

## Work Statistics across a Quantum Phase Transition

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We investigate the statistics of the work performed during a quench across a quantum phase transition using the adiabatic perturbation theory when the system is characterized by independent quasiparticles and the “single-excitation” approximation is assumed. It is shown that all the cumulants of work exhibit universal scaling behavior analogous to the Kibble-Zurek scaling for the average density of defects. Two kinds of transformations are considered: quenches between two gapped phases in which a critical point is traversed, and quenches that end near the critical point. In contrast to the scaling behavior of the density of defects, the scaling behavior of the cumulants of work are shown to be qualitatively different for these two kinds of quenches. However, in both cases the corresponding exponents are fully determined by the dimension of the system and the critical exponents of the transition, as in the traditional Kibble-Zurek mechanism (KZM). Thus, our study deepens our understanding about the nonequilibrium dynamics of a quantum phase transition by revealing the imprint of the KZM on the work statistics.

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*Introduction.*—In cosmology and condensed matter physics the creation of excitations during continuous phase transitions (thermal or quantum) is usually described by the Kibble-Zurek mechanism (KZM) [1–4]. The KZM relates the *average* density  $\langle n_{\text{ex}} \rangle$  of excitations or defects created during a transformation or quench across a critical point to the rate or speed at which the critical region is traversed. This is particularly relevant for adiabatic quantum computation and simulation schemes, where nonadiabatic effects impose a tradeoff between the speed and the fidelity that can be achieved [5–7]. Importantly, the KZM predicts a universal power law dependence of  $\langle n_{\text{ex}} \rangle$  on this rate, with an exponent that is fully determined by the dimension of the system and the critical exponents of the transition. Also, the Kibble-Zurek scaling of the irreversible entropy production has been reported [8]. The traditional heuristic argument behind the KZM, as well as more rigorous derivations based on the adiabatic perturbation theory [9–11], only gives information about the average density of excitations  $\langle n_{\text{ex}} \rangle$ . However, the actual number of excitations created during a particular realization of the quench is a stochastic quantity that will fluctuate from one realization to the next, and thus must be characterized by a probability distribution. It was recently shown by del Campo that in the exactly solvable one dimensional (1D) transverse Ising chain the universal scaling predicted by the KZM also applies to all the cumulants of  $n_{\text{ex}}$  [12], which was recently verified in experiments [13].

Motivated by this finding, we extend previous results in two important aspects. In the first place, we turn our attention away from the density of created excitations and focus instead on the amount of work applied during the quench. Thus, we investigate what are the signatures of the KZM on the characteristic function of work (CFW), which plays an important role in the newly developed field of stochastic thermodynamics [14–16]. In analogy to the partition function, which contains essential information about an equilibrium state, the CFW contains essential information about an arbitrary nonequilibrium process. This interesting quantity has received much attention since it allows us to understand the emergence of irreversibility during a thermodynamic transformation (via the fluctuation relations [17,18]), and is related to other interesting quantities employed to study the nonequilibrium dynamics of complex many-body systems like the Loschmidt echo [18–22]. Second, we provide a general scaling argument, underpinned by the well-known results in the adiabatic perturbation theory [9–11], showing that all the cumulants of the work exhibit a scaling behavior similar to the KZM scaling for systems that can be described in terms of independent quasiparticles. Our predictions are valid in principle for systems in arbitrary dimensions, and are explicitly shown to hold in the exactly solvable 1D quantum transverse Ising model.

*KZM and the adiabatic perturbation theory.*—We first briefly review the basic concepts and heuristic arguments behind the KZM scaling in a quantum phase transition, and

also how to recover (and generalize) the same results using the adiabatic perturbation theory. We consider a second-order quantum phase transition between two gapped phases characterized by the correlation length critical exponent  $\nu$  and the dynamic critical exponent  $z$  [23]. Thus, close to the quantum critical point, the energy gap  $\Delta$  between the ground state and the first relevant excited state, the relaxation time  $\tau$  and the correlation length  $\xi$  scale as [24]

$$\Delta \sim |\lambda|^{z\nu}, \quad \tau \sim |\lambda|^{-z\nu}, \quad \xi \sim |\lambda|^{-\nu}, \quad (1)$$

where  $\lambda$  is a dimensionless parameter which measures the distance from the critical point. We also consider a protocol in which the Hamiltonian of the system is modified in such a way that the parameter  $\lambda(t)$  can be approximated as a linear quench  $\lambda(t) = vt$  near the critical point, where  $v > 0$  is the quench rate. The system is initially prepared in the ground state at  $t_0 \rightarrow -\infty$  and the protocol stops at  $t_1 \rightarrow \infty$ . According to the KZM, the evolution of the system can be divided into three parts: (1)  $t_0 < t < -t^*$ , (2)  $-t^* < t < t^*$ , and (3)  $t^* < t < t_1$ , where the time  $t^*$  is determined by the following argument [3,4,24,25]. During parts (1) and (3), the relaxation time of the system is sufficiently small for its evolution to be considered adiabatic ( $\tau < v^{-1}$ ), since it can always catch up with the change of  $\lambda(t)$  (adiabatic region). In contrast, during part (2) the relaxation time is large compared to  $v^{-1}$  and as a consequence the state of the system is frozen out (impulse region). The freeze-out time  $t^*$  can be estimated by the following relation  $t^* \simeq \tau[\lambda(t^*)]$  and thus we obtain  $t^* \sim v^{-z\nu/(z\nu+1)}$ . Then, the initial state for the adiabatic dynamics of part (3) is approximately the final state of the evolution of part (1), and is therefore characterized by a correlation length  $\xi^* = \xi[\lambda(t^*)] \sim v^{-\nu/(z\nu+1)}$ . This correlation length corresponds to the characteristic length of the system, e.g., the size of the magnetic domains. Thus, the average density of defects or domain walls can be estimated as

$$\langle n_{\text{ex}} \rangle \sim \xi^{*-d} \sim v^{(d\nu)/(z\nu+1)}, \quad (2)$$

where  $d$  is the dimension of the system.

The above results can be reproduced by using the adiabatic perturbation theory [9–11]. For this purpose, we consider a system defined on a  $d$ -dimensional lattice and described by a Hamiltonian  $\hat{H}[\lambda(t)] = \hat{H}_0 + \lambda(t)\hat{V}$ , where  $\hat{H}_0$  is the Hamiltonian at the quantum critical point and  $\lambda(t)$ , now called the work parameter [15], is controlled by an external agent according to the above protocol. Here,  $\hat{V}$  is the driving Hamiltonian. We assume that the system can be described by independent quasiparticles (denoted by mode  $k$ ), and that at the critical point the energy gap vanishes due to the fact that the dispersion relation of low-energy (long wavelength, small  $k$ ) modes exhibits the scaling behavior  $\omega_k = c|k|^z$  ( $\hbar = 1$ ) [23], where  $c$  is a nonzero constant. We also assume that at most one

quasiparticle in every mode can be excited after the quench, which is called the ‘‘single-excitation’’ approximation in this Letter. Then, within the adiabatic perturbation theory, the excitation probability of the  $k$ th-mode quasiparticle  $p_k$  is dominated by (assuming that there is no additional Berry phase) [9,11,24]

$$p_k \approx \left| \int_{\lambda_0}^{\lambda_1} d\lambda \langle 1_k(\lambda) | \partial_\lambda | 0_k(\lambda) \rangle e^{i v^{-1} \int_{\lambda_0}^{\lambda} d\lambda' \omega_k(\lambda')} \right|^2, \quad (3)$$

where  $\partial_\lambda = \partial/\partial\lambda$ ,  $\lambda_0 = \lambda(t_0)$ ,  $\lambda_1 = \lambda(t_1)$ , and  $|n_k(\lambda)\rangle$  denotes the instantaneous energy eigenstate of mode  $k$  of  $\hat{H}(\lambda)$  with the occupation number  $n_k$ . Then, the average density of excitations  $\langle n_{\text{ex}} \rangle$  reads

$$\langle n_{\text{ex}} \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_k p_k = \int \frac{d^d k}{(2\pi)^d} p_k, \quad (4)$$

where  $N$  denotes the number of the lattice points. In order to remove the quantity  $v^{-1}$  in the exponential function in the integral of  $p_k$  [Eq. (3)], we introduce two rescaled quantities,  $\theta$  and  $\phi$ , defined by [9,11,24]

$$\lambda = \theta v^{1/(z\nu+1)}, \quad k = \phi v^{\nu/(z\nu+1)}. \quad (5)$$

Also following Refs. [11,24], we introduce the general scaling argument

$$\begin{aligned} \omega_k(\lambda) &= |\lambda|^{z\nu} F(k/|\lambda|^\nu), \\ \langle 1_k^\lambda | \partial_\lambda | 0_k^\lambda \rangle &= \lambda^{-1} G(k/|\lambda|^\nu), \end{aligned} \quad (6)$$

where  $F$  and  $G$  are two model-dependent scaling functions satisfying  $F(x) \propto x^z$  and  $G(x) \propto x^{-1/\nu}$  for  $|x| \gg 1$ . This is motivated by dimensional considerations and the requirement that the spectrum of the high energy modes should be insensitive to  $\lambda$ . Thus,  $\langle n_{\text{ex}} \rangle$  reads [9,11]

$$\langle n_{\text{ex}} \rangle \approx v^{(d\nu)/(z\nu+1)} \int \frac{d^d \phi}{(2\pi)^d} K(\phi), \quad (7)$$

where

$$K(\phi) = \left| \int_{\theta_0}^{\theta_1} \frac{d\theta}{\theta} G\left(\frac{\phi}{|\theta|^\nu}\right) \exp \left[ i \int_{\theta_0}^{\theta} d\theta' |\theta'|^{z\nu} F\left(\frac{\phi}{|\theta'|^\nu}\right) \right] \right|^2, \quad (8)$$

with  $\lambda_0 = \theta_0 v^{1/(1+z\nu)}$  and  $\lambda_1 = \theta_1 v^{1/(1+z\nu)}$ . For  $d\nu/(z\nu+1) < 2$ , the integral Eq. (8) converges in the limit  $v \rightarrow 0$  and therefore  $\langle n_{\text{ex}} \rangle \sim v^{d\nu/(z\nu+1)}$ , in accordance to the KZM prediction. But for  $d\nu/(z\nu+1) > 2$ , the integral Eq. (8) does not converge, which means that it is not dominated by the low-energy modes [9–11]. The high-energy (ultraviolet) contribution to the integral can be

approximated by the regular analytic adiabatic perturbation theory [9–11], which results in the quadratic scaling  $\langle n_{\text{ex}} \rangle \sim v^2$  [26]. For  $d\nu/(z\nu + 1) = 2$ , an additional logarithmic correction is expected, i.e.,  $\langle n_{\text{ex}} \rangle \sim v^2 \ln v$  [10,11,33]. This concludes our review of the KZM and the adiabatic perturbation theory. In the following, they are applied in analyzing the scaling behaviour of the work distribution during a linear quench.

*Scaling behavior in the characteristic function of work.*—We define the work applied during the quench on the basis of the usual two-time measurement scheme, i.e., as the difference between the results of the projective measurements [34–38] of the system’s energy before and after the linear quench. It is a stochastic quantity with a distribution function  $P(w)$ , and the logarithm of its characteristic function [the Fourier transform of  $P(w)$ ], called the cumulant CFW, reads

$$\begin{aligned} \ln \chi(u) &= \ln \text{Tr}[\hat{U}^\dagger(t_1, t_0) e^{iu\hat{H}(\lambda_1)} \hat{U}(t_1, t_0) e^{-iu\hat{H}(\lambda_0)} \hat{\rho}_0] \\ &= \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} \kappa_n, \end{aligned} \quad (9)$$

where  $\hat{U}(t_1, t_0)$  is the time evolution operator,  $\hat{\rho}_0$  is the initial state and  $\kappa_n$  is the  $n$ th cumulant of work [39]. We assume that  $\lambda(t)$  evolves according to the above protocol, the system is initially prepared in the ground state of  $\hat{H}(\lambda_0)$  and the cumulant CFW satisfies a large deviation principle [40], i.e.,  $\lim_{N \rightarrow \infty} N^{-1} \ln \chi(u)$  exists.

For the adiabatic driving ( $v \rightarrow 0$ ), the system is in the ground state all the time. Hence, we have  $\kappa_1 = \sum_k [\varepsilon_k^0(\lambda_1) - \varepsilon_k^0(\lambda_0)] \equiv N\mu$ , where  $\varepsilon_k^0(\lambda)$  denotes the zero-point energy of the  $k$ th mode of  $\hat{H}(\lambda)$  and  $\kappa_n = 0$  for  $n \geq 2$  due to the definite measurement results. Thus, the cumulant CFW for the adiabatic process should be  $\ln \chi_a(u) = Niu\mu$  according to Eq. (9). For nonadiabatic driving ( $v$  is small but nonzero), since the nonadiabatic corrections to the cumulant CFW come from the impulse region of the KZM, we expect these corrections to exhibit the following scaling relation, i.e.,  $\kappa_1 = N(\mu + v^{\delta_1} f_1)$  and  $\kappa_n = Nv^{\delta_n} f_n$ ,  $n \geq 2$ , where  $f_n$  are model-dependent scaling functions and  $\delta_n$  are the corresponding exponents characterizing the scaling behavior of each cumulant.

Every exponent  $\delta_n$  can be determined as follows. According to Eq. (9) and in the single-excitation approximation, the scaling behavior of  $\kappa_n$  should be the scaling behavior of  $N^{-1} \sum_k [\omega_k(\lambda_1)]^n p_k$  because the excitations of quasiparticles in different modes are independent, i.e.,  $\kappa_1 \approx N[\mu + N^{-1} \sum_k \omega_k(\lambda_1) p_k]$ ,  $\kappa_n \approx \sum_k [\omega_k(\lambda_1)]^n p_k$  (see Supplemental Material [26]). Now, by utilizing the expressions of  $\kappa_1$  and  $\kappa_n$  and following the same procedure as in the last section, we obtain

$$v^{\delta_n} f_n \approx v^{(d\nu)/(z\nu+1)} \int \frac{d^d \phi}{(2\pi)^d} [\omega_{k=\phi v^{1/(z\nu+1)}}(\lambda_1)]^n K(\phi). \quad (10)$$

If we fix  $\lambda_0$  and  $\lambda_1$  when varying  $v$  and  $\lambda_1$  is away from the critical point,  $\omega_k(\lambda_1)$  is a nonzero constant when  $k \rightarrow 0$ . Also because only low-energy modes can be excited after the quench, we obtain

$$v^{\delta_n} f_n \approx [\omega_{k=0}(\lambda_1)]^n v^{(d\nu)/(z\nu+1)} \int \frac{d^d \phi}{(2\pi)^d} K(\phi). \quad (11)$$

Following the same analysis as that after Eq. (7), the exponents in the cumulant CFW read

$$v^{\delta_n} = \begin{cases} v^{d\nu/(z\nu+1)} & d\nu/(z\nu+1) < 2, \\ v^2 \ln v & d\nu/(z\nu+1) = 2, \\ v^2 & d\nu/(z\nu+1) > 2. \end{cases} \quad (12)$$

Finally, according to Eq. (9), since in this case  $\delta_n$  is independent of  $n$ ,  $\ln \chi(u)$  reads

$$\ln \chi(u) = \ln \chi_a(u) + Nv^{\delta_n} f(iu), \quad (13)$$

where  $f(iu) \equiv \sum_{n=1}^{\infty} (iu)^n f_n/n!$ . We would like to emphasize that the scaling behavior exists not only in the CFW but also in the work distribution [see Eq. (14)]. According to the Gartner-Ellis theorem, the distribution of the work per lattice site  $P_N(w) \equiv P(Nw)$  also takes on the large deviation form,  $\lim_{N \rightarrow \infty} N^{-1} P_N(w) = -I(w)$ . Here, the rate function  $I(w)$  is obtained by the Legendre-Fenchel transform [40] via

$$I(w) = \sup_{u \in \mathbb{R}} \{u(w - \mu) - v^{\delta_n} f(u)\}. \quad (14)$$

Now, let us consider a second case:  $\lambda_1$  is near the critical point. Since in this case  $\omega_k(\lambda_1) = c|k|^z$  when  $k \rightarrow 0$ , according to Eq. (10), we have

$$v^{\delta_n} f_n \approx c^n v^{[(d+nz)\nu]/[z\nu+1]} \int \frac{d^d \phi}{(2\pi)^d} |\phi|^{nz} K(\phi). \quad (15)$$

Similar to the discussion about  $\langle n_{\text{ex}} \rangle$ , we obtain

$$v^{\delta_n} = \begin{cases} v^{(d+nz)\nu/(z\nu+1)} & (d+nz)\nu/(z\nu+1) < 2, \\ v^2 \ln v & (d+nz)\nu/(z\nu+1) = 2, \\ v^2 & (d+nz)\nu/(z\nu+1) > 2. \end{cases} \quad (16)$$

We note that the quantity in Eq. (15) for  $n = 1$  is called the excess energy in Refs. [10,11]. If  $\delta_1 < 2$ , to a good approximation, we can cut off the sum in Eq. (9) to the first order ( $n = 1$ ) for sufficiently small  $v$  and obtain

$$\ln \chi(u) \approx \ln \chi_a(u) + Nv^{\delta_1} iu f_1. \quad (17)$$

Accordingly,  $P_N(w)$  is a Dirac delta distribution located at  $\mu + v^{\delta_1} f_1$ .

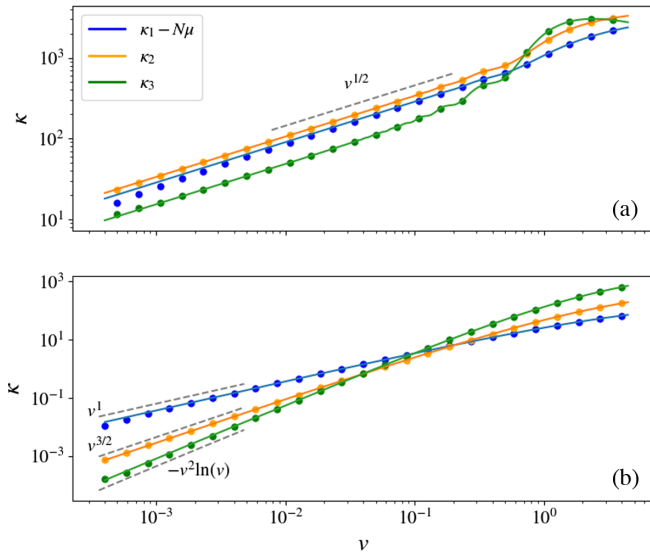


FIG. 1. The first three cumulants of the work distribution as a function of the quenching rate for a 1D transverse Ising chain. Solid lines correspond to the exact analytic solution in the macroscopic limit [42] and dots to an exact numerical simulation of a chain of  $N$  spins. (a) Quench between  $\lambda_0 = -4$  and  $\lambda_1 = 1$ , for which all the exponents are  $1/2$ . In this case we have  $N = 4000$  and the deviations at low  $v$  are due to the finite size effects [12]. (b) Quench between  $\lambda_0 = -4$  and  $\lambda_1 = 0$ , for which our theory predicts  $\kappa_1 - N\mu \propto v$ ,  $\kappa_2 \propto v^{3/2}$  and  $\kappa_3 \propto v^2 \ln v$ . Even in this case of a shorter chain ( $N = 1000$ ), finite size effects are less relevant. This is natural given the scale invariance at the critical point, which is the ending point of the protocol.

In summary, our analysis shows that the scaling of the work cumulants is qualitatively different depending on whether  $\lambda_1 = 0$ . If  $\lambda_1 \neq 0$ , all the cumulants (for whatever  $n$ ) have the same scaling exponent, while if  $\lambda_1 = 0$ , they do not. This is illustrated in Fig. 1 by the exact numerical simulation of the 1D transverse Ising model [41], which is also studied analytically in the following. It is important to note that this difference between the two kinds of quenches is not observed for the density of excitations  $n_{\text{ex}}$ , which displays the same scaling behavior irrelevant to the ending point of the protocol.

*Example.*—We calculate the CFW of the 1D transverse Ising model to demonstrate our results since it is solvable and the KZM is valid in this model [25,41]. The Hamiltonian of a chain of  $N$  spins in a transverse magnetic field reads

$$\hat{H}(\lambda) = -J \sum_{l=1}^N [\hat{\sigma}_l^z \hat{\sigma}_{l+1}^z + (\lambda - 1) \hat{\sigma}_l^x], \quad (18)$$

with the Born–von Kármán periodic boundary condition. Here,  $\hat{\sigma}_l^{x,y,z}$  denote the Pauli matrices on site  $l$ , and  $J$  denotes the energy scale. The critical points are at  $\lambda = 0, 2$ . Moreover,  $d = z = \nu = 1$  [24,25]. For the critical point  $\lambda = 0$ , we choose  $\lambda(t) = vt$ ,  $\lambda_0 < 0$  and  $0 < \lambda_1 < 2$ .

According to Ref. [45], when  $N \rightarrow \infty$ , the cumulant CFW reads

$$\ln \chi(u) = \frac{N}{\pi} \int_0^\pi dk \ln \frac{g_k(u)}{g_k(0)}, \quad (19)$$

where

$$g_k(u) = \{1 + \cos(u\omega_k^1) \cos[(u - i\beta)\omega_k^0] + Q_k \sin(u\omega_k^1) \sin[(u - i\beta)\omega_k^0]\}^{1/2}. \quad (20)$$

Here,  $\beta = (k_B T)^{-1}$  is the inverse temperature of the canonical initial state,  $\omega_k^0 = \omega_k(\lambda_0)$ ,  $\omega_k^1 = \omega_k(\lambda_1)$ , where  $\omega_k(\lambda) = 2J\sqrt{(\lambda - 1 + \cos k)^2 + \sin^2 k}$  is the energy of the  $k$ th mode. Also,  $Q_k = 1 - 2p_k$ , where  $p_k \approx e^{-2\pi Jk^2/v}$  is the excitation probability in the corresponding Landau-Zener model for mode  $k$  [41]. From Eq. (19), we obtain the first and the second cumulants of work

$$\begin{aligned} \kappa_1 &= \frac{N}{2\pi} \int_0^\pi dk (\omega_k^0 - Q_k \omega_k^1) \tanh\left(\frac{\beta\omega_k^0}{2}\right), \\ \kappa_2 &= \frac{N}{4\pi} \int_0^\pi dk [(\omega_k^0 - Q_k \omega_k^1)^2 + (1 - Q_k^2)(\omega_k^1)^2 \cosh(\beta\omega_k^0)] \\ &\quad \times \text{sech}^2\left(\frac{\beta\omega_k^0}{2}\right). \end{aligned} \quad (21)$$

Quantum phase transitions occur at the absolute zero. Hence, we consider that case in which the initial state is chosen to be the ground state of  $\hat{H}(\lambda_0)$ . From Eq. (19), we have

$$\begin{aligned} \ln \chi(u) &= N \int_0^\pi \frac{dk}{2\pi} \{iu(\omega_k^0 - \omega_k^1) + \ln[1 + p_k(e^{2iu\omega_k^1} - 1)]\}, \\ &= \ln \chi_a(u) + N \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_0^\pi \frac{dk}{2\pi} p_k^n (e^{2iu\omega_k^1} - 1)^n, \end{aligned} \quad (22)$$

where  $\ln \chi_a(u) = Niu\mu = Niu(2\pi)^{-1} \int_0^\pi dk (\omega_k^0 - \omega_k^1)$  is the cumulant CFW for the adiabatic process. The sum in the last equation is convergent under the condition  $|p_k(e^{2iu\omega_k^1} - 1)| = 2p_k |\sin(u\omega_k^1)| < 1$  [46]. Also, for  $T = 0$ , from Eq. (22), we have

$$\begin{aligned} \kappa_1 &= N(\mu + \frac{1}{\pi} \int_0^\pi dk \omega_k^1 p_k), \\ \kappa_2 &= \frac{2N}{\pi} \int_0^\pi dk (\omega_k^1)^2 p_k (1 - p_k), \\ \kappa_3 &= \frac{4N}{\pi} \int_0^\pi dk (\omega_k^1)^3 p_k (1 - p_k)(1 - 2p_k). \end{aligned} \quad (23)$$

Due to the exponential decay of  $p_k$ , only low-energy modes can get excited. Thus, we extend the upper limit of the

integral to  $\infty$  and we approximately have  $p_k e^{2i\omega_k^1} \approx p_k e^{2i\omega_{k=0}^1} = p_k e^{4iuJ\lambda_1}$ . In this way, from Eqs. (22) and (23), we obtain

$$\begin{aligned} \frac{1}{N} \ln \frac{\chi(u)}{\chi_a(u)} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{(e^{4iuJ\lambda_1} - 1)^n}{2\pi} \int_0^{\infty} dk e^{-2n\pi Jk^2/v}, \\ &= \frac{-v^{1/2} J^{-1/2} \sqrt{2} \text{Li}_{3/2}(1 - e^{4iuJ\lambda_1})}{8\pi}, \end{aligned} \quad (24)$$

and

$$\begin{aligned} \kappa_1 &= N \left( \mu + \frac{v^{1/2} J^{1/2} \lambda_1 \sqrt{2}}{2\pi} \right), \\ \kappa_2 &= N \frac{v^{1/2} J^{3/2} \lambda_1^2 (2\sqrt{2} - 2)}{\pi}, \\ \kappa_3 &= N \frac{v^{1/2} J^{5/2} \lambda_1^3 4(6\sqrt{2} - 18 + 4\sqrt{6})}{3\pi}, \end{aligned} \quad (25)$$

where  $\text{Li}_s(z) = \sum_{l=1}^{\infty} z^l/l^s$  is the polylogarithm function. Also, we have  $f(u) = -\sqrt{2/J} \text{Li}_{3/2}(1 - e^{4Ju\lambda_1})/(8\pi)$  and  $\delta_n = 1/2 = d\nu/(z\nu + 1)$ . Obviously, the 1D transverse Ising model verifies our predictions in Eqs. (12) and (13).

We also would like to do some quantitative analysis about the work distribution function.  $f(u)$  is a monotonic function with the following asymptotic behavior: for  $u \rightarrow -\infty$ ,  $f(u) = -\sqrt{2/J} \zeta(3/2)/(8\pi)$ , where  $\zeta(z)$  is the Riemann zeta function; for  $u \rightarrow \infty$  [the domain of  $f(u)$  has been extended to the real axis by applying analytic continuation],  $f(u) = J(\lambda_1 u)^{3/2} \sqrt{2}/[\pi \Gamma(5/2)]$ , where  $\Gamma(z)$  is the Gamma function. Hence, from the asymptotic behavior of  $f(u)$  and by applying the Legendre-Fenchel transform, we find that for  $w < \mu$ ,  $P_N(w) = 0$  which is consistent with the initial ground state condition. A confusion may arise when we consider  $w > w_m \equiv \mu + \int_0^{\pi} dk \omega_k^1/\pi$  since now  $P_N(w)$  is the probability of unphysical events. This is a consequence of the approximation in which we extend the upper limit of the integral to  $\infty$ . Actually, it can be proved that when  $w > w_m$ ,  $I(w) > I(w_m) \propto v^{-1}$ . Because  $P_N(w) \propto e^{-NI(w)}$ ,  $P_N(w) < P_N(w_m) \ll 1$ , which indicates that the probabilities of the unphysical events are sufficiently small and our approximation is still reasonable.

If  $\lambda_1$  is near the critical point,  $\omega_k^1 = 2Jk$  when  $k \rightarrow 0$ . And for every mode, the dynamics corresponds to a half Landau-Zener problem [11,24,43,44], where  $p_k$  reads

$$\begin{aligned} p_k &= 1 - 2 \frac{e^{-\pi\alpha_k/8} \sinh\left(\frac{\pi\alpha_k}{4}\right)}{\pi\alpha_k} \\ &\times \left| \Gamma\left(1 + \frac{i\alpha_k}{8}\right) + \sqrt{\frac{\alpha_k}{8}} \Gamma\left(\frac{1}{2} + \frac{i\alpha_k}{8}\right) e^{i\pi/4} \right|^2, \end{aligned} \quad (26)$$

with  $\alpha_k = 4Jk^2/v$ . This function has the following asymptotic behavior: for  $\alpha_k \rightarrow 0$ ,  $p_k = 1/2$ ; for  $\alpha_k \rightarrow \infty$ ,  $p_k = 1/(2\alpha_k)^2$ . Because only low-energy modes can get excited after the quench, we have

$$\begin{aligned} \kappa_1 &= N \left( \mu + \frac{2J}{\pi} \int_0^{\infty} dk k p_k \right) \approx N(\mu + 0.038v), \\ \kappa_2 &= \frac{8NJ^2}{\pi} \int_0^{\infty} dk k^2 p_k (1 - p_k) \approx 0.092Nv^{3/2}J^{1/2}. \end{aligned} \quad (27)$$

For  $n \geq 3$ , the upper limit of the integral cannot be extended to  $\infty$  due to the power-law decay of  $p_k$ . After some careful analysis, we find for  $n > 3$ ,  $\kappa_n \sim v^2$  is reproduced [47]. Moreover, for  $n = 3$ , the logarithmic correction appears:  $\kappa_3 \sim v^2 \ln v$ . These results again verify our predictions in Eq. (16).

*Conclusions.*—In this Letter, we have studied the statistics of the work applied across a quantum phase transition in systems characterized by independent quasiparticles. We have shown that all the cumulants of the work distribution exhibit a scaling behavior for small quench rates, and that the scaling exponents are determined by the dimension of the system and the critical exponents of the transition. This is in analogy to the predictions of the KZM, although there are qualitative differences in quenches ending close to and away from the critical point. In addition, we are also able to determine the scaling exponents  $\delta_n$  when (1) the energy spectrum is always gapped during the protocol, (2) the initial state is not the ground state, or (3) the protocol is a sudden quench protocol near the critical point (see the Supplemental Material [26]). We also show that although the cumulant CFW for slow linear quenches traversing a critical point is analytic for  $u \simeq 0$  (which allows to properly define the cumulants), it has nonanalyticities at certain values of  $u$ . This is related to the phenomenon of dynamical quantum phase transitions [26], which has been previously reported for the case of sudden quenches [18,19,48].

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