Editors' Suggestion

Stationary Black Holes and Light Rings

Pedro V. P. Cunha[1] and Carlos A. R. Herdeiro[2]
[1]*Max Planck Institute for Gravitational Physics—Albert Einstein Institute, Am Mühlenberg 1, Potsdam 14476, Germany*
[2]*Departamento de Matemática da Universidade de Aveiro and CIDMA, Campus de Santiago, 3810-183 Aveiro, Portugal*

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The ringdown and shadow of the astrophysically significant Kerr black hole (BH) are both intimately connected to a special set of bound null orbits known as light rings (LRs). Does it hold that a *generic* equilibrium BH *must* possess such orbits? In this Letter we prove the following theorem. A stationary, axisymmetric, asymptotically flat black hole spacetime in $1+3$ dimensions, with a nonextremal, topologically spherical, Killing horizon admits, at least, one standard LR outside the horizon for each rotation sense. The proof relies on a topological argument and assumes C^2 smoothness and circularity, but makes no use of the field equations. The argument is also adapted to recover a previous theorem establishing that a horizonless ultracompact object must admit an even number of nondegenerate LRs, one of which is stable.

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Introduction.—The second decade of the 21st century will be celebrated as the dawn of precision strong gravity. New observational data is testing, in particular, the true nature of astrophysical black holes (BHs). Both gravitational wave observations [1,2], notably through the ringdown phase, and electromagnetic observations, in particular through the shadow imaging [3–5], are expected to provide hitherto inaccessible information on the BH spacetime geometry.

The ringdown and shadow observables are both intimately connected to a special set of bound null orbits for test particles [6,7]. When planar, these orbits are known as *light rings* (LRs). They are an extreme form of light deflection, such that the path of light closes over itself. In the general nonplanar case these light paths are dubbed fundamental photon orbits (FPOs) [8]. For a spherical BH, such as the Schwarzschild solution, all FPOs are LRs. This is not so for axisymmetric, but nonspherical, BHs. In the special case of the Kerr spacetime, the FPOs are known as spherical photon orbits [9], all of which are unstable (in the radial direction) outside the horizon and reduce, in two appropriate limits, to LRs. The latter correspond to equatorial photon orbits which are corotating or counterrotating with the Kerr horizon.

The close connection between LRs and the aforementioned key observables raises the following question: Does an equilibrium BH spacetime always possess LRs? This is the case for the paradigmatic electro-vacuum BHs of General Relativity (GR), but can one safely extrapolate to BHs with generic matter contents or modified gravity?

In this Letter we shall provide a generic and robust answer to these questions using a topological argument. Concretely, under reasonable assumptions, we shall establish the following theorem: A stationary, axisymmetric, asymptotically flat, $1+3$ dimensional BH spacetime, $(\mathcal{M},g)_{\rm BH}$, with a nonextremal, topologically spherical Killing horizon, $\mathcal{H}$, admits at least one standard LR outside the horizon for each rotation sense.

The spacetime.—We assume an equilibrium BH spacetime under the conditions of the last paragraph. No assumption is made on the field equations $(\mathcal{M},g)_{\rm BH}$ solves. This spacetime possesses two Killing vectors $\{\xi,\eta\}$, associated, respectively, to stationarity and axisymmetry. Asymptotic flatness implies $\{\xi,\eta\}$ must commute [10]. Then, coordinates (t,φ) adapted to the Killing vectors $\xi=\partial_t, \eta=\partial_\varphi$ can be chosen. In addition, we assume that the metric is at least C^2-smooth on and outside $\mathcal{H}$, and circular. The latter, together with asymptotic flatness, implies the spacetime admits a 2-space orthogonal to $\{\partial_t,\partial_\varphi\}$—see, e.g., Theorem 7.11 in Ref. [11]. This means the metric g possesses a discrete symmetry $(t,\varphi)\to(-t,-\varphi)$ [12].

In the orthogonal 2-space one can introduce spherical-like coordinates (r,θ). The sections of $\mathcal{H}$ are assumed to be topologically spherical. A gauge choice guarantees the horizon is located at a constant (positive) radial coordinate $r=r_H$. The polar coordinate θ is chosen to be always orthogonal to r. In such a gauge, $g_{r\theta}=0$, $g_{rr}>0$ and $g_{\theta\theta}>0$ outside $\mathcal{H}$. One can further require that (r,θ) reduce to standard spherical coordinates in the asymptotically flat limit $r\to\infty$. The coordinates' range is then, outside the horizon, $r\in[r_H,\infty[$, $\theta\in[0,\pi]$ with $\theta=\{0,\pi\}$ at the rotation axis, $\varphi\in[0,2\pi[$ and $t\in]-\infty,+\infty[$. Outside $\mathcal{H}$, causality requires $g_{\varphi\varphi}\geq 0$. The metric, which has a Lorenzian signature $(-,+,+,+)$, thus reads $ds^2=g_{tt}dt^2+2g_{t\varphi}dtd\varphi+g_{\varphi\varphi}d\varphi^2+g_{rr}dr^2+g_{\theta\theta}d\theta^2$.

The Killing horizon.—The existence of $\mathcal{H}$ means there is a Killing vector field, $\chi=\partial_t+\omega_H\partial_\varphi$, ($\omega_H=$ const) that is null on $\mathcal{H}$, $(\chi^\mu\chi_\mu)|_{\mathcal{H}}=0$. Then, χ is the horizon null generator. For stationary BHs, one can further introduce a (positive) constant quantity on $\mathcal{H}$, the surface gravity κ, defined via the following relation computed at the horizon $[\nabla_\mu(\chi^2) = -2\kappa\chi_\mu]|_{\mathcal{H}}$. Taking $\mu \in \{t, \varphi\}$, one obtains $0 = (g_{\mu t} + g_{\mu\varphi}\omega_H)|_{\mathcal{H}}$. This implies that $\omega_H = -(g_{t\varphi}/g_{\varphi\varphi})|_{\mathcal{H}}$, for the horizon angular velocity ω_H, and $D|_{\mathcal{H}} = 0$, where we have defined $D \equiv (g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi})$. Thus, D vanishes on $\mathcal{H}$; in fact, it is positive outside the horizon and away from the axis [13].

LRs and a topological charge.—For diagnosing the occurrence of LRs in $(\mathcal{M}, g)_{\rm BH}$, one must consider the null geodesic flow. Following Ref. [14], LRs are identified by considering the effective potentials on the orthogonal 2-space, $H_\pm$:

$$H_\pm(r,\theta) \equiv \frac{-g_{t\varphi} \pm \sqrt{D}}{g_{\varphi\varphi}}. \tag{1}$$

LRs are critical points of $H_\pm$ [15]; a LR obeys either $\partial_\mu H_+ = 0$ or $\partial_\mu H_- = 0$ or both simultaneously (e.g., for static spacetimes) [16]. The $\pm$ sign is typically associated with the two possible rotation senses (see Supplemental Material [17], Sec. I).

We can associate a topological charge to LRs. First, introduce a field $\mathbf{v} = (v_r, v_\theta)$ as a normalized gradient of $H_\pm$:

$$v_r \equiv \frac{\partial_r H_\pm}{\sqrt{g_{rr}}}, \qquad v_\theta \equiv \frac{\partial_\theta H_\pm}{\sqrt{g_{\theta\theta}}}. \tag{2}$$

If follows that $\partial^\mu H_\pm \partial_\mu H_\pm = v_r^2 + v_\theta^2 \equiv v^2$. Hence, in terms of $\mathbf{v}$, a LR occurs if and only if $\mathbf{v} = 0 \Leftrightarrow v = 0$.

Second, define an angle Ω such that $v_r = v\cos\Omega$, $v_\theta = v\sin\Omega$. Then, Ω together with the "norm" v, parameterizes the auxiliary 2-space spanned by $\mathbf{v}$, denoted $\mathcal{V}$.

Third, in the physical orthogonal 2-space (r,θ), consider a simple closed curve C, that is piecewise smooth and positive oriented. Since C is closed, the angle Ω after a full revolution must be the same, modulo 2π. Hence,

$$\oint_C d\Omega = 2\pi w, \qquad w \in \mathbb{Z}. \tag{3}$$

In the physical (r,θ) space w counts the winding number of $\mathbf{v}$ as C is circulated in the positive sense. When C encloses a single (nondegenerate [14,18]) LR, the integer w is the topological charge of the LR. Indeed, the curve C, in the physical (r,θ) space, defines a curve $\tilde{C}$ in $\mathcal{V}$, via Eq. (2). In $\mathcal{V}$, w is the winding number of $\tilde{C}$ around the origin ($v = 0$), which corresponds to a LR. Thus, in $\mathcal{V}$, w constitutes a well-defined topological quantity [19]: deforming $\tilde{C}$ without crossing the origin does not change w. Consequently, in the physical (r,θ) space, deforming C without crossing a LR does not change w.

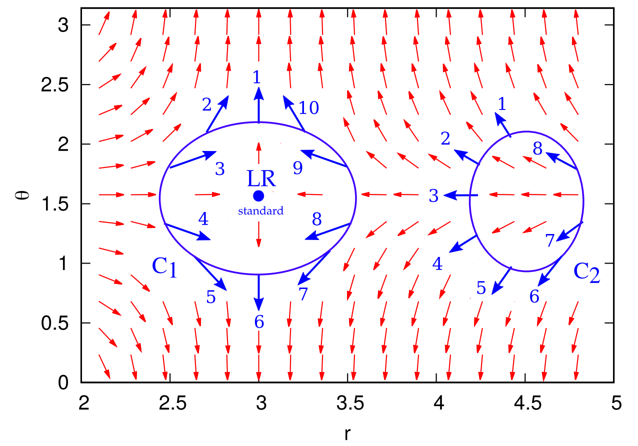

FIG. 1. The red arrows represent $\mathbf{v}$ (normalized to unity), defined from Eq. (2) with $H_+ = \sqrt{1-2/r}/(r\sin\theta)$, on a portion of the (r,θ) plane for the Schwarzschild BH with unit mass, in standard coordinates. The LR sits at $r = 3$, $\theta = \pi/2$. Circulating the contour C_1 (or any contour that encloses the LR) anticlockwise, $\mathbf{v}$ winds once clockwise (follow the blue arrows $1 \to 10$). Thus $w = -1$. By contrast, circulating the contour C_2 (or any contour that does not enclose the LR) anticlockwise, $\mathbf{v}$ has no winding. Thus $w = 0$. Observe two important properties that will be general. (1) $\mathbf{v}$ becomes vertical at $\theta = 0$ ($\theta = \pi$) and downwards (upwards) directed; (2) v_r is positive (negative) as the horizon (asymptotic infinity) is approached. The signs are reversed for H_-.

Figure 1 exhibits $\mathbf{v}$ for a Schwarzschild BH. It illustrates that $w = -1$ ($w = 0$) for any contour that encloses (does not enclose) the Schwarzschild LR. In general, if C encloses a single saddle point (maximum or minimum) of the potential $H_\pm(r,\theta)$, then $w = -1$ ($w = +1$). A LR with $w = -1$ ($w = +1$) is dubbed standard (exotic). LRs in Schwarzschild and Kerr are standard. Furthermore, for any C, the total w is the sum of the individual LR charges within C. In particular, if there are no LRs within C, then $w = 0$.

Our task is to show that the total LR topological charge in the region outside a BH (under the assumptions stated above) is $w = -1$, regardless of choosing H_+ or H_-. This implies that at least one standard LR must exist within that region, for each rotation sense of the BH, and establishes the theorem. To achieve this we must select an appropriate contour.

The contour.—For our generic $(\mathcal{M}, g)_{\rm BH}$, we define a contour C that encompasses a subregion $\mathcal{I}$ of the orthogonal 2-space exterior to the horizon. Then, taking appropriate limits, $\mathcal{I}$ becomes the full exterior region.

The region $\mathcal{I}$ is shown in Fig. 2 and it is defined as $r_0 \leq r \leq R$ and $\delta \leq \theta \leq \pi - \delta$. The constants $\{r_0, R, \delta\}$ are such that $r_H < r_0 \ll R$ and $0 < \delta \ll 1$.

$\mathcal{I}$ is the region enclosed by the curve C (see Fig. 2), which is defined as the union of four line segments: $\{r = R, \delta \leq \theta \leq \pi - \delta\} \cup \{\theta = \pi - \delta, r_0 \leq r \leq R\} \cup \{r = r_0, \delta \leq \theta \leq \pi - \delta\} \cup \{\theta = \delta, r_0 \leq r \leq R\}$.

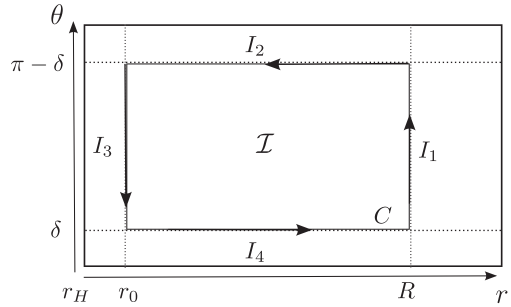

FIG. 2. Representation of the contour C (which encloses $\mathcal{I}$) on the (r, θ) plane. The curve C has positive orientation and it is composed by four line segments.

The topological charge of $\mathcal{I}$ is computed from Eq. (3), decomposed as $2\pi w_{\mathcal{I}} = I_1 + I_3 + I_2 + I_4$, where

$$I_1 = \left[\int_{\delta}^{\pi-\delta} \frac{d\Omega}{d\theta} d\theta\right]_{r=R}, \quad I_2 = \left[\int_{R}^{r_0} \frac{d\Omega}{dr} dr\right]_{\theta=\pi-\delta}, \tag{4}$$

$$I_3 = \left[\int_{\pi-\delta}^{\delta} \frac{d\Omega}{d\theta} d\theta\right]_{r=r_0}, \qquad I_4 = \left[\int_{r_0}^{R} \frac{d\Omega}{dr} dr\right]_{\theta=\delta}. \tag{5}$$

To obtain the total topological charge of the exterior region, we take first $\delta \to 0$ (axis limit), and only then $r_0 \to r_H$ (horizon limit) and $R \to +\infty$ (asymptotic limit):

$$w = \lim_{R\to+\infty} \lim_{r_0 \to r_H} (\lim_{\delta\to 0} w_{\mathcal{I}}). \tag{6}$$

These limits must be taken with care, as we now discuss.

Axis limit.—The axis is the set of points for which $g_{\varphi\varphi} = \eta \cdot \eta = 0 = \eta \cdot \xi = g_{t\varphi}$. To approach the axis, introduce a local coordinate ρ, defined as $\rho \equiv \sqrt{g_{\varphi\varphi}}$ (recall $g_{\varphi\varphi} > 0$ outside $\mathcal{H}$). Clearly, $d\rho/d\theta$ is positive (negative) as $\theta \to 0$ $(\theta \to \pi)$. Then, consider a small ρ expansion close to the axis:

$$g_{\varphi\varphi} = \rho^2, \qquad g_{t\varphi} \simeq b_o \rho^n + \mathcal{O}(\rho^{n+1}), \tag{7}$$

$$g_{tt} \simeq g_{tt}^0 + \mathcal{O}(\rho), \qquad g_{\rho\rho} \simeq g_{\rho\rho}^0 + \mathcal{O}(\rho), \tag{8}$$

where $n \in \mathbb{N}$ and some constants were introduced. By assuming C^2 smoothness and regularity (e.g., a nondiverging Ricci scalar) close to the axis $g_{\varphi\varphi}$ cannot go to zero faster than $g_{t\varphi}$ in the axis limit (see Supplemental Material [17], Sec. II and Ref. [20]). Then $2 \le n$ and $\rho^{2n} \ll \rho^2$. It follows from the definition of D that $\sqrt{D} \simeq \rho\sqrt{-g_{tt}^0}$. Hence, from Eq. (1):

$$H_\pm \simeq \pm \frac{\sqrt{-g_{tt}^0}}{\rho}. \tag{9}$$

One can now estimate $\mathbf{v}$ from Eq. (2). In particular, using $g_{\rho\rho} d\rho^2 \simeq g_{\theta\theta} d\theta^2$ at zeroth order in ρ:

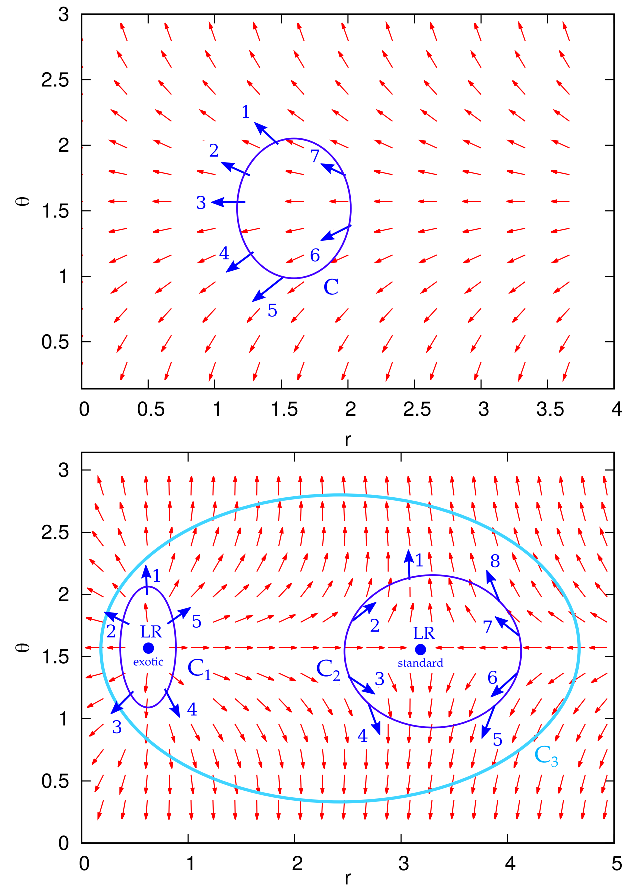

FIG. 3. Top: The red arrows represent $\mathbf{v}$, defined from Eq. (2) with $H_+ = 1/(r\sin\theta)$, on a portion of the (r, θ) plane for flat spacetime, in standard coordinates. There are no LRs. Observe the key difference with respect to Fig. 1. Here, v_r is negative (positive for H_-) as the left boundary of the domain is approached, which is now a regular origin at $r = 0$, rather than a horizon. Any contour C will have $w = 0$. Bottom: $\mathbf{v}$ (defined from H_+) for a horizonless ultracompact object (rotating boson star). There are two LRs with opposite topological charge. By circulating the contour C_1 (C_2) anticlockwise, $\mathbf{v}$ winds once in the positive (negative) sense (follow the numbered blue arrows). By contrast, after a full circulation along C_3, which encompasses both LRs, $\mathbf{v}$ winds up zero times $(w = 0)$.

$$v_\theta \simeq \mathrm{sign}\left(\frac{d\rho}{d\theta}\right) \frac{\partial_\rho H_\pm}{\sqrt{g_{\rho\rho}}} \sim \mp\, \mathrm{sign}\left(\frac{d\rho}{d\theta}\right) \frac{1}{\rho^2}. \tag{10}$$

Since $v_\theta \sim \rho^{-2}$ and $v_r \sim \rho^{-1}$, then $v_\theta^2 \gg v_r^2$, and so $v \simeq |v_\theta|$. Hence as $\rho \to 0$ one obtains $v_\theta / v \to \mp\, \mathrm{sign}(d\rho/d\theta)$. Consequently,

$$\Omega = \arcsin\left(\frac{v_\theta}{v}\right)\Big|_{0,\pi} \to \begin{cases} \pm\pi/2 & \text{for } \theta \to \pi \\ \mp\pi/2 & \text{for } \theta \to 0. \end{cases} \tag{11}$$

The axis limit is $\lim_{\delta\to 0} C$, which implies $\rho \to 0$ along the integration paths of $\{I_2, I_4\}$. Thus, the bottom line is that Ω becomes constant along the integration path.

Consequently, the contribution of $\{I_2, I_4\}$ to w *vanishes* as $\delta \to 0$.

This result can be interpreted as follows. In a generic BH spacetime, the arrows analogous to those in Fig. 1 become vertical along $\{I_2, I_4\}$ as $\delta \to 0$, directed upwards (downwards) at $\theta = \pi$ and downwards (upwards) at $\theta = 0$, for H_+ (H_-). Hence, the integration along these paths does not contribute to the winding of $\mathbf{v}$, as C is circulated.

Horizon limit.—To address the horizon limit ($r_0 \to r_H$) we observe that, as discussed in Ref. [21], the metric near the Killing horizon of a generic stationary and axially symmetric BH is fairly constrained if we require regularity (e.g., finite Ricci scalar at horizon). If the BH is not extremal ($\kappa \neq 0$), we can set a local radial coordinate x such that $g_{xx} = 1$ and $x|_{\mathcal{H}} = 0$ at the horizon. We also define $N = \sqrt{D/g_{\varphi\varphi}}$ and $\omega = -g_{t\varphi}/g_{\varphi\varphi}$, which yields $H_\pm = \omega \pm N/\sqrt{g_{\varphi\varphi}}$. Then, near the horizon [21]

$$\omega \simeq \omega_H + \mathcal{O}(x^2), \quad N \simeq \kappa x + \mathcal{O}(x^3), \quad g_{\varphi\varphi} \simeq g^H_{\varphi\varphi} + \mathcal{O}(x^2). \tag{12}$$

This leads to

$$\partial_x H_\pm \simeq \pm \frac{\kappa}{\sqrt{g^H_{\varphi\varphi}}} + \mathcal{O}(x). \tag{13}$$

Since $(1/\sqrt{g_{xx}})(\partial/\partial x) = (1/\sqrt{g_{rr}})(\partial/\partial r)$, then near the horizon ($x \simeq 0$):

$$v_r = \frac{\partial_r H_\pm}{\sqrt{g_{rr}}} \simeq \pm \frac{\kappa}{\sqrt{g^H_{\varphi\varphi}}}. \tag{14}$$

Thus, we have the following horizon limit:

$$\mathrm{sign}(v_r)|_{\mathcal{H}} = \pm 1. \tag{15}$$

This is sufficient for our purpose. It means that $\mathbf{v}$ has a positive (negative) radial component along I_3 for H_+ (H_-), in the horizon limit. By continuity, along I_3 $\mathbf{v}$ interpolates between an upwards (downwards) directed $\mathbf{v}$ at the intersection with I_2—see Fig. 1—and a downwards (upwards) directed $\mathbf{v}$ at the intersection with I_4, for H_+ (H_-). Its positive (negative) radial component along I_3, means $\mathbf{v}$ winds in the negative, i.e., clockwise, direction along I_3, producing half of a full winding. Thus

$$\Omega^{\mathcal{H}}_{\theta=0} - \Omega^{\mathcal{H}}_{\theta=\pi} = -\pi. \tag{16}$$

Asymptotic limit.—Finally consider the limit $R \to \infty$ (integration path of I_1). One reaches flat spacetime in standard spherical coordinates, yielding

$$v_r \simeq \mp \frac{1}{r^2 \sin\theta} \Rightarrow \mathrm{sign}(v_r)|_\infty = \mp 1. \tag{17}$$

Again, this information suffices: $\mathbf{v}$ has a negative (positive) radial component along I_1 for H_+ (H_-). A similar reasoning to that discussed above for the horizon limit, means $\mathbf{v}$ winds in the negative (i.e., clockwise) direction along I_1, when C is circulated in the positive (i.e., counterclockwise) direction, producing another half of a full winding. This means

$$\Omega^\infty_{\theta=\pi} - \Omega^\infty_{\theta=0} = -\pi. \tag{18}$$

Total topological charge in the exterior region.—The limits discussed above imply that the topological charge within $\mathcal{I}$, computed from Eq. (6) is $w = -1$, corresponding to a full winding of $\mathbf{v}$ in the negative sense as the contour delimiting $\mathcal{I}$ is circulated in the positive sense. Indeed, Eq. (6) reduces to

$$w = \frac{1}{2\pi}\left[\int_0^\pi d\Omega\right]_{r=\infty} + \frac{1}{2\pi}\left[\int_\pi^0 d\Omega\right]_{r=r_H}, \tag{19}$$

or

$$w = \frac{1}{2\pi}(\Omega^\infty_\pi - \Omega^\infty_0 + \Omega^{\mathcal{H}}_0 - \Omega^{\mathcal{H}}_\pi) = -1, \tag{20}$$

where Eqs. (16) and (18) were used in the last equality. This holds for both $H_\pm$ and means that there exists at least one standard LR (saddle point of $H_\pm$) for each rotation sense, in the exterior of the BH. Thus, the theorem is proved.

Absence of a horizon.—To understand the key importance of the horizon $\mathcal{H}$, consider the potential $H_\pm$ for flat spacetime—see top row of Fig. 3. As expected the essential difference occurs near the left edge of Fig. 3 (top row). The absence of a horizon means $\mathbf{v}$ keeps flowing towards the left in the whole domain, i.e., $v_r = -1/(r^2 \sin\theta) < 0$, with the sole exception of the axis limit, where it becomes vertical, since $v_\theta/v_r = \cot\theta \to \pm\infty$ at $\theta = 0, \pi$, respectively. It is the presence of a horizon that introduces the $v_r > 0$ boundary behavior at the left boundary of the (r, θ) domain. As our theorem shows, this new boundary behavior *must* introduce (at least) one LR for each rotation sense.

For flat spacetime, $w = 0$ for any contour, and, in particular, one that encloses the full (r, θ) plane, as it is clear from Fig. 3 (top row). This is true, in fact, as long as the behavior at all boundaries is kept, even for a curved spacetime. Thus, smoothness at the origin and at the axis, together with asymptotic flatness guarantees that the total topological charge will remain zero $w = 0$, for any axisymmetric, stationary spacetime, which is smoothly deformable into flat spacetime (and circular). Nonetheless, in such generic smooth horizonless spacetime $\mathbf{v}$ may be locally deformed in the bulk so that LRs emerge. LRs *do not* require a horizon. The individual LR charges, however, must add up to zero. In particular, for each standard LR (a saddle point of $H_\pm$, thus with $w = -1$) there must be a nonstandard LR (maximum or minimum, thus with $w = +1$). This is the theorem in Ref. [14]. Moreover, if the null energy condition is obeyed, the nonstandard LRs must be stable. Thus, horizonless, asymptotically flat spacetimes with LRs must have a stable LR as long as they are a smooth deformation from flat spacetime, like those originating from an incomplete gravitational collapse [14]. This is illustrated in Fig. 3 (bottom) where $\mathbf{v}$ is exhibited for an ultracompact rotating boson star, a horizonless object in Einstein-Klein-Gordon theory [7,22]. Observe that $w = \{+1, -1, 0\}$, respectively, for the contours $\{C_1, C_2, C_3\}$.

Discussion.—Our theorem puts on a firm ground the hitherto unproved expectation that *generic* equilibrium BHs must have one standard LR (for each rotation sense), (see also Refs. [23,24]). In addition, it suggests possible ways to circumvent this result. For instance, by dropping: (i) the circularity of the metric. Spacetime circularity holds in vacuum GR BHs but there are reasonable scenarios wherein it can be violated (e.g., toroidal magnetic fields [25]). There is no fundamental reason for circularity to hold for astrophysical BHs; (ii) asymptotic flatness. Changing the asymptotic behavior of the spacetime may change the boundary behavior [Eq. (17)] and hence the whole result. The powerful tool of contour integration and topological LR charge may help us understand more general situations. It seems possible to tackle extremal BHs or nonspherical (e.g., toroidal) horizons in a similar way. Astrophysically one does not expect extremal BHs, which are thus not the focus of this work. Moreover, recall that for extremal Kerr BHs, the Boyer-Lindquist radial coordinate of the corotating LR coincides with that of the horizon. This is a coordinate artifact, but it suggests that the analysis of extremal BHs introduces subtleties.

Finally, some of our assumptions are implied if one focuses on GR with physical matter. For instance, assuming a GR stationary BH spacetime that is asymptotically flat and regularly predictable, with matter satisfying the dominant energy condition, then by Hawking's theorem [26] the cross section of the event horizon has to be *topologically spherical* (S^2), and the event horizon is a *Killing horizon*. By further assuming that the spacetime is analytic, nonstatic and with the ergosphere intersecting the horizon, the spacetime is then required to be *axially symmetric* by Hawking's rigidity theorem.

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