network and its properties have not been considered so far.

Here we propose a model to simulate the quantum internet assuming that it is going to be built from the current network of optical fibers (see for instance [17,18]). Our goal is twofold: first, to predict large-scale properties of typical photonic networks, such as their connectivity, nodes distance, and aggregation, and (ii) to provide quantitative estimates for useful quantities such as the minimum density of nodes needed to have a fully connected network and the typical network distances between nodes. Our findings predict a phase transition in the network connectivity as a function of the density of nodes: there is a critical density above which the network changes from being disconnected to presenting a giant connected cluster. We estimate the value of this critical density and the critical exponents characterizing the phase transition. Nicely, few nodes are needed to make photonic networks of realistic sizes fully connected. However, as opposed to the current internet [19], the quantum internet does not present the small world property. Notwithstanding, the typical network distances between nodes are small, implying that few entanglement swappings have to be employed to distribute entanglement between any two nodes.

Formally, a network model is defined by a set of N nodes being connected by vertices according to a given probabilistic rule. The central goal of network science is to understand the asymptotic properties of networks as the number of nodes increases. A particularly relevant example is given by random networks [20,21], defined by a model where every pair of node is connected with probability p [22]. The characteristic trait of random networks is that for sufficiently large number of nodes N, the probability of finding a node with k connections $P(k)$, called the degree distribution, can be approximated by a Poisson distribution $P(k) = [e^{-\langle k \rangle}\langle k \rangle^k/k!]$, where $\langle k \rangle = p(N-1)$ is the average connectivity of the network. Despite being very simple, the random network model presents very rich statistical phenomena. For instance, it displays a phase transition: there is a critical probability p_c such that if $p < p_c$ the network is composed by small and disconnected clusters and, if $p > p_c$ a giant cluster with size of same order of the whole network is present. Another striking feature is the appearance of a phenomenon known as *small world*. This refers to the fact that the average shortest path length (i.e., the shortest path between two nodes) scales logarithmically with N, meaning that the typical distances between pairs of nodes is very short compared to the size of the network.

Another important property of networks is the average clustering coefficient. It captures how the neighbors of each node are connected between them on average. Let us first define the local clustering coefficient of node i as $C_i = [2n_i/k_i(k_i - 1)]$, where n_i the number of edges between the k_i neighbors of the site i and $k_i(k_i - 1)/2$ is total possible number of edges between them. If $C_i = 0$ there is no links between the neighbors of i, while $C_i = 1$ indicates that the neighbors of i form a fully connected graph. The average clustering coefficient is defined as $\langle C \rangle = (1/N)\sum_i C_i$. For random networks $\langle C \rangle = \langle k \rangle/N$, showing a decrease with the network size.

In what follows we will propose a model to simulate the quantum internet and use it to predict these properties for photonic networks. As we will see, these networks present similarities and differences with the random networks.

Our model considers a network built from optical fibers, the main candidate to carry quantum information encoded in photons. Other technologies, such as quantum satellites, are also being considered and will probably be combined with the fiber-optics infrastructure [23,24]. Thus, the results presented here can be seen as benchmark to be improved by additional technologies. Our model is defined by the following steps: step 0—Nodes distribution. We first distribute N nodes uniformly in a disk of radius R (points at Fig. 1) [25]. Step 1—Fiber-optics network simulation. Following [26], we simulate how the optical fibers are distributed among these nodes using the Waxman model [27], a soft random geometric graph (RGG) with two sources of randomness [28]. Each pair of nodes i and j are

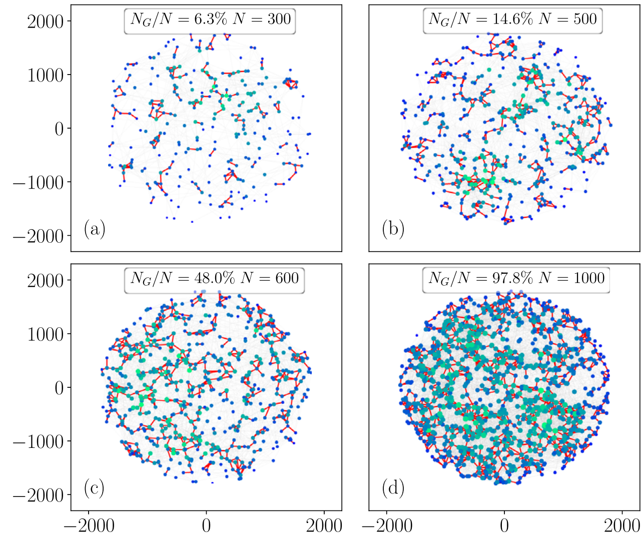

FIG. 1. Samples from the quantum internet. The grey edges represents the fiber-optics networks generated at step 1 (see main text). The red edges show the photonic links established in step 2. Greener (bluer) nodes are more (less) connected, following a Poisson distribution. N_G refers to the number of nodes belonging to the biggest cluster in the network, and N the total number of nodes. The plots considered $R = 1800$ km (giving, approximately, the U.S. area) and show that the biggest cluster consists of 97,8% of the nodes when $N = 1000$.

connected by a fiber (grey lines at Fig. 1) with probability given by $\Pi_{ij} = \beta e^{-d_{ij}/\alpha L}$, where d_{ij} is the Euclidean distance between i and j, L is the maximum distance between any two nodes, the parameter $\alpha > 0$ controls the typical edge length of the network (the maximum distance of two nodes directly connected), and $0 < \beta \leq 1$ controls the average degree of the network (see [29] for further generalizations of this connection rule). The constants have been estimated for particular optical fiber networks, such as for the U.S. fiber-optics network where $\alpha L = 226$ km and $\beta = 1$ [26,30], the values we use in the numerical simulations presented here. Notice, in particular, that L and α will depend on the radius R and the node's density ρ but $\alpha L = 226$ km is kept constant.

Step 2—Photonic network simulation. Once we generate the fiber-optics network we simulate the transmission of photons through it. It turns out that photonic losses increase exponentially with the fiber length [31]. More precisely, the transmissivity determining the fraction of energy received at the output of a fiber link connecting nodes i and j is given by $q_{ij} = 10^{-\gamma d_{ij}/10}$, where d_{ij} (km) is the Euclidean distance between i and j and the value of the fiber loss γ depends on the photon wavelength. For instance, for the silicon fiber, losses are minimized at the wavelength of 1550 nm, achieving $\gamma \simeq 0.2$ dB/km, the value we consider in our simulations. Even with further advances, the intrinsic physical loss limit of the silica optical fibers is estimated to be between 0.095 to 0.13 dB/km [32].

Given that two nodes share an optical fiber, we define the probability p_{ij} that two nodes are connected as

$$p_{ij} = 1 - (1 - q_{ij})^{n_p}. \quad (1)$$

The free parameter n_p controls how many photons are sent between each node in the attempt of generating a photonic link, i.e., two nodes are connected if at least one out of n_p photons is transmitted between them. Notice that, depending on the protocol one is interested in performing, extra parameters can enter in the analysis, such as the performance of quantum memories or computers. For an illustrative matter, we chose $n_p = 1000$ in the figures depicted here, as this value guarantees that connections over 100 km, the order of the state-of-art quantum communication experiments, are established. We highlight, however, that extensive simulations have been also performed with different values (see the Supplemental Material [33]), showing that the qualitative features described below of the photonic networks are universal and independent of the value of n_p.

Clearly, we are assuming that one cannot actively add new optical fibers to the existing network, thus our photonic quantum network is constructed in an inherently passive manner. In contrast, protocols for entanglement distribution [10–16] have a more active flavor since the strategies can be adapted depending on the specific network topology or channel capacity of a given link. Notice, however, that such protocols assume an underlying quantum network, over which they operate. To our knowledge, however, there is no specific model for such quantum network. That is precisely what Steps 0–2 above achieve.

All in all, the model described above defines a network where the nodes are uniformly distributed and connected with probability $P_{ij} = \Pi_{ij} p_{ij}$, i.e., each pair of nodes are connected if there is an optical fiber liking them and at least one photon is transmitted through it. So in essence, our model can be considered as two soft RGGs build one over the other. Considering the asymptotic regime $N \to \infty$, properties of particular RGGs (different from the two layers we consider here) have been widely studied [28,42–46]. Since we are interested in real finite networks, we have to rely on numerical simulations. In our simulations we sample 10^3 of such networks and calculate its typical properties which agree with known results in the asymptotic limit. Some samples of the networks generated by this algorithm are shown in the Fig. 1.

The first property we analyze is the degree distribution $P(k)$. As shown in Fig. 2 (top panels) and in agreement with the asymptotic results in [28], the degree distribution can be perfectly fitted a Poisson distribution that depends solely on the density of nodes ρ:

$$P(k) = \frac{e^{-A\rho}(A\rho)^k}{k!}, \quad (2)$$

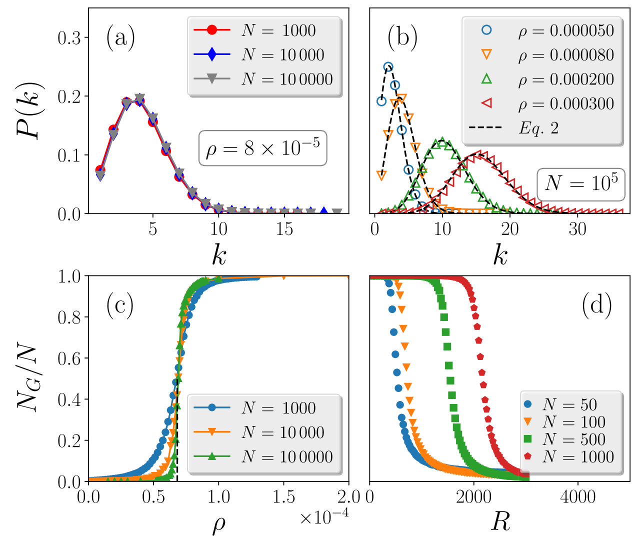

FIG. 2. Degree distribution and emergence of the giant cluster. (Top panel) (a) The degree distribution $P(k)$ for a fixed density value of $\rho = 8 \times 10^{-5}$ and several values of N. (b) $P(k)$ for a fixed value of N and several values of ρ. (Bottom panel) Relative size of the giant cluster as a function of (c) ρ (density). We see a clear phase transition at $\rho_c \approx 6.82 \times 10^{-5}$ corresponding to $\langle k \rangle_c \approx 3.56$. (d) The appearance of the giant cluster for moderate size networks covering a relatively large area.

where $A = 5.2 \times 10^4$ (see [33]). In this sense the quantum internet has a similar structure of a random network. In particular, the majority of nodes are connected to few other nodes, while very few nodes can make many connections.

Another similarity with random networks model is the existence of a phase transition (see bottom panels of Fig. 2): if the density of nodes is below a certain critical value, the network belongs to a disconnected phase, where only disconnected clusters of nodes are present. In this phase, nodes that belong to different clusters can not perform quantum protocols between them. However, if the density of nodes is above the critical value then a giant cluster containing all the nodes is present. In this phase, since there is a path between any pair of nodes, entanglement can in principle be distributed through entanglement swapping between the whole network. We have estimated the critical density to be $\rho_c \approx 6.82 \times 10^{-5}$ [see Fig. 2(c)], which corresponds to $\langle k \rangle_c \approx 3.5$ (as opposed to $\langle k \rangle_c = 1$ in random networks). Nicely, this critical density is quite small, implying that large areas can be connected by few nodes. For instance, Fig. 2(d) shows that the minimum number of nodes necessary to have a connected network in areas comparable to the U.S. or Europe are of the order of 1000 nodes. More generally, our results can be readily applied to infer what are the requirements of a specific quantum network for the appearance of the giant cluster. For instance, from Figs. 2(c) and 2(d) and given a certain number of nodes, one can extract what is the minimum density (alternatively, the maximum area) required for a fully connected network. If a given network is below the critical density, our results immediately show how many more nodes should be added to such a network (either using the existing optical fibers or perhaps adding new ones).

As showed in the Fig. 2(c), the relative size of the giant cluster displays a second order phase transition with respect to the density ρ. In [33] we show that $m \equiv \langle N_G \rangle / N$, at the critical density $\rho_c \simeq 6.82 \times 10^{-5}$, exhibits a power law behavior given by $m \sim (\rho - \rho_c)^{\beta}$, with the associated critical exponent $\beta \simeq 0.2$. Furthermore, we also analyzed the standard deviation of the size of the largest cluster, analogous to the susceptibility, defined by $\chi \equiv \sqrt{\langle N_G^2 \rangle - \langle N_G \rangle^2}$, the characteristic cluster size s^*, the cluster size distribution $n(s)$ and computed the associated critical exponents (see [33] for more details).

In spite of the previously discussed similarities between the quantum internet and random networks, we have observed two important differences. First, as shown in Fig. 3(a), the photonic quantum network does not display the small world property, since $\langle l \rangle$, the average shortest path length between pairs of nodes, grows faster than $\ln N$. We estimated that $\langle l \rangle$ depends of ρ and N following the relation $\langle l \rangle \simeq bN^{\alpha}/\rho$ with $b = 5 \times 10^{-5}$ and $\alpha \approx 1/2$ as can be seen in the fit of the Fig. 3(a). This result is in agreement and provides numerical confirmation for the conjecture in [47] about $\langle l \rangle$ in planar random networks. Nevertheless, as shown in Fig. 3(b), for moderate network sizes (expected from the first stages of implementation of the quantum internet) $\langle l \rangle$ is still small. This is very relevant, for instance, in entanglement distribution. As discussed in more details below, the smaller is this path, the smaller is the number of entanglement swappings required to connect any two nodes and thus the better is the amount of entanglement established among them.

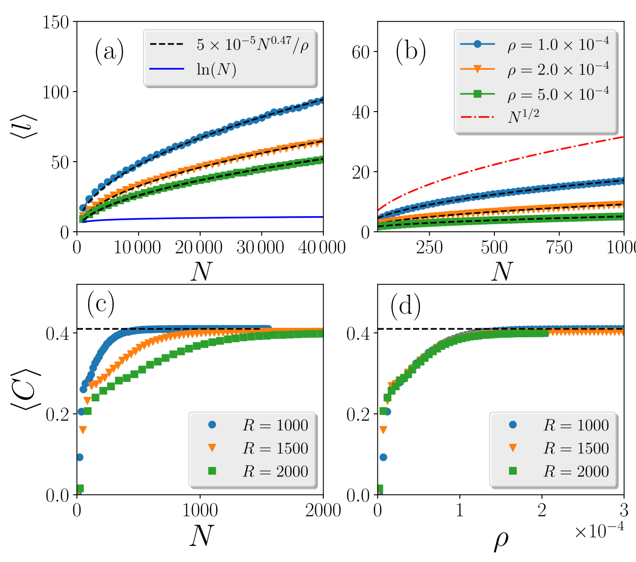

FIG. 3. Average shortest path and average clustering coefficient. (Top panel) (a) $\langle l \rangle$ as a function of N for various values of ρ. $\langle l \rangle$ grows faster than $\ln N$, showing no small-world phenomenon as expected for random networks. (b) However, the average shortest path for moderate size networks can be relatively small. (Bottom panel) (c) Average clustering coefficient $\langle C \rangle$ as a function of N for fixed values of R. $\langle C \rangle$ grows with N and decreases with R. (d) Plotting $\langle C \rangle$ as a function of ρ we see that all curves collapse, showing a universal behavior.

The average clustering coefficient of the photonic quantum networks also differs from the random network case. As we can see in Fig. 3(c), $\langle C \rangle$ increases with N independently of the radius R and reach a maximum value $\langle C \rangle \simeq 0.41$, in agreement with [48]. This means that the photonic quantum networks can be classified as very aggregated [1]. As consequence, if a given node i is connected to another distant node j, then with a very high probability any of the nodes in the local vicinity of i will also be connected to j through i. Furthermore, as shown in Fig. 3(d), all curves collapses when we plot $\langle C \rangle$ as function of ρ, pointing out the emergence of a universal behavior. This means that we can describe any curve of the clustering coefficient for any value of R with the same function.

In this Letter, we have proposed a model to study the properties of a quantum internet based on optical fiber technology. Using this model we predicted a phase transition, where there is a critical network density at which a giant cluster suddenly emerges. Crucially, the critical density separating the two phases is quite small, implying that few nodes are needed to hold a fully connected network in realistic areas. We also showed that, even though the generated networks are very aggregated locally, they do not lead to the small-world property. Although this might seem as a negative result, we also showed that for realistic networks sizes, the typical network distances between nodes are small. For instance, in a disk of radius $R = 800$ km, we would need around $N = 1000$ nodes to have a connected network, while keeping the average shortest path length of $\langle l \rangle \approx 5$.

Our findings have important implications for entanglement distribution. Suppose that the photonic links generating the networks are used to establish entanglement between the nodes (i.e., each link can be seen as an entangled pair of photons). In this case, it is possible to generate entanglement between two nodes only if there is a path connecting them (by performing entanglement swapping on intermediate nodes), that is, only if the nodes belong to the same cluster. Our results show that in the connected phase, any node can in principle become entangled to any other node, since all of them belong to the same giant cluster. Notice that, in practice, each entanglement swapping has the effect of damaging the final amount of entanglement between the end nodes. Thus, the fact that the number of intermediate nodes between any pair is small means that few entanglement swappings are necessary, which suggests the feasibility of entanglement distribution in such networks.

Our results give a novel perspective to analyze the quantum internet, providing an interdisciplinary bridge between quantum information and network theory. The present contribution should be seen as a starting point towards more complicated models. For instance, it would be interesting to see how the performance of other components such as quantum memories and quantum processors may affect the quantum network [49]. Another layer of complexity would be to consider the quantum features of the arriving photons, such as coherence and entanglement [50] and understand how entanglement distribution protocols [10–16] perform over our model for the quantum internet. Finally, it would be interesting to consider other technologies such as the use of satellites for quantum communication [23,51,52].

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