Wave-Turbulence Origin of the Instability of Anderson Localization against Many-Body Interactions

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Whether Anderson localization is robust against many-body interactions and, closely related, whether a disordered many-body system can be thermalized are long outstanding issues. In this Letter, we address these issues with the wave-turbulence theory. We show that, in general, the thermalization time in one-dimensional disordered lattice systems is inversely proportional to the squared interaction strength in the thermodynamic limit. It leads to the conclusion that such systems can always be thermalized by arbitrarily weak many-body interactions and thus the localized states are unstable.

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Introduction.—Anderson localization was originally derived for noninteracting disordered electron systems [1–3] and later also for disordered phonon systems with harmonic interactions [4,5], but whether it can survive nonlinear interactions [6–9] is still an outstanding problem. Most studies have suggested the existence of a threshold in the interaction strength [10–14] below which localization remains, while a few other works have suggested the absence of such a threshold [15–17]. Recently, it has been realized that the size of the system may play an essential role. In particular, it has been found that introducing the nonlinear interactions to a disordered electronic system can induce the "dephasing catastrophe" in the thermodynamic limit [18] and thus destroy many-body localization. Numerical evidence that many-body localization becomes unstable in the thermodynamic limit has also been found in studies of disordered spin chains [19,20] and a classical disordered nonlinear system [21].

This issue is closely related to the energy equipartition hypothesis (EEH) in statistical physics, which states that, in the thermodynamic limit, arbitrarily weak interactions may result in energy equipartition among all degrees of freedom. In a lattice system, it follows equivalently, that the energy initially distributed over a portion of normal modes will eventually spread over all normal modes. Obviously, the EEH implies localized states in a general system must be unstable. Conversely, if localized states are stable, the EEH fails.

Checking the validity of the EEH is of fundamental importance. It was initiated by the seminal work by Fermi *et al.* in the 1950s [22], and since then, extensive and intensive investigations have been done [23–42]. However, in spite of all these efforts, it is still inconclusive whether and under what conditions the EEH holds. In this Letter, we study one-dimensional (1D) disordered systems. We show analytically and verify numerically that the EEH is valid. As a result, Anderson localization is unstable against many-body interactions no matter how weak they are, given the system is large enough.

The Hamiltonian of our systems is

$$H = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m_i} + \frac{(q_{i+1} - q_i)^2}{2} + \frac{\lambda}{n}(q_{i+1} - q_i)^n \right), \quad (1)$$

where p_i and q_i, respectively, represent the momentum and the displacement from the equilibrium position of the ith atom of mass m_j. For convenience below, we rescale Hamiltonian (1) by energy density ε: $q_i = \tilde{q}_i \varepsilon^{1/2}$ and $H = \varepsilon \tilde{H}$, so that the parameter λ and ε has a scaling relation $\tilde{\lambda} = \lambda \varepsilon^{(n-2)/2}$. Here $\tilde{\lambda}$ represents the interaction strength. We adopt this model for two reasons. First, a general interaction potential can be expanded as the Taylor series of polynomial terms $(q_{i+1} - q_i)^n$. The theoretical prediction based on this model thus applies to a general 1D disordered lattice. For instance, we will provide in the following the numerical results for the disordered Lennard-Jones (LJ) lattice system with $V(x) = [1/(1+x)^6 - 1]^2/72$, which is frequently adopted for modeling real lattice systems. It will be seen that they are in good agreement with the predictions based on Eq. (1). Second, the polynomial potentials allow us to deal with the issue analytically by use of the wave-turbulence theory. Thanks to this advantage, we are able to obtain our key result; i.e., the equipartition time T_{eq} is inversely proportional to the squared interaction strength, $T_{\mathrm{eq}} \propto \tilde{\lambda}^{-2}$, in the thermodynamic limit.

A brief introduction to the wave-turbulence theory.—The wave-turbulence theory deals with a nonequilibrium statistical system of many randomly interacting waves. It was pioneered by Peierls in 1929 [43] and got its name, "wave turbulence" or sometimes "weak turbulence," in the 1960s [44,45]. For a detailed introduction, see Refs. [46,47].

In recent years, the wave-turbulence approach has been taken for studying energy equipartition in 1D homogeneous systems [31–36]. This approach has three key ingredients. First, the p-wave resonance conditions

$$k_1 \pm k_2 \pm \cdots \pm k_p \mod N = 0 \tag{2}$$

for the wave vectors and

$$\omega_1 \pm \omega_2 \pm \cdots \pm \omega_p = 0 \tag{3}$$

for the frequencies should be satisfied simultaneously. Here k_i and ω_i represent, respectively, the wave number and the frequency of the ith normal mode. Second, these p-wave resonances should be nontrivial and connected to form a network involving all N normal modes. Under such conditions, the irreversible transfer of energy among all normal modes will occur. Resonances satisfying the above conditions, but resulting in only a nonlinear frequency shift and not contributing to energy transfer among modes, as discussed, for example, in Ref. [31], are referred to as trivial resonances. Third, the nontrivial resonances of the p wave should dominate. As a result, energy equipartition is attributed to the p-wave nontrivial resonance, which distinguishes the wave-turbulence approach from other methods, such as the Chirikov resonance [28–30]. Note that the allowed resonances of the lowest order in (1) are n-wave resonances, but following the wave-turbulence approach they are not necessarily nontrivial. In the case of small system size, it is found that the size of the nontrivial resonance for homogeneous systems of $n = 3$ and $n = 4$ is $p = 6$ [31,32]. As a result, the scaling law of equipartition time is $T_{\mathrm{eq}} \propto \tilde{\lambda}^{-8}$ and $T_{\mathrm{eq}} \propto \tilde{\lambda}^{-4}$, respectively. In the thermodynamic limit, it has been shown that the dominant resonances are the n-wave ones ($p = n$) for homogeneous systems (1) of $n \geq 4$, which leads to $T_{\mathrm{eq}} \propto \tilde{\lambda}^{-2}$ [34–36]. Nevertheless, for homogeneous systems of $n = 3$, three-wave resonances are forbidden due to the dispersion relation, and thus the resonance of the lowest order in the thermodynamic limit is the four-wave resonance ($p = 4$), which leads to $T_{\mathrm{eq}} \propto \tilde{\lambda}^{-4}$. It has also been found that, in the case of an odd n, some deviations from wave-turbulence approach are observed [34,36].

Wave-turbulence approach for disordered many-body systems: Theoretical analysis.—Disorders enter the system (1) via random masses m_i, which fluctuate around $\langle m_i \rangle = 1$. In the present Letter, m_i is chosen independently and identically from a uniform distribution between $1 \pm \delta m$; the strength of disorder is thus characterized by δm. In general, normal modes of disordered systems can be obtained by diagonalizing the harmonic matrix, which is defined as

$$\boldsymbol{\Phi} = \Phi_{ij} = \frac{1}{\sqrt{m_i m_j}} \frac{\partial^2 H}{\partial q_i \partial q_j}\bigg|_{\mathbf{q}=\mathbf{0}}. \tag{4}$$

There exists a unitary transformation matrix $\mathbf{U}$, whose columns are the normal modes u^k, such that

$$\mathbf{U}^\dagger \boldsymbol{\Phi} \mathbf{U} = \boldsymbol{\Omega}^2, \tag{5}$$

where $\boldsymbol{\Omega}$ is a diagonal matrix whose elements are the normal mode frequencies, namely, $\Omega_{kl} = \omega_k \delta_{kl}$. Spectral index k follows an ascending order so that $\omega_k \leq \omega_{k+1}$.

We introduce the direct and inverse discrete transformation of the q_j variables,

$$\begin{aligned} Q_k &= \sum_j \sqrt{m_j}\, q_j u_j^k, \\ q_j &= \sum_k Q_k u_j^k / \sqrt{m_j}. \end{aligned} \tag{6}$$

With this transformation, the complex amplitude of a normal mode $a_k(t)$ is

$$a_k(t) = \frac{1}{\sqrt{2\omega_k}} (P_k - i\omega_k Q_k), \tag{7}$$

where $P_k = \dot{Q}_k$. Noting $a_{-k} = a_k$ and substituting Eqs. (6) and (7) into Eq. (1), we obtain

$$\tilde{H} = \sum_k \omega_k a_k a_k^* + \frac{\tilde{\lambda}}{n} \sum_{k_1,\ldots,k_n} A_{1,\ldots,n} \prod_{s=1}^{n} (a_{k_s}^* + a_{k_s}), \tag{8}$$

where the matrix $A_{1,\ldots,n}$ weights the transfer of energy among modes $k_1, k_2, \ldots, k_n$ and is given precisely by

$$A_{1,\ldots,n} = (-i)^n \tilde{A}_{1,\ldots,n} \prod_{s=1}^{n} \frac{\sqrt{2\omega_{k_s}}}{2\omega_{k_s}}, \tag{9}$$

where

$$\tilde{A}_{1,\ldots,n} = \sum_j \prod_{s=1}^{n} \left(\frac{u_{j+1}^{k_s}}{\sqrt{m_{j+1}}} - \frac{u_j^{k_s}}{\sqrt{m_j}} \right). \tag{10}$$

Then, the equation of motion for the k_1th complex normal mode reduces to

$$i\dot{a}_{k_1} = \omega_{k_1} a_{k_1} + \tilde{\lambda} \sum A_{1,\ldots,n} \prod_{s=2}^{n} (a_{k_s}^* + a_{k_s}). \tag{11}$$

Equation (11) has a Hamiltonian structure with canonical variables $\{ia_k, a_k^*\}$, describing the time evolution of the amplitudes of the normal modes of the system. To evaluate the equipartition time, it is convenient to introduce the wave action spectral density $D_i \delta_i^j = \langle a_{k_i} a_{k_j}^* \rangle$. Following the wave-resonance approach, we then obtain the n-wave kinetic equation in the thermodynamic limit and in the weak-nonlinearity limit (see Supplemental Material [48]),

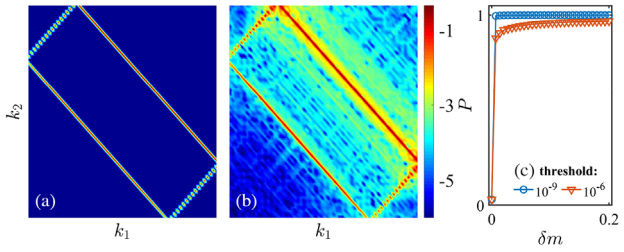

FIG. 1. The logarithm of $|\tilde{A}_{1,2,3}|$ in k_1-k_2 plane at $k_3 = 48$. (a) The homogeneous lattice systems and (b) the disordered systems. (c) The probability of nonvanishing $\tilde{A}_{1,2,3}$ versus the disorder strength.

$$\dot{D}_1 = \tilde{\lambda}^2 \int_{-1}^{1} |A_{1,\ldots,n}|^2 \mathfrak{F}(D_{1,n}) \delta(\omega_{1,n}) dk_2 \cdots dk_n, \quad (12)$$

where $\mathfrak{F}(D_{1,n})$ is a function of $D_1, D_2, \ldots, D_n$, and $\delta(\omega_{1,n})$ is the shorthand notation of delta function $\delta(\omega_{k_1} \pm \omega_{k_2} \pm \cdots \pm \omega_{k_n})$. One can easily show that the term $A_{1,\ldots,n}$ reduces to $\delta(k_{1,n})$ multiplying by a constant if the system is homogeneous, where the integral does not vanish when the spectral indices (namely, the wave numbers of homogeneous lattices) satisfy the resonance condition (2) with $p = n$. Otherwise, by applying the spatial translation invariance and the normal mode solutions to Eq. (10), one can prove that the integral must vanish. However, in disordered systems, the spatial translation invariance is broken and thus $A_{1,\ldots,n}$ does not vanish, in general. We take the system with $n = 3$ as an example to illustrate this fact. In Fig. 1(a) we plot $|\tilde{A}_{1,2,3}|$ calculated by Eq. (10) in k_1-k_2 plane at fixed $k_3 = 48$, in the case of a homogeneous lattice with $N = 64$. The amplitude of $|\tilde{A}_{1,2,3}|$ is represented by the gray level in the logarithm scale. These plots confirm that $|\tilde{A}_{1,2,3}|$ does not vanish only when the condition (2) is satisfied. Note that, due to the restriction of the dispersion relation, the conditions (2) and (3) cannot be satisfied simultaneously, resulting in the vanishing of the integral in Eq. (12). Therefore, the three-wave resonance is forbidden for the homogeneous lattice of $n = 3$.

In contrast, by introducing disorder with $\delta m = 0.2$, $\tilde{A}_{1,2,3}$ turns out to be nonzero almost in the entire k_1-k_2 plane, as shown in Fig. 1(b). In Fig. 1(c), we plot further the probability P of nonvanishing $\tilde{A}_{1,2,3}$ as a function of disorder strength δm. The probability is calculated by checking the amplitude of $\tilde{A}_{1,2,3}$ for all of the combinations of k_1, k_2, and k_3. When defining 10^{-9} to be the threshold below which $|\tilde{A}_{1,2,3}|$ is considered vanishing, we find that the probability jumps from $P = 0$ to $P = 1$ at $\delta m \neq 0$. Even when the threshold is increased up to 10^{-6}, the jump from $P = 0$ to nonzero P is still clearly seen. Therefore, once the spatial translation invariance is removed, $\tilde{A}_{1,2,3}$ does not vanish.

So only one restriction exists, which is the resonance condition (3) for the frequencies. For lattice systems, the normal frequencies are bounded. Specific to our system (1), frequencies are confined in the interval $[0, 2/\sqrt{1-\delta m}]$. Therefore, in the thermodynamic limit, the frequencies are dense. As such, the resonance condition (3) is always satisfied for $p = n$ and, based on it, all the modes form a connected network. For example, for $n = 3$, $\omega_1 - \omega_2 - \omega_3 = 0$ is expected for any given mode of ω_1 if $\omega_2 = \omega_1/2 + \delta$ and $\omega_3 = \omega_1/2 - \delta$, where δ is a constant. As functions of δ, ω_2 and ω_3 cover the interval of $[0, \omega_1]$ as δ varies. In other words, modes in this interval are all connected through ω_1. Hence, all the modes of the system are connected in this three-wave resonance network, which can be concluded by considering the case that ω_1 tends to the largest frequency $\omega_{\max} = 2/\sqrt{1-\delta m}$. So the condition required by the wave-turbulence approach is fully satisfied.

These analyses show that the n-wave resonances dominate the irreversible transfer of energy of disordered systems (1) in the thermodynamic limit. To get our main results, we rewrite Eq. (12) as

$$\dot{D}_1 = \eta_1 - \gamma_1 D_1, \quad (13)$$

where η_1 and γ_1 are D_1-independent constants proportional to $\tilde{\lambda}^2$. A detailed derivation of this formula is given in the Supplemental Material [48]. The satisfying of the resonance condition guarantees a nonzero γ_1, which leads to $D_1(t) \sim \exp(-\gamma_1 t)$ (note that $\eta_1 \sim 0$ in the weak interaction regime of small $\tilde{\lambda}$). Then, by defining the equipartition time T_{eq} as the characteristic time of relaxation, we obtain

$$T_{\text{eq}} \propto \tilde{\lambda}^{-2} \quad (14)$$

for $n \geq 3$. This is the key result of our study. It means that for a given $\tilde{\lambda}$ there is a finite T_{eq} above which energy equipartition is reached.

We would like to emphasize that the above approach is applicable to a general system with analytical potential. Indeed, by performing Taylor expansion of the potential, the second term on the right-hand side of Eq. (8) is replaced by a sum over n, so we can repeat the subsequent treatment straightforwardly. As a result, in such a case, though all of the multiwave resonances coexist, they do not change the scaling law. Meanwhile, we would like to point out that Eq. (14) cannot be promised for the models with external potentials (i.e., the so-called on-site potentials), because there is a truncation frequency $\omega_{\min}$ and in the extreme case that $\omega_{\min} > \omega_{\max}/2$, three-wave resonances are forbidden. However, one can expect it approximately holds when $\omega_{\min}$ is small enough.

Numerical verification.—Now we put the scaling law of T_{eq} into a numerical test. Strictly speaking, the above conditions for a finite system may not be met since frequencies of normal modes are discrete. However, in the presence of the nonlinearity, each frequency is broadened. When the system size is large enough, the frequency

interval can still be densely covered. Our numerical simulation below is to show that the scaling law (14) can be perfectly approached with the increase of the system size. Note that Eq. (14) can be written as $T_{\rm eq} \propto \tilde{\lambda}^{-2} = \lambda^{-2}\varepsilon^{(n-2)}$. For the sake of convenience, we fix λ to $\lambda = 1$ and investigate the dependence of $T_{\rm eq}$ on ε.

We adopt the method presented in Ref. [37] to calculate $T_{\rm eq}$. The energy of the kth mode is $E_k = (P_k^2 + \omega_k^2 Q_k^2)/2$. Thermalization is probed by $\xi(t) = 2\tilde{\xi}(t)e^{\eta(t)}/N$, where $\eta(t) = -\sum_{k=N/2}^{N} w_k(t)\log w_k(t)$ is the spectral entropy, where $w_k = E_k(t)/\sum_{l=N/2}^{N} E_l(t)$, $\tilde{\xi}(t) = 2\sum_{N/2}^{N} \bar{E}_k(t)/\sum_{1}^{N} E_k(t)$, and $\bar{E}_k(T) = (1/[(1-\mu)T])\int_{\mu T}^{T} E_k(P(t), Q(t))dt$ is the average energy of the kth normal mode. Here, parameter μ controls the size of the time window for averaging and is fixed at $\mu = 2/3$ in simulation. The equipartition time $T_{\rm eq}$ is measured as the time when $\xi(T_{\rm eq}) = 1/2$.

In simulations, we use the eighth-order Yoshida method [49] to integrate the equations of motion. The typical integration time step is set to be $\Delta t = 0.1$. In order to reduce the fluctuations, as done in Ref. [37], an average over 120 random initial states is performed for each realization of disorder. In the following, when no confusion arises, we use $\bar{E}_k(t)$ or $\xi(t)$ to denote a random variable and its average on initial states.

Figure 2 shows the energy $\bar{E}_k(T)$ at different times. It is for the system of $n = 3$ with $N = 1023$, $\delta m = 0.2$, and $\varepsilon = 10^{-4}$. Energy initially concentrates on 10% of modes of the lowest and highest frequencies, respectively, in Figs. 2(a) and 2(b). Note that the normal modes of high frequency are localized, while those of low frequency are extended. We see that in both cases equipartition eventually occurs. The metastable state, in which $\bar{E}_k(T)$ is hardly changed in a very broad range of time and has been found in the homogeneous Fermi-Pasta-Ulam-Tsingou lattice [38] and φ^4 model [39], is not found here. This phenomenon is similar to what was found in the homogeneous Frenkel-Kontorova model [40]. Furthermore, equipartition is seen to occur even when only an extended mode or a localized mode is excited. Figures 2(c) and 2(d) show, respectively, the results of the energy spectrum when we initially excite the mode of the lowest and the highest frequency.

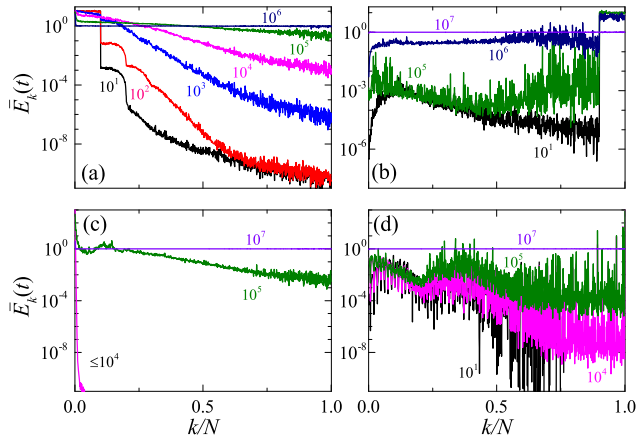

FIG. 2. The function $\bar{E}_k(T)$ versus k/N at different times for a system of $n = 3$ with $N = 1023$, $\delta m = 0.2$, and $\varepsilon = 10^{-4}$. Energy is initially distributed among (a) 10% of modes of the lowest frequency, (b) 10% of modes of the highest frequency, (c) on a mode of the lowest frequency, and (d) on a mode of the highest frequency, respectively.

In Fig. 3(a), we show $T_{\rm eq}$ as a function of ε for disordered systems of $n = 3$, 4, and 5 with $\delta m = 0.2$ and different system sizes, when energy initially concentrates on the 10% of modes of the lowest frequencies. It can be seen that, the larger the system size is, the better the scaling law: $T_{\rm eq} \propto \varepsilon^{-(n-2)}$ agrees with the data, meanwhile, the lower the energy density is, the larger the size must be to converge to the theoretical prediction. These facts lead us to conclude that the scaling law (14) is exact for arbitrarily low-energy density or arbitrarily weak nonlinearity in the thermodynamic limit. For a finite system, the deviation from the universal scaling law may appear with the decrease of energy density (see Supplemental Material [48] for more evidence). This deviation is similar to that observed in homogeneous lattices [32,33,50]; it suggests that a threshold for equipartition might exist. Note that $T_{\rm eq}$ increases as a function of n since in the low-energy density the amplitude of the variable is smaller than 1 and thus the perturbation amplitude decreases with the increase of n.

Finally, we report the result of the disordered LJ model. Figure 3(b) indicates that $T_{\rm eq} \propto \varepsilon^{-1}$, which is consistent with the prediction for $n = 3$. This follows from the fact that the lowest order of nonlinearity in the Taylor expansion of its potential is the cubic term. In the low-energy density regime, the cubic term is much larger than higher-order terms and thus dominates the equipartition process. We see that this leads to a key difference from the case in the homogeneous counterpart, where $T_{\rm eq} \propto \varepsilon^{-2}$ due to the

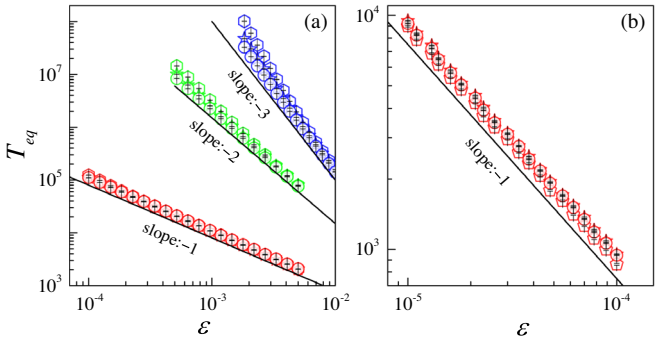

FIG. 3. The equipartition time $T_{\rm eq}$ as a function of ε in log-log scale for (a) systems (1) of $n = 5$ (top), 4 (middle), 3 (bottom), and (b) LJ lattice systems, at system sizes of $N = 511$ (hexagon), 1023 (star), 2047 (circle). All simulation results are obtained with $\delta m = 0.2$. Energy is initially distributed among 10% of modes of the lowest frequencies.

absence of the three-wave resonance [34]. This implies, contrary to the common belief, that the disorder can accelerate the process of thermalization in the weak-nonlinearity regime.

Conclusion and discussion.—In summary, we have shown that, in a general 1D disordered lattice system, localized states are unstable in the presence of interactions and EEH holds. Moreover, in the frame of wave-turbulence theory, thermalization is dominated by multiwave resonances, thus the scaling exponent of thermalization time is determined by the number of the waves and is independent of the spatial dimension. Therefore, energy equipartition can be expected in higher-dimensional systems because the scaling law of the thermalization time we have obtained should also be valid. It is important to note that the thermodynamic limit is a key premise of wave-turbulence analysis. Besides, it is valid for weak nonlinearity and in the models without on-site potentials. This fact implies three situations where stable localized states might be observed, i.e., in a system of a finite size and/or with strong interactions or in a system with an on-site potential. These possibilities deserve further investigation. In addition, as a theory of interacting waves, the applications of the wave-turbulence analysis to other important and interesting issues, such as heat transport in the classical lattice and thermalization of the quantum systems, are also desired.

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