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Topology-Bounded Superfluid Weight in Twisted Bilayer Graphene

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While regular flat bands are good for enhancing the density of states and hence the gap, they are detrimental to the superfluid weight. We show that the predicted nontrivial topology of the two lowest flat bands of twisted bilayer graphene (TBLG) plays an important role in the enhancement of the superfluid weight and hence of superconductivity. We derive the superfluid weight (phase stiffness) of the TBLG superconducting flat bands with a uniform pairing, and show that it can be expressed as an integral of the Fubini-Study metric of the flat bands. This mirrors results already obtained for nonzero Chern number bands even though the TBLG flat bands have zero Chern number. We further show that the metric integral is lower bounded by the topological $C_{2z}T$ Wilson loop winding number of TBLG flat bands, which renders that the superfluid weight is also bounded by this topological index. In contrast, trivial flat bands have a zero superfluid weight. The superfluid weight is crucial in determining the Berezinskii-Kosterlitz-Thouless transition temperature of the superconductor. Based on the transition temperature measured in TBLG experiments, we estimate the topological contribution of the superfluid weight in TBLG.

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The recently discovered superconducting phase in twisted bilayer graphene has received extensive attention 43–1]]. The topology of the lowest two bands (per spin and valley) of twisted bilayer graphene (TBLG) is currently under debate [22,26–28,44]. Although theoretical models suggest a nontrivial topological number of these bands, the experimentally measurable effects through which one could prove or falsify the predicted nontrivial topology are scarce. Currently, one viable experimentally observable effect [45] predicts that the single-particle magnetic field spectrum of a topologically nontrivial band can cross the single-particle gap, in stark contrast to conventional knowledge and to the in-field spectrum of trivial bands. We here present another effect of a set of topologically nontrivial bands observable at zero field (of the kind present in TBLG) that appears when these bands become superconducting. It has been shown in Ref. [1] that the superfluid weight in the superconducting state is the sum of two terms: a conventional term, which vanishes when the bands are perfectly flat, and a topological term, which we will prove that it is bounded from below by the Wilson loop winding number of the $C_{2z}T$ protected topology in TBLG.

This Letter is organized as follows. First, we show that by assuming perfectly flat bands and s wave pairing, the superfluid weight can be written as the integral of Fubini-Study metric over the Brillouin zone (BZ), and show that it is lower bounded by the Wilson loop winding. Second, by applying this result to TBLG, we estimate the topological contribution of superfluid weight and explain the relatively high transition temperature.

The two characterizing features of superconductors are the zero dc resistance and Meissner effect. Both of these properties are captured by the celebrated London equation [46]. It tells us that the electric current in a superconductor $\mathbf{j}$ is proportional to the gauge potential $\mathbf{A}$ under Coulomb gauge:

$$j_i = -[D_s]_{ij} A_j, \tag{1}$$

in which the coefficient $[D_s]_{ij}$ is called the superfluid weight. Some spacial symmetry, such as C_{3z}, requires it to be isotropic in 2D. In some works, such as Ref. [47], it is called "phase stiffness" (describing energy susceptibility with respect to phase twists). The London equation has two kinds of consequences. One is Meissner effect, and the other one is the frequency dependence of ac conductance. In 2D, the superfluid weight is also related with the transition temperature. The phase coherence will disappear at a temperature $T = T_c$ given by $[\hbar^2 D_s(T_c)/e^2 k_B T_c] = (8/\pi)$ (known as the Berezinskii-Kosterlitz-Thouless [BKT] transition [48,49]), because of the creation of vortex-antivortex pairs. Usually $D_s(T)$ decreases with increasing temperature, so the transition temperature T_c is always lower than $[\pi\hbar^2 D_s(0)/8e^2 k_B]$. Thus a small D_s at zero temperature leads to a low transition temperature.

In experiments, D_s is related with ac conductance $\sigma(\omega)$ [50], which can be measured by the time-domain transmission spectroscopy without any contact with the sample [51]. For example, at zero temperature, the stiffness temperature of $Bi_2Sr_2CaCu_2O_{8+\delta}$ is measured to be $T_\theta = 55$ K, and the corresponding superfluid weight is $D_s = e^2 k_B T_\theta/\hbar^2 = 1.8 \times 10^9\ \mathrm{H}^{-1}$ [47]. As another example, the superfluid weight of MoGe thin film is measured to be $D_s = 5 \times 10^8\ \mathrm{H}^{-1}$ [52].

In Landau-Ginzburg (LG) theory of conventional superconductivity, the superfluid weight is given by $D_s \approx e^2 n_s/m^*$ where m^* is the band effective mass and n_s is the superfluid density which is temperature dependent [46,53]. If the band is exactly flat, m^* will become infinity, and LG theory tells us the superfluid weight can be zero even when Cooper pairing happens. We use the Bistritzer-MacDonald (BM) model [3] to estimate the bandwidth and the conventional contribution of superfluid weight in TBLG. In the magic angle BM model, a lattice relaxation (around $0.8 \sim 0.9$ in reality [54]) is needed for the flat bands to be gapped from the higher bands, so that the topology of the flat bands is well defined [26,28]. Around the magic angle, the flat bands mostly lie in an energy range $|\epsilon| < \mathrm{W} \approx 0.5$ meV [50]. Hence the effective mass is approximately $m^* \approx \hbar^2 K_M^2/2W$, where K_M is the distance between Γ and K in Moiré BZ. Thus the conventional superfluid weight is $[D_s]_{\mathrm{tri}} \approx e^2 n_s/m^* \approx 2e^2 WN/\hbar^2 \Omega_c K_M^2 = 3\sqrt{3} e^2 WN/4\pi^2\hbar^2$, where Ω_c is the area of the Moiré unit cell, and N is the number of electrons per Moiré unit cell. Here we assume that the superfluid density is given by the total electron density, which is the upper limit of n_s. As an order-of-magnitude estimation, we take $\nu = 1/4$ or equivalently $N = 2$ [55], the value of superfluid weight will be $[D_s]_{\mathrm{tri}} \approx 5 \times 10^7\ \mathrm{H}^{-1}$, and the corresponding BKT transition temperature will not be higher than 0.6 K. However, LG theory is valid only when the band is trivial, as the spreading of its Wannier function has a nonzero lower bound, therefore the estimation based on LG theory in this paragraph is not enough [1,56,57]. As a result, we show that even in the exactly flat band limit, as long as the Wannier functions of the flat bands have some overlap, the Cooper pairing may acquire nonlocal phase correlations and thus a nonzero superfluid weight, which will give rise to a higher transition temperature.

To obtain the contribution of nontrivial band topology to the superfluid weight, we consider a mean-field Bogoliubov-de-Gennes (BdG) Hamiltonian:

$$H_{\mathrm{BdG}} = \frac{1}{2}\sum_{\mathbf{k}} \Psi_{\mathbf{k}}^\dagger \begin{pmatrix} \mathcal{H}(\mathbf{k}) - \mu & \Delta(\mathbf{k}) \\ \Delta^\dagger(\mathbf{k}) & -\mathcal{H}^*(-\mathbf{k}) + \mu \end{pmatrix} \Psi_{\mathbf{k}} + \frac{1}{2}\sum_{\mathbf{k}} \mathrm{Tr}[\mathcal{H}(\mathbf{k}) - \mu]. \quad (2)$$

We use Ω_0 to denote the ground state energy of H_{BdG}, which is also the free energy at zero temperature. We substitute $\mathbf{k} \to \mathbf{k} - e\mathbf{A}$ by Peierls substitution [58] when a nonzero uniform gauge potential $\mathbf{A}$ is turned on, and the free energy becomes a function of $\mathbf{A}$. We can then expand $\Omega(\mathbf{A})$ to the second order of A_i and obtain $\Omega(\mathbf{A}) \approx \Omega_0 + \frac{1}{2} V[D_s]_{ij} A_i A_j$, where V is the area of the sample, and the second order coefficient $[D_s]_{ij}$ is the superfluid weight. The first order derivative $\partial_{\mathbf{A}}\Omega(\mathbf{A})$ gives us the electric current, which agrees with the London equation in Eq. (1).

The free energy $\Omega(\mathbf{A})$ and thus the superfluid weight can be derived from the BdG Hamiltonian. The general expression of the superfluid weight was first derived in Ref. [1], and can be found in Eqs. (S18), (S19), and (S28) in the Supplemental Material [50], Sec. II. The first term in Eq. (S18) corresponds to the Landau-Ginzburg contribution, while Eqs. (S19) and (S28) are additional contributions due to the $\mathbf{k}$ dependence of the flat band wave functions. The convensional contribution vanishes in flat bands, but the wave function contributions Eqs. (S19) and (S28) still exist, and can be related to the band topology as we show below.

Before we start our discussion about TBLG, we briefly review the superconductivity in the spin Chern insulator with exactly flat bands studied in Ref. [1]. In this model, $\mathcal{H}(\mathbf{k})$ has both spinful time reversal symmetry and s_z conservation, which allows one to define a spin Chern number C. The order parameter $\Delta(\mathbf{k}) = i s_y \Delta$ (in which s_y is the y direction spin Pauli matrix) is momentum independent, and one can show that the superfluid weight given by sum of Eqs. (S18), (S19), and (S28) can be reduced to the following integral in the BZ:

$$[D_s]_{ij} = \frac{8e^2\Delta}{\hbar^2}\sqrt{\nu(1-\nu)} \int \frac{d^2k}{(2\pi)^2} g_{ij}(\mathbf{k}), \quad (3)$$

where ν is the filling ratio of the spinful flat bands, and $g_{ij}(\mathbf{k})$ is the Fubini-Study metric evaluated from the Bloch wave function of the spin $\uparrow$ flat band:

$$g_{ij}(\mathbf{k}) = \frac{1}{2}[\partial_{k_i} u^\dagger(\mathbf{k}) \partial_{k_j} u(\mathbf{k}) + \partial_{k_j} u^\dagger(\mathbf{k}) \partial_{k_i} u(\mathbf{k})] + u^\dagger(\mathbf{k}) \partial_{k_i} u(\mathbf{k}) u^\dagger(\mathbf{k}) \partial_{k_j} u(\mathbf{k}), \quad (4)$$

where $u(\mathbf{k})$ is the Bloch wave function at momentum $\mathbf{k}$ of the spin up flat band. The integral of metric can be nonzero even for topological trivial flat bands [59].

The Fubini-Study metric defines a distance on the BZ torus: two momentum points are close to each other if their wave functions have a large overlap [60]. The integral of $\mathrm{tr} g = g_{xx} + g_{yy}$ also corresponds to the gauge invariant part of the "Wannier function localization functional," which has been studied in detail in previous research [61,62]. The metric is also related to Berry curvature through the quantum geometric tensor defined by

$\mathfrak{G}_{ij} = \partial_{k_i} u^\dagger(\mathbf{k})[1 - u(\mathbf{k})u^\dagger(\mathbf{k})]\partial_{k_j} u(\mathbf{k})$. The real part of $\mathfrak{G}_{ij}$ is the metric g_{ij} and the imaginary part is the Berry curvature. One of the most important properties of $\mathfrak{G}_{ij}$ is its *positive definiteness*. It can be shown that for arbitrary complex vectors $\{c_i\}$, the inequality $\sum_{ij} c_i^\dagger \mathfrak{G}_{ij} c_j \geq 0$ always holds [50]. If we choose $c_x = 1$ and $c_y = i$, we will find $\mathrm{tr}g = g_{xx} + g_{yy} \geq -\mathcal{F}_{xy}$; similarly, we choose $c_x = 1$ and $c_y = -i$, and we will obtain $\mathrm{tr}g \geq \mathcal{F}_{xy}$. Therefore we prove that the metric is bounded by the absolute value of curvature $\mathrm{tr}g \geq |\mathcal{F}_{xy}|$. From the expression of D_s one can easily notice that $\mathrm{tr}D_s$ is bounded by the spin Chern number C. In TBLG, the (spin) Chern number is zero, and the system is multiband, likely with more complicated pairing symmetry. Hence a new bound or limit (if any exists) for the superfluid weight must be obtained.

We first generalize the result of Ref. [1] to multiband systems with a more realistic pairing. The free fermion Hamiltonian $\mathcal{H}(\mathbf{k})$ is assumed to be invariant under spinful time reversal transformation, which is represented by $\mathcal{T} = U_T\mathcal{K}$, where U_T is a real unitary matrix and $\mathcal{K}$ is complex conjugation operator. We do not (any longer) assume momentum independent pairing. $\mathcal{H}(\mathbf{k})$ is diagonalized by $U(\mathbf{k})$ as $\varepsilon_\mathbf{k} = U^\dagger(\mathbf{k})\mathcal{H}(\mathbf{k})U(\mathbf{k})$ where $\varepsilon_\mathbf{k}$ is a diagonal matrix, and we assume that it has N_F flat bands at same energy. We also assume the band gap between these flat bands and any other bands is larger than the bandwidth of flat bands, and the interaction between the electrons. In the following discussion we use a time reversal symmetric pairing between Kramers pairs as follows:

$$\Delta(\mathbf{k}) = [\Delta_1 \tilde{U}_\mathbf{k} \tilde{U}_\mathbf{k}^\dagger + \Delta_2(\mathbb{1} - \tilde{U}_\mathbf{k}\tilde{U}_\mathbf{k}^\dagger)]U_T, \tag{5}$$

in which $\Delta_{1,2} \in \mathbb{R}$ because of time reversal symmetry. Here $\tilde{U}_\mathbf{k} = [u_1(\mathbf{k}), u_2(\mathbf{k}) \cdots u_{N_F}(\mathbf{k})]$ is the projection of $U(\mathbf{k})$ into these N_F flat bands and $u_i(\mathbf{k})$ are the eigenvectors of matrix $\mathcal{H}(\mathbf{k})$. This ansatz implies that s wave pairing happens between Kramers pairs and the pairing strength in the flat bands and in all other bands are given by Δ_1 and Δ_2, respectively. Therefore we have the following three important assumptions in total: (1) the free fermion Hamiltonian with time reversal symmetry has N_F flat bands at the same energy near the Fermi level; (2) there is a large band gap between flat bands and other bands (in TBLG this can be satisfied by introducing lattice relaxation, as we mentioned earlier); (3) the pairing order parameter satisfies Eq. (5). For flat bands with attractive interaction between Kramers pairs, the mean field BCS theory is a good approximation to the ground state ([50], Sec. VIII). Because of this large band gap between flat bands and other bands, we can project the BdG Hamiltonian into the flat bands of the free fermion model and then derive the superfluid weight. The result is given by

$$[D_s]_{ij} = \frac{2e^2|\Delta_1|}{\hbar^2}\left(1 + \frac{\Delta_2}{\Delta_1}\right)\sqrt{\nu(1-\nu)}\int \frac{d^2k}{(2\pi)^2} g_{ij}(\mathbf{k}) \tag{6}$$

$$g_{ij}(\mathbf{k}) = \mathrm{Tr}\left[\frac{1}{2}(\partial_{k_i}\tilde{U}_\mathbf{k}^\dagger \partial_{k_j}\tilde{U}_\mathbf{k} + \partial_{k_j}\tilde{U}_\mathbf{k}^\dagger \partial_{k_i}\tilde{U}_\mathbf{k}) + (\tilde{U}_\mathbf{k}^\dagger \partial_{k_i}\tilde{U}_\mathbf{k}\tilde{U}_\mathbf{k}^\dagger \partial_{k_j}\tilde{U}_\mathbf{k})\right], \tag{7}$$

in which $\mathrm{Tr}(X) = \sum_{n=1}^{N_F}(X_{nn})$ stands for the trace over all the flat band indices. Equation (7) is the generalization of Fubini-Study metric in Eq. (4) to multiband systems, which is also positive definite [50]. The result in Eq. (3) derived in Ref. [1] is a special case of our result in Eq. (6) when the time reversal transformation is represented by $U_T = is_y$, the spin z component is conserved, and $\Delta = \Delta_1 = \Delta_2$.

We also notice that if we assume $(\Delta_2/\Delta_1) \leq -1$, the superfluid weight will become zero, or even a negative number. A negative superfluid weight is unphysical, denoting that the BCS wave function of such a pairing is not a stable ground state. This also indicates that if there is no constraint on the order parameter $\Delta(\mathbf{k})$, the superfluid weight will not be bounded. However we expect a weaker pairing strength in the bands which are farther away from the Fermi level, or $|\Delta_2| < |\Delta_1|$. If the pairing in higher bands are much stronger than pairing in the flat bands—a physically impossible situation—our projection of the BdG Hamiltonian into the flat bands may also become invalid. Hence we later set $\Delta_2 = 0$ in order to estimate the topological contribution of D_s.

Now we apply Eq. (6) to TBLG, and show that the fragile topology of TBLG flat bands yields a finite lower bound of the superfluid weight although it has zero spin Chern number. The Bistritzer-MacDonald (BM) model [3] has $C_{2z}T$, C_{2x}, and C_{3z} symmetries, in which T stands for the spinless time reversal transformation. If all the spins and valleys are considered here, we will have well-defined time reversal symmetry, although the BM model itself does not. The $C_{2z}T$ symmetry is crucial for the flat bands' topology [22,26,28,44]. Because of this symmetry, the two eigenvalues of the non-Abelian Wilson loop have to be complex conjugation to each other [26,28]. A winding number can be defined by Wilson loop eigenvalues. $C_{2z}T$ symmetry gives a constraint not only to the Wilson loop but also to the Berry connection and Berry curvature. It can be shown [26,28] that the non-Abelian Berry connection and curvature of the two flat bands can always be written as $\mathbf{A}(\mathbf{k}) = -\mathbf{a}(\mathbf{k})\sigma_2$ and $\mathcal{F}_{xy}(\mathbf{k}) = -f_{xy}(\mathbf{k})\sigma_2$ under a proper *local* gauge choice on a patch in the Brillouin zone (although a *global* gauge choice which satisfies this condition might not exist [63]). In the Supplemental Material [50], Sec. V, we prove that the Wilson loop winding number [26], or the "Euler class" in Ref. [28] denoted by e_2, of the two topological bands, which is an integer, is given by the integral of f_{xy} over the whole BZ (with Dirac points removed from the integral area):

$$e_2 = \frac{1}{2\pi} \int d^2 k f_{xy}. \qquad (8)$$

The wave function of the two flat bands in TBLG (per spin per valley—which are good quantum numbers for small twist angles) also can be used to define the positive-definite non-Abelian quantum geometric tensor $\mathfrak{G}_{ij}$ (which is a 2×2 complex matrix) and Fubini-Study metric $g_{ij} = \frac{1}{2}\mathrm{Tr}(\mathfrak{G}_{ij} + \mathfrak{G}_{ij}^\dagger)$. For arbitrary complex vectors $c_i \in \mathbb{C}^2$, the inequality $\sum_{ij} c_i^\dagger \mathfrak{G}_{ij} c_j \geq 0$ always holds. By choosing vectors c_x and c_y properly [50], we find that the metric is bounded by the "Abelian part" f_{xy} of the non-Abelian Berry curvature $\mathcal{F}_{xy}$: $\mathrm{tr} g \geq 2|f_{xy}|$. The derivation of band topology and metric of the TBLG can be found in the Supplemental Material [50], Sec. V.

In small angle TBLG, all bands are fourfold degenerate with respect to spin $\uparrow, \downarrow$ and with respect to original graphene valley K, K'. The order parameter ansatz in Eq. (5) corresponds to the pairing between opposite spins and valleys, because time reversal transformation in TBLG will flip both spin $\uparrow, \downarrow$ and valley K, K'. If we assume no pairing in higher bands ($\Delta_1 = \Delta, \Delta_2 = 0$), which is physically reasonable, the superfluid weight in the exact flat band limit will be the same as the expression shown in Eq. (3). However, both the pairing strength and metric have different meanings. $g_{ij}(\mathbf{k})$ stands for the Fubini-Study metric derived from the wave functions of two flat bands (with spin $\uparrow$ and valley K, all other degenerate spin and valley bands have identical contribution to the superfluid weight). Also Δ_1 is no longer the pairing strength in all the bands but only in these flat bands. Because of the two band winding number protected by $C_{2z}T$ symmetry, we have a new lower bound. The inequality $\mathrm{tr} g > 2|f_{xy}|$ naturally leads to the lower bound of the trace of the superfluid weight $\mathrm{tr} D_s \geq \frac{8e^2\Delta_1}{\pi\hbar^2}\sqrt{\nu(1-\nu)}$ even though the (spin) Chern number here vanishes. Here we used the fact that the winding number of TBLG flat bands is $e_2 = 1$. Due to the C_{3z} symmetry, superfluid weight must be isotropic, therefore $D_s = \frac{1}{2}\mathrm{tr}[D_s]_{ij}$. Then we can use the following equation to estimate the topological contribution of D_s:

$$D_s \approx \frac{4e^2\Delta_1}{\pi\hbar^2}\sqrt{\nu(1-\nu)}. \qquad (9)$$

From this equation, we find that a nonzero superfluid weight is possible even when the bands are exactly flat, as long as the Cooper pairing gap is developed ($\Delta_1 \neq 0$). The parameter Δ_1 can be estimated from the measured T_c. The filling ratio ν is determined by the carrier density. Below we use experimental data to estimate the value of superfluid weight at zero temperature [64].

As previously mentioned, in 2D superconductors, the transition temperature T_c measured in experiment is the Berezinskii-Kosterlitz-Thouless temperature (when phase

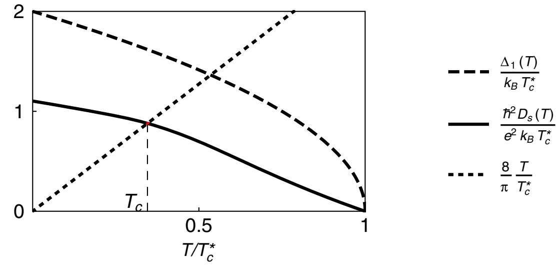

FIG. 1. The gap function $\Delta_1(T)$ and superfluid weight $D_s(T)$. In the flat band limit the BKT temperature is $T_c \approx 0.35 T_c^*$ when $\nu = 1/4$.

coherence disappears) instead of the BCS mean field transition temperature $T_c^* \approx \Delta_1/2k_B$ (when Cooper pairing vanishes). The BKT transition temperature is given by the universal relation $[\hbar^2 D_s(T_c)/e^2 k_B T_c] = (8/\pi)$ [49], and it is generically lower than the BCS mean field transition temperature T_c^*. To derive the BKT temperature of TBLG in the flat band limit, we can generalize the topological superfluid weight expression in Eq. (9) to finite temperatures, which however has no simple analytical expression. By assuming $\Delta_1(T) \approx 2k_B T_c^*(1 - T/T_c^*)^{1/2}$ [46], we can numerically calculate the temperature dependence of the superfluid weight. As an example, we have plotted D_s in Fig. 1 as a function of T_c/T_c^* for filling ratio $\nu = 1/4$ (two electrons per Moiré unit cell), from which we find $T_c/T_c^* = 0.35$. See the Supplemental Material [50], Sec. VII, for more detailed calculations.

We now estimate the TBLG topological superfluid weight D_s at zero temperature. When the bandwidth is small (zero), one expects the superfluid weight to be dominated by the band topology contribution. The experimental transition temperature [4,5] is $T_c = 1.5$ K, thus the order parameter can be estimated to be $\Delta_1 = 2k_B T_c^* \approx 0.74$ meV. By using $\nu = 1/4$, the topological superfluid weight is $[D_s]_{\mathrm{top}} \approx (4e^2\Delta_1/\pi\hbar^2)\sqrt{\nu(1-\nu)}$. One notices that this is an order of magnitude smaller than the superfluid weight in conventional materials, such as BSCCO and MoGe [66,67]. However in both cuprates and MoGe thin films, the typical carrier density is around $n \approx 10^{14}$ cm^{-2} [68] two orders of magnitude larger than that in TBLG (where $n \approx 10^{12}$ cm^{-2}), hence such a superfluid weight is already large for TBLG.

Moving away from completely flat bands, we find that the conventional term in the superfluid weight mostly depends on the bandwidth W, while the topological term mostly depends on the pairing order parameter Δ_1. In TBLG, the bandwidth W and the transition temperature T_c have a similar magnitude, and we expect the topological term to have an important contribution. In the strong pairing limit where $\Delta > W$, the superfluid weight will be underestimated if by only the conventional contribution [34].

Recently, superconductivity has been observed in other Moiré systems, including twisted double bilayer graphene, and multilayer graphene and boron nitride heterostructure [69–77]. In these systems, a displacement electric field is necessary for superconductivity. This yields gapped flat bands with a nonzero valley Chern number [69,74]. The Chern number can be larger than one, which could lead to a larger topological lower bound of superfluid weight. In twisted multilayer graphene, the bandwidth of flat bands is even smaller than that in TBLG. A higher transition temperature (larger Δ) was also observed [75,76]. We expect the topological lower bound plays a significant role in the superfluid weight of these systems. Moreover, multilayer systems with $C_{2z}T$ symmetry can realize larger Euler e_2 invariants, hence increasing the superfluid weight.

In summary, we proved that the fragile topology can yield a nonzero superfluid weight which is lower bounded by its Wilson loop winding number. We also note that our lower bound is consistent with the upper bound studied in Ref. [78]. For topological flat bands, the upper bound in Ref. [78] is around the energy gap E_g between the flat bands and other bands, while our lower bound is proportional to the order parameter Δ, which is derived under the assumption that $|\Delta| \ll E_g$. However, all the above discussion is based on superconducting mean field theory. Competing order parameters might be able to break symmetry which protects the topology, and further break the topology-bounded quantities.

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Note added.—Recently, two related works [79,80] also appeared on arXiv preprint server, which support our conclusion.

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