Ground State Properties of the Diluted Sherrington-Kirkpatrick Spin Glass

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We present a numerical study of ground states of the dilute versions of the Sherrington-Kirkpatrick (SK) mean-field spin glass. In contrast to so-called "sparse" mean-field spin glasses that have been studied widely on random networks of finite (average or regular) degree, the networks studied here are randomly bond diluted to an overall density p, such that the average degree diverges as $\sim pN$ with the system size N. Ground state energies are obtained with high accuracy for random instances over a wide range of fixed p. Since this is an NP-hard combinatorial problem, we employ the extremal optimization heuristic to that end. We find that the exponent describing the finite-size corrections ω varies continuously with p, a somewhat surprising result, as one would not expect that gradual bond dilution would change the $T = 0$ universality class of a statistical model. For $p \to 1$, the familiar result of $\omega(p = 1) \approx \frac{2}{3}$ for the SK model is obtained.

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The Sherrington-Kirkpatrick (SK) model [1] was devised as the mean-field limit of finite-dimensional Ising spin glasses, first introduced by Edwards and Anderson (EA) [2], to describe the unusual phenomenology [3] of disorder in the interaction between classical dipolar magnets in certain materials. Despite the dramatic simplification that such a limit entails, i.e., replacing the lattice with a dense network of bonds between all mutual pairs of spins, the SK model proved so intricate that it took several years and a herculean effort by Parisi to reveal its full structure, referred to as replica symmetry breaking (RSB) [4–6]. RSB was verified rigorously only 30 years later [7,8]. Over the years, the importance of these Ising spin glass models has significantly increased as a most concise conceptualization of systems with disorder and frustration and the complex structure and dynamics that emerges [6,9]. Far beyond its origins in materials science, the SK model has inspired notions of learning in neural networks and artificial intelligence [10], actual neurons [11], facilitated optimization of hard combinatorial problems in operations research and engineering [6,12–14], elucidated the nature of energy landscapes [15], and made connections to biological evolution [16], social dynamics [17], etc. Ironically, in most of these applications, the unstructured mean-field version, such as the SK glass, is far more realistic than the lattice geometry of EA. Moreover, many of these problems, like optimization and learning, concern the low-temperature limit, instead of the physically pertinent phase transition at some finite critical temperature T_c: As long as $T_c > 0$, a glassy phase exists at $T \to 0$. In fact, in mean field there is an entire critical line extending from T_c to $T = 0$ [18]. Notably, $T = 0$ is its own fixed point [19] in the renormalization group sense [20], with its own set of scaling relations, yet to be completed [21–25], connecting domain-wall excitations, ground state energy fluctuations, and finite-size corrections (FSC).

Extending RSB to glassy systems on sparse networks, i.e., random graphs [26] of finite average or fixed degree (Bethe lattices, BL), constituted another major breakthrough [27]. More recently, the one-dimensional long-range model [28] has gained popularity [29–32] for the ability to interpolate between the SK and the EA model (but on a 1D-ring geometry) based on the range of interactions. That model has effective upper and lower dimensions, but all results obtained are numerical.

It is thus surprising that after so many years of studying mean-field spin glasses in the thermodynamic limit on fully connected (SK) or on sparse networks (BL), there has been no consideration given to dense but *dilute* systems. (Reference [33], concerning optimal graph bipartitioning, a problem closely related to spin glasses [34], might pose a rare exception.) For BL, the average or fixed number c of other spins that any one spin is randomly bonded with, i.e., its "degree," is held constant for all network sizes $N \to \infty$. In contrast, it is the average bond density,

$$p = \frac{c}{N-1}, \tag{1}$$

that is held constant in a dilute system. Clearly, in the SK model each spin has a bond to every one of the other spins, i.e., $c_{\mathrm{SK}} = N - 1$ and $p = 1$, while at some general $0 < p \leq 1$, the degree for each spin diverges as $c \sim pN$ in the thermodynamic limit $N \to \infty$. Thus, the dilute SK model presents a true alternative to BL, for which $p \sim 1/N \to 0$, likely resulting in an alternative RSB analysis. These connections are illustrated in Fig. 1. In this Letter, we provide some tantalizing numerical evidence that such an analysis might be quite distinct and potentially more fruitful

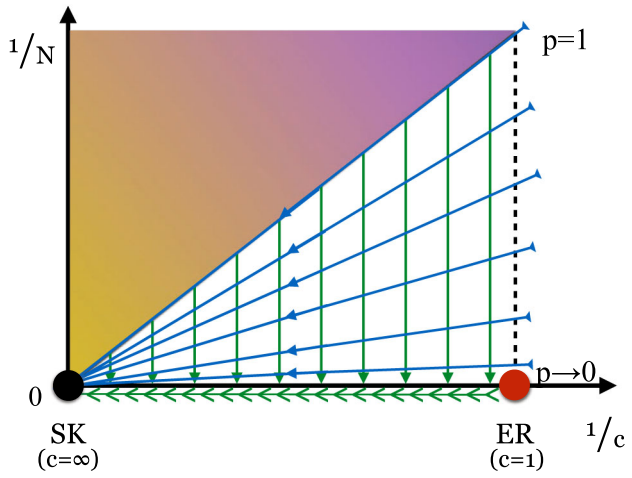

FIG. 1. Depiction of alternative ways to approach the thermodynamic limit $N \to \infty$ (or $1/N \to 0$) for mean-field spin glass models of (average or fixed) spin degree c. Previous work had been focused on constant c while $1/N \to 0$ (green down arrows), referred to as "Bethe lattices" due to their locally treelike structure [27]. In Refs. [36,37], it was shown that the thermodynamic limit of their ground state energy densities $\langle e_0 \rangle_{N=\infty}^{\text{Bethe}}$ can be connected (horizontal green arrows) to that of the SK model (black dot) via $\langle e_0 \rangle_{N=\infty}^{\text{Bethe}} \sim c^{1/2} e_{\text{Parisi}}$, at least for $c \gg 1$, i.e., above the Erdös-Rényi percolation transition for sparse random graphs [26] (red dot). This study explores a diluted SK system, in which system size N and connectivity c both evolve such that $p \sim c/N$ remains constant (blue rays).

in revealing, for instance, the nature of finite-size corrections that occur when $N \to \infty$, which have remained beyond the scope of RSB.

Understanding the nature of FSC for $N \to \infty$ is an essential ingredient in the proper interpretation of numerical data obtained from thermodynamic systems [35]. To reach the thermodynamic limit with data derived from, inevitably, finite-size simulations usually requires a certain degree of extrapolation [36–43]. Here, we will specifically focus on FSC to the ensemble average of the ground state energy density, assuming the form

$$\langle e_0 \rangle_N \sim \langle e_0 \rangle_\infty + \frac{A}{N^\omega}, \qquad (N \to \infty), \tag{2}$$

defining the energy density in the thermodynamic limit $\langle e_0 \rangle_\infty$. In many disordered systems, such as for spin glasses in the low-temperature limit exhibiting RSB, those FSC are dogged by (unknown) subextensive transients [21,24], i.e., transients that diminish slower than the bulk, $\omega < 1$, which at times obscure the physical interpretation to a point of arbitrariness [25]. Even in mean field, exact results for scaling properties of the glassy phase short of the thermodynamic limit are few [23,44–47]. Finding an accessible problem as a model to make conceptual inroads on determining FSC would thus constitute a major advance for RSB.

Numerical simulations, in fact, have provided numerous insights into the nature of FSC in Ising spin glass models. It was found that ground state energies (and entropies) for mean-field systems of N spins have FSC decaying to excellent approximation with $N^{-2/3}$. This was observed first for BL with bimodal bonds [36,37] and subsequently [41–43,48] also for the SK model. (BL with Gaussian bonds exhibit FSC with $\omega \approx 0.8$ [40,49].) For finite-dimensional Ising spin glasses (EA), FSC collapse of domain-wall excitations at $T \to 0$ allowed an accurate determination of the stiffness exponent θ in dimensions $d = 3, \ldots, 7$ [50]. This exponent is fundamental to many aspects of the glassy state [3]; for instance, $\theta(d_l) = 0$ defines the lower critical dimension, which appears close to $d_l = 2.5$ [22,51,52], while its determination for $d \geq 6$ allowed a direct check on mean-field predictions [23]. In particular, FSC were shown to decay consistently as in Eq. (2), applied to hypercubic lattices of size $N = L^d$ with $\omega = 1 - \theta/d$ [24], suggesting the importance of domain-wall excitations for FSC [21]. Recently, we have proposed to use FSC analysis to assess the quality and scalability of optimization heuristics for hard combinatorial problems [14].

In the present study, we generate $N \times N$ symmetric bond matrices with entries from a dilute bond distribution,

$$P(J) = p\delta\left(J^2 - \frac{1}{pN}\right) + (1-p)\delta(J), \tag{3}$$

such as to minimize the SK Hamiltonian [1],

$$H_J = -\sum_{i>j} J_{ij}\sigma_i\sigma_j, \tag{4}$$

over the set of N Ising spin variables, $\sigma_i = \pm 1$. We thus approximate the ground state energy density, $e_0(N,p) = (1/N)\min_{\vec{\sigma}} H_J$, for each instance J. Like at $p = 1$, $P(J)$ is symmetric with variance $\langle J^2 \rangle = 1/N$ but higher moments for the dilute SK model diverge for $p \to 0$, i.e., $\langle J^{2n} \rangle = p/(pN)^n$ for $n = 2, 3, \ldots$. For each bond density p $(0 < p \leq 1)$, we sample ensemble averages $\langle e_0 \rangle_N$ of the ground state energies over a range of sizes N, where $P(J)$ in Eq. (3) ensures that the average thermodynamic ground state energy is universal [8,53], $\langle e_0 \rangle_\infty = e_{\text{Parisi}} = -0.7631667265\ldots$, first approximated by Parisi [4]. Here, we report on the results for a range of values $p < 1$ and find surprisingly nontrivial behavior in the continuous dependence of $\omega(p)$. As the topology of the diagram in Fig. 1 suggests, RSB should remain in effect for all p, possibly even in the limit $p \to 0$, where a solution should become trivial. (Even for the smallest constant p, there is a neighborhood of the thermodynamic limit, for sizes $1/p \ll N < \infty$, where the dilute system is dense enough to be above the percolation transition for sparse random graphs at $c = 1$, i.e., $p_c \sim 1/N$ [26]; see Fig. 1.)

The following results are obtained with the extremal optimization (EO) heuristic [54–56]. For a generic combinatorial optimization problem, EO performs a local search [56,57] on an existing configuration of N variables by changing preferentially those of poor *local* arrangement. For example, in the case of the spin glass model in Eq. (4), it assigns to each spin variable a "fitness" $\lambda_i = \sigma_i \sum_{j=1}^{N} J_{i,j}\sigma_j$, corresponding to the negative of the local energy of each spin, so that $H = -\frac{1}{2}\sum_{i=1}^{N} \lambda_i$ reproduces the SK Hamiltonian in Eq. (4). A local search with EO requires the ranking of these fitnesses λ_i from worst to best, $\lambda_{\Pi(1)} \leq \lambda_{\Pi(2)} \leq \cdots \leq \lambda_{\Pi(N)}$, where $\Pi(k) = i$ is the index for the kth-ranked variable σ_i. In the basic version of EO, it always updates the lowest rank, $k = 1$ [54,58,59]. Instead, τ-EO as used here selects the kth-ranked variable with a *scale-free* probability $P_k \propto k^{-\tau}$. The selected variable is updated *unconditionally*, and its fitness and that of its neighboring variables are reevaluated. This update is repeated as long as desired, where the unconditional update ensures significant fluctuations, yet, sufficient incentive to return to near-optimal solutions due to *selection against* variables with poor fitness, for the right choice of τ. Clearly, for finite τ, EO never "freezes" into a single configuration; it instead records one (or even an extensive set [37,60]) of the

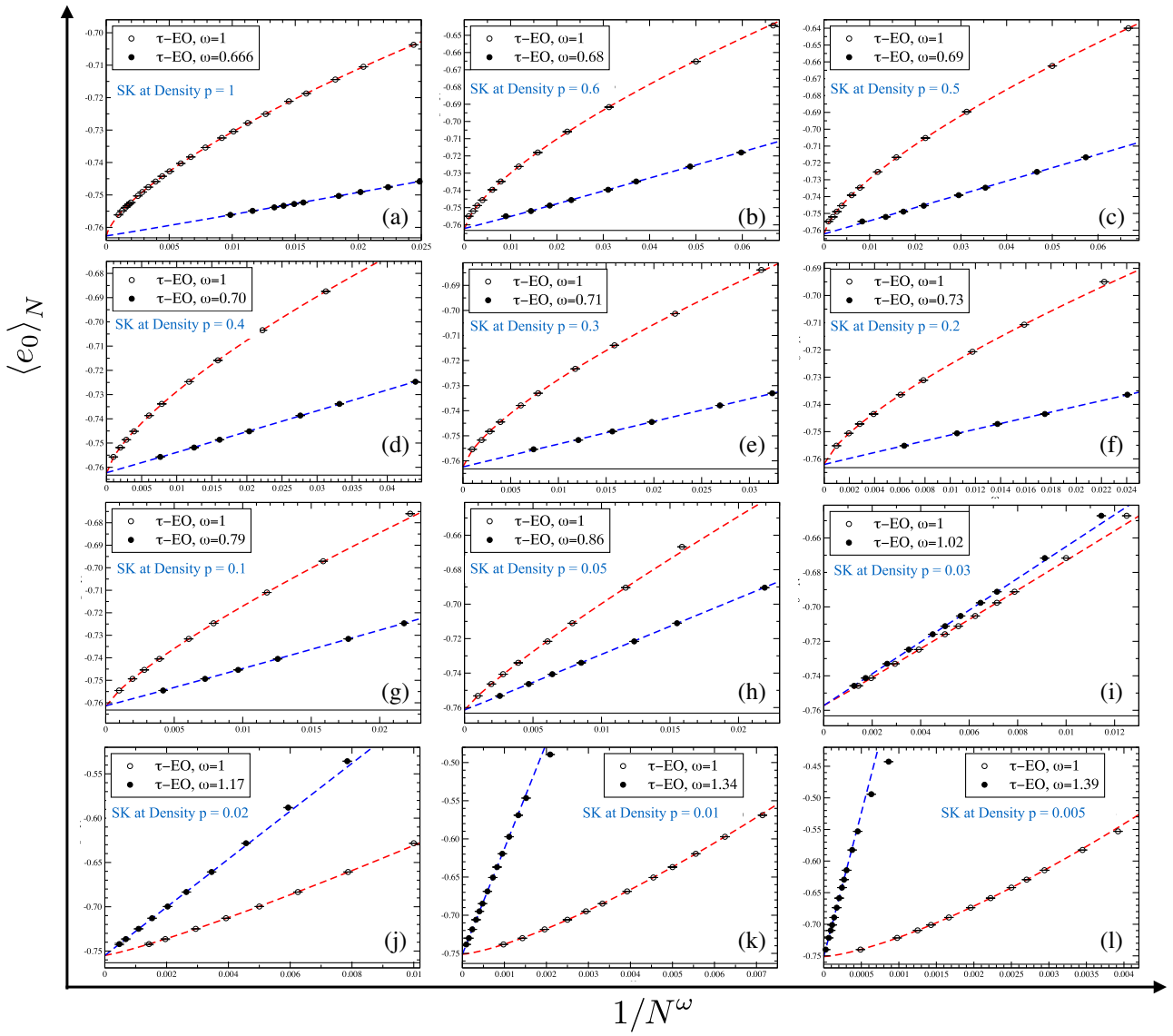

FIG. 2. Extrapolation for the rescaled ground state energy densities $\langle e_0 \rangle_N$ of the diluted SK model of bond density p at different sizes N, where each data point is plotted once for $1/N$ (i.e., $\omega = 1$, open symbols) and a second time for $1/N^{\omega}$ with a value of ω chosen such that the extrapolation to the thermodynamic limit at the intercept $1/N^{\omega} \to 0$ is asymptotically linear (closed symbols). Each panel (a)–(l) depicts a different density p, where data are fitted to the asymptotic form in Eq. (2) (drawn as either red or blue dashed lines, respectively). Each fit obtains the exponent ω and the thermodynamic ground state energy density $\langle e_0 \rangle_{N=\infty}$, which should approach the Parisi energy density, $e_{\text{Parisi}} = \langle e_0 \rangle_{N\to\infty}$, for all p [53] (horizontal line), listed in Table I.

best configurations in passing. Our specific implementation of τ-EO for the SK model proceeds is described in Ref. [42].

EO is implemented [61] for denser instances ($p \geq 0.05$) as described in Refs. [41,42], for sparser instances ($p \leq 0.05$) as described in Refs. [36,37]; we have obtained statistically identical results for *both* at $p = 0.05$. For any given value of p, we generate a large number of instances over a large range of sizes N (from 10^5 instances for all $N < 200$ to 2×10^3 at $N \approx 1000$, to 10^2–10^3 for $N > 1000$) and average the obtained ground state energies $\langle e_0 \rangle_N$ plotted as a function of N in Fig. 2(a)–2(l). It is well known that finding solutions of lowest energy for each instance corresponds an NP-hard combinatorial problem (Max-Cut [62]), and a significant effort must be undertaken to minimize systematic errors in the approximation of ground states. Luckily, we can gauge the accuracy of EO (and any other heuristic [14]) using the theoretical predictions already obtained with RSB. For instance, in Fig. 2(a), pertaining to the SK model ($p = 1$) as previously studied in Ref. [41], the EO data were extrapolated to the thermodynamic limit and fit according to Eq. (2) to reproduce the RSB prediction for e_{Parisi} to 5 digits of accuracy. Similarly, EO applied to sparse networks [36,37] reproduced the RSB prediction for BL of fixed degree $c = 3$ from Ref. [27] to 4 digits of accuracy. Further application of EO to BL of fixed degrees $c = 4, \ldots, 26$ provided predictions for thermodynamic $\langle e_0^{(c)} \rangle_{N=\infty}$, which themselves extrapolate consistently for $c \to \infty$ such that $c^{-1/2} \langle e_0^{(c)} \rangle_\infty \sim e_{\mathrm{Parisi}}$. Thus, the extrapolation plot, i.e., the very fact that a scaling according to Eq. (2) can be consistently applied, becomes a bootstrap measure of validation in its own right [14].

In Table I we list all parameters obtained from the data displayed in Fig. 2 for each value of p asymptotically for large N to Eq. (2). We observe that the dependence of the FSC exponent ω on p, shown in Fig. 3, is quite remarkable. While the SK model [41–43,48] as well as sparse networks [21,36,37] with bimodal bonds have consistently exhibited FSC with $\omega \approx \frac{2}{3}$, independent of degree c, for fixed $p < 1$ in the dilute SK model we find significant variation in $\omega(p)$. For decreasing p, $\omega(p)$ rises from its SK value at $p = 1$ with what appears to be a continuous hyperbolic form, roughly $\omega - \frac{2}{3} \sim \frac{1}{p}$, for about two decades, $0.03 \leq p \leq 1$, as the inset of Fig. 3 suggests. Leaving $\langle e_0 \rangle_\infty$ as a fitting paramter, the exact result, $\langle e_0 \rangle_\infty = e_{\mathrm{Parisi}}$, is reproduced within errors for $p > 0.03$; see Table I. Significant deviations only arise for the smallest values of p studied here, and it is not obvious whether these are due to systematic errors in EO or due to the assumptions underlying Eq. (2) Fixing $\langle e_0 \rangle_\infty = e_{\mathrm{Parisi}}$ for the fit has virtually no effect on our key result, the variation of $\omega(p)$, as listed in Table I, for $p > 0.03$. However, the data for smaller p no longer fit to any FSC we considered, such as higher order corrections to Eq. (2), logarithmic corrections, etc., unless we assume large systematic errors and discount EO data for larger N. Not only does that contradict aforementioned results in Refs. [36,37], it renders any such fit arbitrary. It is interesting that this transition occurs at a value of p where the fitted value of $\omega(p)$ just about passes unity, suggesting that a bulk $(1/N)$ correction, subdominant in Eq. (2) for $\omega < 1$, might interfere. (At $p = 1$, such a correction has been ruled out for the SK model in Ref. [41].) The breakdown of simple FSC in Eq. (2) is also signaled by the diverging amplitude $A(p)$ in Table I.

An analytic study of the dilute SK model in the limit of $p \to 0$ should be able to reveal whether the limit for ω is

TABLE I. List of the fitted values for the average ground state energies $\langle e_0 \rangle_{N=\infty}$, the correction amplitude A, and the FSC exponent ω of the SK model at various bond densities p, obtained by fitting the numerical data displayed in Fig. 2 to the asymptotic form in Eq. (2). That fit was conducted over the specified range of system sizes N.

p	$\langle e_0 \rangle_\infty$	ω	A	N range
0.005	−0.751(1)	1.39(1)	448(5)	340–2047
0.01	−0.752(1)	1.32(1)	125(5)	165–2047
0.02	−0.755(1)	1.16(1)	26(3)	255–1023
0.03	−0.757(1)	1.02(1)	9(1)	180–512
0.05	−0.761(1)	0.86(1)	3.3(5)	165–512
0.1	−0.762(1)	0.79(1)	1.7(1)	63–1023
0.2	−0.762(1)	0.73(1)	1.04(7)	63–1023
0.3	−0.762(1)	0.71(1)	0.91(5)	63–1023
0.4	−0.762(1)	0.70(1)	0.86(5)	63–1023
0.5	−0.762(1)	0.69(1)	0.80(4)	45–1023
0.6	−0.762(1)	0.68(1)	0.75(3)	45–1023
1.0	−0.763 23(5)	0.666(3)	0.71(1)	80–2047

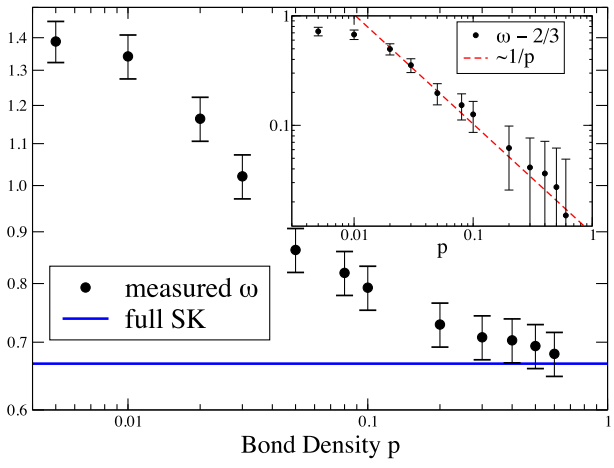

FIG. 3. Plot of the fitted values for the exponent ω controlling the FSC in the extrapolation of the ground state energies shown in Fig. 2 for the bond-diluted SK model as a function of bond density p. The data for ω can be found in Table I. Inset: Except for the smallest values of p, the exponent subtracted by its SK value ($\omega_{\mathrm{SK}} \approx \frac{2}{3}$ at $p = 1$), i.e., $\omega - \frac{2}{3}$, appears to approach the SK value roughly hyperbolically, $\sim 1/p$.

regular or singular. A perturbative expansion around that limit might also shed light on the nature of FSC in RSB, since there does not appear to be a transition at any finite p from RSB near $p = 1$ to a simple replica-symmetric phase, at least at $T = 0$. Thus, future studies should explore the properties of the dilute SK model for finite T. But even at the ground state level, we intend to explore the behavior of other characteristic features, such as the ensemble fluctuations in the ground state energies [23,41,44]. As there is expected to be a scaling relation between the FSC exponent ω and the exponent describing such fluctuations [21], investigating their relation while evolving with p should be very revealing about the nature of universality at $T = 0$.

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