Quantum Generalized Hydrodynamics

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Physical systems made of many interacting quantum particles can often be described by Euler hydrodynamic equations in the limit of long wavelengths and low frequencies. Recently such a classical hydrodynamic framework, now dubbed generalized hydrodynamics (GHD), was found for quantum integrable models in one spatial dimension. Despite its great predictive power, GHD, like any Euler hydrodynamic equation, misses important quantum effects, such as quantum fluctuations leading to nonzero equal-time correlations between fluid cells at different positions. Focusing on the one-dimensional gas of bosons with delta repulsion, and on states of zero entropy, for which quantum fluctuations are larger, we reconstruct such quantum effects by quantizing GHD. The resulting theory of quantum GHD can be viewed as a multicomponent Luttinger liquid theory, with a small set of effective parameters that are fixed by the thermodynamic Bethe ansatz. It describes quantum fluctuations of truly nonequilibrium systems where conventional Luttinger liquid theory fails.

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The behavior of fluids at very low temperatures is usually peculiar, as the quantum nature of their constituents dominates over thermal fluctuations. To describe collective quantum effects, it is customary to start from classical hydrodynamic equations, and to quantize them. This path was taken by Landau in 1941 [1] in his development of the theory of superfluid helium [2,3]. Since then, similar approaches have been developed for various other quantum liquids [4,5], including, for instance, Bose-Einstein condensates where quantum fluctuations are captured by the Bogoliubov theory [6–8], quantum Hall liquids [9–11], or one-dimensional (1D) quantum fluids described by Luttinger liquid theory [12–14].

The purpose of this Letter is to apply a similar program to the classical hydrodynamics of one-dimensional quantum integrable models introduced in 2016 [15,16], now dubbed generalized hydrodynamics (GHD). At equilibrium, the Luttinger liquid theory—which is the quantized hydrodynamic theory of 1D fluids [17] with few conserved quantities such as charge, magnetization, energy, momentum—is enough to capture quantum fluctuations of 1D gapless integrable models [12–14]. However, when dealing with true out-of-equilibrium situations, like the quantum Newton's cradle setup [18]—whose hydrodynamics description must keep track of all higher conservations laws [19], and is provided by GHD [20]—quantum fluctuations must be given by a more general quantum hydrodynamics theory, obtainable by quantizing GHD. Here our goal is to identify that theory.

Our starting point is GHD, which, on the technical side, relies on the formalism of the thermodynamic Bethe ansatz [21,22]. Thermodynamically large integrable systems are described by densities of different species of quasiparticles. For simplicity, in this Letter we formulate our results in a specific model: the 1D Bose gas with delta repulsion [23–25]. This model is singled out because of its experimental relevance—it is routinely used for describing contemporary cold atom experiments in 1D [26–30]—and because of its simple thermodynamics involving a *single* species of quasiparticles. Our approach can be straightforwardly generalized to other integrable systems with a GHD description [31–60], including, for instance, the *XXZ* chain; we defer mathematical formulas for the general multispecies case to the Supplemental Material (SM) [61].

At the microscopic level, the 1D Bose gas with delta repulsion is defined by the Hamiltonian for N bosons $H=\sum_{i=1}^{N}[-(\hbar^2/2)\partial_{x_i}^2+V(x_i)]+\hbar\bar{g}\sum_{i<j}\delta(x_i-x_j)$, where $g=\hbar\bar{g}>0$ is the repulsion strength between the bosons and $V(x)$ is an external trapping potential. We set the mass of the bosons to 1.

GHD is formulated at the Euler scale, where space-time scales of observations and length scales of external potentials are simultaneously sent to infinity; at the Euler scale diffusion is absent (but subleading diffusive corrections to GHD are also known [42–44]). In the 1D Bose gas the Euler scale is equivalently expressed as the classical limit [64–66]:

$$\hbar \to 0, \quad \text{keeping}\, \hbar N, V(x), \bar{g}\, \text{fixed}. \tag{1}$$

For the gas starting at zero temperature, its evolution under GHD takes a particularly simple form [34]. Indeed, an initial zero-temperature state has zero entropy and entropy is conserved by Euler equations such as GHD. (This is generally true for Euler hydrodynamic equations away from shocks, and it is known that GHD does not admit shocks [34,67,68].) Thus the Bose fluid remains locally in a macrostate with zero entropy at all times. In the Bose gas with delta repulsion, the presence of higher conservation laws allows for a large space of macrostates with zero entropy: the split Fermi seas [69–72]. They can be labeled by a set of Fermi rapidities $\{\theta_a\}_{1\le a\le 2q}$ such that the Fermi factor $n(\theta)$—the number of Bethe quasiparticles with rapidity in $[\theta, \theta + d\theta]$ divided by the number of available states in that interval (see, e.g., Chap. 1 in Ref. [25] for an introduction to that formalism)—is

$$n(\theta) = \begin{cases} 1 & \text{if } \theta \in [\theta_1, \theta_2] \cup \ldots \cup [\theta_{2q-1}, \theta_{2q}] \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The local macrostate is then assumed to be a function of position x and of time t. The global state of the system at time t is best represented by the Fermi contour Γ_t (see Fig. 1), which is defined such that the Fermi factor $n(x, \theta)$ is 1 for all points (x, θ) inside the contour, and 0 outside. For simplicity we restrict to situations where Γ_t is a simple closed curve, parametrized by a function $s \mapsto (x_t(s), \theta_t(s))$,

$$\Gamma_t = \{(x_t(s), \theta_t(s)), s \in \mathbb{R}/2\pi\mathbb{Z}\}. \tag{3}$$

According to GHD, the time evolution of the contour Γ_t is given by the classical equation [34]

$$\frac{d}{dt}\begin{pmatrix} x_t(s) \\ \theta_t(s) \end{pmatrix} = \begin{pmatrix} v^{\mathrm{eff}}(x_t(s), \theta_t(s)) \\ a^{\mathrm{eff}}(x_t(s), \theta_t(s)) \end{pmatrix}, \tag{4a}$$

which expresses the fact that quasiparticles inside the contour move at an effective velocity $v^{\mathrm{eff}}(x, \theta)$ and are accelerated at an effective acceleration $a^{\mathrm{eff}}(x, \theta)$, both of which depend in general on the local Hamiltonian and macrostate, hence on the Fermi points at x. Equation (4a) is complemented by a closed formula for the effective velocity [15,16,46,73] and acceleration [74],

$$v^{\mathrm{eff}} = (\partial_\theta E)^{\mathrm{dr}}/1^{\mathrm{dr}}, \qquad a^{\mathrm{eff}} = -(\partial_x E)^{\mathrm{dr}}/1^{\mathrm{dr}}, \tag{4b}$$

where $E(x, \theta)$ is the bare energy of a quasiparticle with respect to the local Hamiltonian, $1(\theta) = 1$, and the dressing of a function $f(\theta)$ in the local macrostate is defined by the integral equation $f^{\mathrm{dr}}(\theta) = f(\theta) + \int (d\theta'/2\pi)[d\phi(\theta - \theta')/d\theta] n(\theta') f^{\mathrm{dr}}(\theta')$. Here $\phi(\theta - \theta') = 2 \arctan[(\theta - \theta')/\bar{g}]$ is the two-body scattering phase for the delta Bose gas

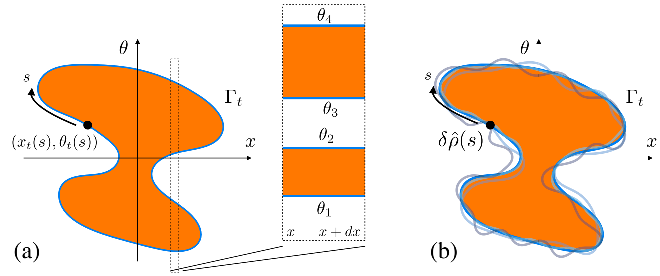

FIG. 1. (a) Zero-entropy GHD describes the motion of the Fermi contour Γ_t, parametrized as in Eq. (3), which separates the regions in phase space where the Fermi factor $n(x, \theta)$ is one (orange) or zero (white) at a given time t. In any small interval $[x, x + dx]$ the fluid is in a state called split Fermi sea [69–72] labeled by Fermi rapidities $\theta_1 < \theta_2 < \ldots < \theta_{2q}$; the number of fluid components q is a piecewise constant function of x and t. (b) In this Letter the contour Γ_t is allowed to have quantum fluctuations around the classical solution to the zero-entropy GHD equations (4a) and (4b). The quantum fluctuations are captured by a chiral boson with density $\delta\hat{\rho}(s)$ living along the contour.

[23–25]. In the present case, $E(x, \theta) = \theta^2/2 + V(x)$, and the effective acceleration simplifies to give Newton's second law [74]:

$$a^{\mathrm{eff}} = -\partial_x V(x). \tag{5}$$

Notice that $\hbar$ is completely absent from Eqs. (4a) and (4b), which is consistent with our claim that zero-entropy GHD corresponds to the classical limit Eq. (1) in the microscopic model.

Goal of this Letter.—Because it is a classical hydrodynamic description, GHD misses certain quantum effects, such as quantum entanglement or correlations between the different parts of the fluid at a given time. Such effects appear as subleading orders in an expansion at small $\hbar$ in the limit Eq. (1). Here we initiate the development of a theory of quantum fluctuations around GHD. Analogously to Bogoliubov theory [6–8], our strategy is to start from linear sound waves propagating on top of a background configuration $(x_t(s), \theta_t(s))$ which solves the GHD equation (Fig. 1), and then find a way to quantize those. We find that the resulting theory takes the form of a time-dependent, spatially inhomogeneous, multicomponent Luttinger liquid, which generalizes the effective theory of (homogeneous, time-independent) split Fermi seas developed recently by Eliëns and Caux [70]; see also Refs. [69,71,72]. It also generalizes the theory of inhomogeneous Luttinger liquids (see, e.g., Refs. [14,17,64–66,75,76]) to truly out-of-equilibrium situations, like the situation depicted in Fig. 2 (see the discussion below).

Sound waves in zero-entropy GHD.—Linearly propagating waves are consequences of the conservation laws of hydrodynamics. By fluctuation dissipation, they are subject

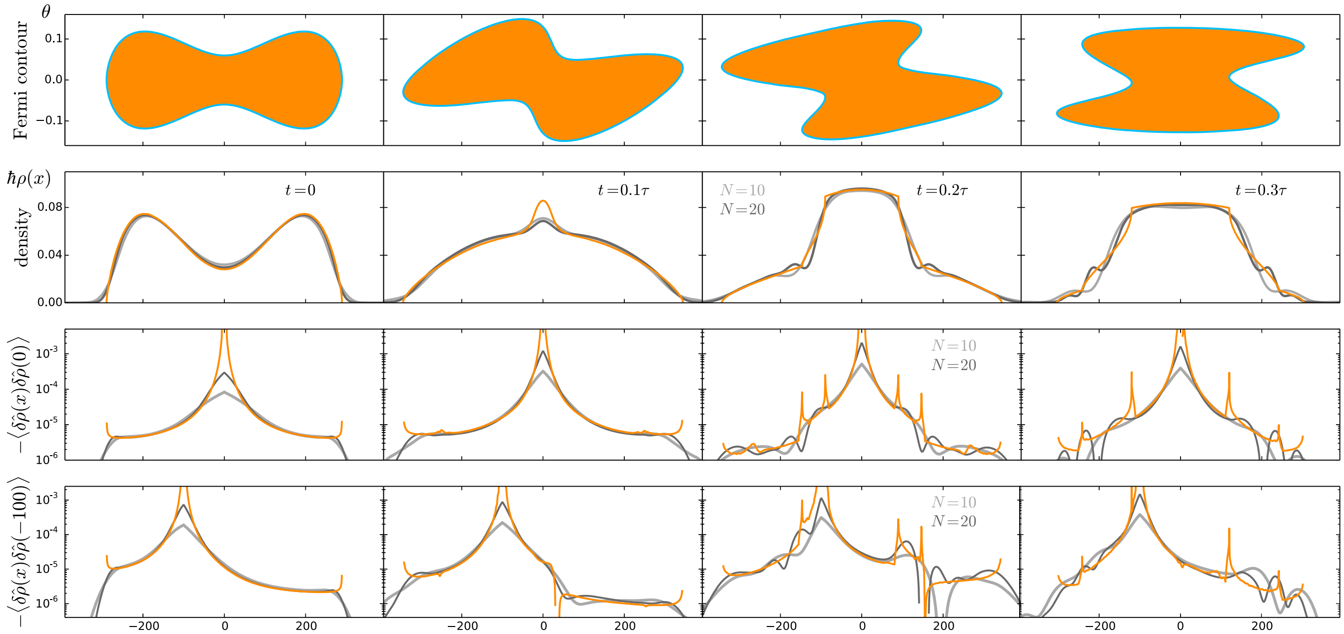

FIG. 2. Quantum quench from double to single well in the 1D Bose gas with delta repulsion. We compare the predictions of GHD and QGHD (orange curves) to time-dependent DMRG simulation for $N = 10$ (light gray) and $N = 20$ particles (dark gray). First row: Fermi contour evolved with GHD. Second row: Density profile predicted by GHD, compared with DMRG. Third and fourth row: Connected density-density correlator $\langle\hat\rho(x)\hat\rho(x_0)\rangle$ predicted by QGHD and compared with DMRG, for two different positions x_0. Each row shows the corresponding quantity as a function of the spatial coordinate x, at different times, expressed as a fraction of the period τ (from $t = 0$ in the first column to $t = 0.3\tau$ in the last). For the DMRG simulation we work with particles on a lattice at very low density. The parameters are as follows: repulsion strength $\bar g = 0.1$; $L = 800$ lattice sites; number of particles $N = 10, 20$; $\hbar = 30/N$; prequench potential $V_0(x) = (x/L)^4 - 0.12(x/L)^2$; postquench potential $V(x) = \omega^2 x^2/2$ with $\omega = 0.3/L$ (and period $\tau = 2\pi/\omega$). The dimensionless Lieb parameter $\gamma = \bar g/\hbar\rho$ is of order 1, so we are far from both the Gross-Pitaevski limit and the Tonks-Girardeau limit.

to correlations due to microscopic fluctuations. Thus, our first task is to find conserved fluid modes and their linear-response evolution. Let us parametrize locally the contour Γ_t by the Fermi points $\theta_a(x,t)$ $(1 \le a \le 2q)$. Fluctuations can be expressed as deformations of the contour $\theta_a(x,t) \to \theta_a(x,t) + \delta\theta_a(x,t)$. Plugging this into Eqs. (4a) and (4b), one would arrive at an evolution equation for $\delta\theta_a(x,t)$, describing the propagation of sound waves on top of the background solution $(x_t(s), \theta_t(s))$. These, however, do not take the form of conservation equations.

Instead, we consider the momentum and energy of an excitation [25], $p(\theta) = \theta + \int(d\theta'/2\pi)\phi(\theta-\theta')n(\theta')1^{\rm dr}(\theta')$ and $\epsilon(\theta) = E(x,\theta) + \int(d\theta'/2\pi)\phi(\theta-\theta')n(\theta')1^{\rm dr}(\theta')\times v^{\rm eff}(\theta')$, respectively. The dispersion relation of such an excitation is the effective velocity, $\partial_\theta\epsilon/\partial_\theta p = v^{\rm eff}$. In the theory of GHD [15,16], any conserved charge of the form $q = \int(d\theta'/2\pi)f(\theta')n(\theta')1^{\rm dr}(\theta')$, which counts $f(\theta')$ for every quasiparticle θ', satisfies a continuity equation with the current $j = \int(d\theta'/2\pi)f(\theta')n(\theta')1^{\rm dr}(\theta')v^{\rm eff}(\theta')$. In an external potential, the continuity equation includes the effective acceleration Eq. (5), see Ref. [74]: $\partial_t q + \partial_x j = \int(d\theta'/2\pi)f'(\theta')n(\theta')1^{\rm dr}(\theta')a^{\rm eff}(\theta')$. We observe that the second terms in the expressions of $p(\theta)$, $\epsilon(\theta)$ are precisely of the form q, j [with $f(\theta') = \phi(\theta-\theta')$ [77]]; therefore, the conservation equation with force term holds,

$$\partial_t p + \partial_x\epsilon + a^{\rm eff}1^{\rm dr} = 0. \tag{6}$$

Excitations in a zero-entropy state occur at the Fermi points. Combining the dispersion relation, Eqs. (6) and (4), one finds an exact conservation law for their momentum $p_a = p(\theta_a)$ and energy $\epsilon_a = \epsilon(\theta_a)$ (see Supplemental Material for detailed derivation [61]):

$$\partial_t p_a + \partial_x\epsilon_a = 0. \tag{7}$$

Then the small fluctuations obey, at first order,

$$\partial_t\delta p_a + \sum_b\partial_x[\mathsf{A}_a^b\delta p_b] = 0, \tag{8}$$

where $\mathsf{A}_a^b = \partial\epsilon_a/\partial p_b$ is the flux Jacobian. This is the propagation equation we were looking for: it is an equation for linear sound waves which takes the form of a conservation equation.

Quantization of sound waves.—The conserved modes δp_a can now be given quantum fluctuations, $\delta p_a \to \delta\hat p_a$. In quantized fluid theory one assumes that there is a classical hydrodynamic action $S = S(\{p_a\})$, whose minimum gives rise to the fluid equation, and which provides the quantum fluctuations and long-range correlations simply by quadratic expansion:

$$e^{iS} \approx e^{iS_{\text{classical}} + i\sum_{ab} S^{(2)}_{ab}\delta p_a \delta p_b}. \tag{9}$$

Passing to the Hamiltonian formalism, there must be a symplectic structure and a Hamiltonian, quadratic in hydrodynamic wave operators $\delta\hat{k}_a = \delta\hat{p}_a/\hbar$, which reproduces Eq. (8).

To identify those, consider the measure $dp = 1^{\text{dr}} d\theta$, which takes into account the density of allowed states 1^{dr} [25], and the phase-space volume form it induces, $dx \wedge dp = 1^{\text{dr}} dx \wedge d\theta$. This volume form is preserved by GHD [78]. Therefore, the fluctuations at zero entropy are fluctuations of an incompressible region in the (x, p) plane. A first consequence is that small volume variations $dp^a = \sigma_a dp_a$, where $\sigma_a = (-1)^a$ is the chirality of the volume boundary, are thermodynamic potentials, leading to an Onsager reciprocity relation (see SM [61]):

$$\mathsf{A}^{ab} = \mathsf{A}^{ba} \qquad (\mathsf{A}^{ab} = \partial\epsilon_a/\partial k^b = \sigma_b \mathsf{A}^b_a). \tag{10}$$

That is, the diagonal matrix $\sigma = \text{diag}(\{\sigma_a\}_{1\le a\le 2q})$ gives a symplectic structure under which the flux Jacobian is symmetric. Second, the problem of quantizing fluctuations of incompressible regions is well known in the literature on the quantum Hall effect [79–82]. Parametrizing the boundary of that region as $(x(s), p(s))$ and introducing a density operator which measures the excess number of occupied states around $(x(s), p(s))$, $\delta\hat{\rho}(s) = (1/2\pi\hbar)(dx/ds)\times \delta\hat{p}(x)$, the commutation relation is the one of a chiral U(1) current algebra:

$$[\delta\hat{\rho}(s), \delta\hat{\rho}(s')] = \frac{1}{2\pi i}\delta'(s - s'). \tag{11a}$$

Equivalently, with the local parametrization $\delta\hat{p}_a(x)$:

$$[\delta\hat{p}_a(x), \delta\hat{p}_b(y)] = -i\sigma_a 2\pi\hbar^2 \delta_{ab}\delta'(x - y). \tag{11b}$$

Using this symplectic structure, the Hamiltonian generating Eq. (8) can be taken as

$$\hat{H}[\Gamma_t] = \frac{1}{4\pi\hbar}\int dx \sum_{a,b} \delta\hat{p}_a(x)\mathsf{A}^{ab}\delta\hat{p}_b(x). \tag{12}$$

Indeed, together with the commutation relation (11b), the Heisenberg equation,

$$\frac{d}{dt}\delta\hat{p}_a(x) = \frac{i}{\hbar}[\hat{H}[\Gamma_t], \delta\hat{p}_a(x)], \tag{13}$$

reproduces the equation for sound waves (8).

The dependence of $\hat{H}[\Gamma_t]$ on Γ_t is via that of A^{ab} on the Fermi points $\{\theta_c(x, t)\}$. The contour-dependent Hamiltonian (12) is the most important result of this Letter, and we refer to it as the QGHD Hamiltonian. Crucially, QGHD is a quadratic theory, so correlation functions can be calculated easily, at least numerically. (Higher-derivative and higher-order terms would lead to a generalization of the nonlinear Luttinger liquid [83,84] or nonlinear bosonization [85–88], which are beyond the scope of this Letter.)

QGHD is the theory of a multicomponent, spatially inhomogeneous, time-dependent, quantum fluctuating liquid with (locally) q coupled components. Importantly, in the particular case of homogeneous time-independent split Fermi seas, we have checked (see SM [61]) that it coincides with the multicomponent quadratic Hamiltonian of Eliëns and Caux [70,72] (see also Refs. [69,71]). As noted by these authors, the case of a single component $q = 1$ is nothing but the standard Luttinger liquid theory.

An example, and numerical check.—To illustrate the possibilities offered by QGHD, we consider the dynamics of the 1D Bose gas after a quench of the trapping potential from double well, $V_0(x) = a_4x^4 - a_2x^2$, to harmonic, $V(x) = \omega^2x^2/2$. The gas is initially in its ground state in $V_0(x)$, with a single pair of Fermi points (i.e., $q = 1$) everywhere. At time $t > 0$, after some fraction of the period of the trap $\tau = (2\pi/\omega)$, the contour Γ_t gets deformed and a region appears near the boundaries with a split Fermi sea $q = 2$. Hence this is a true out-of-equilibrium situation, not describable by standard hydrodynamics. This protocol mimics the famous quantum Newton's cradle [18] and it can be realized experimentally (see, e.g., Refs. [29,89]).

We focus on the equal-time density-density correlation function (Fig. 2). At a point x, the fluctuations of the particle density are measured by the operator,

$$\delta\hat{\rho}(x, t) = \sum_s \left|\frac{ds}{dx}\right|\delta\hat{\rho}(s) = \sum_a \frac{1}{2\pi\hbar}\delta\hat{p}_a, \tag{14}$$

which is a sum over the $2q$ Fermi points at (x, t). Its two-point function at time t is

$$\langle\delta\hat{\rho}(x, t)\delta\hat{\rho}(x', t)\rangle = \sum_s\sum_{s'}\left|\frac{ds}{dx}\right|\left|\frac{ds'}{dx'}\right| G((s, t), (s', t)),$$

where $G((s, t), (s', t'))$ is the Green's function along the contour $G((s, t), (s', t')) = \langle\delta\hat{\rho}(s, t)\delta\hat{\rho}(s', t')\rangle$. At $t = t' = 0$, $G((s, 0), (s', 0))$ is the ground state correlation in the Hamiltonian $\hat{H}[\Gamma_0]$. At later times, $G((s, t), (s', t'))$ satisfies the evolution equation derived from

$$\frac{d}{dt}\delta\hat{\rho}(s, t) = \partial_s(v(s)\delta\hat{\rho}(s, t)) + \frac{i}{\hbar}[\hat{H}[\Gamma_t], \delta\hat{\rho}(s, t)], \tag{15}$$

where $v(s) = v^{\text{eff}}(\theta_a)(dx/ds)$ if a labels the local Fermi point with parameter s. Importantly, $G((s, t), (s', t'))$ is of order O(1) in the limit Eq. (1), so we see that QGHD captures the first correction to the classical result (which is zero):

$$\frac{\langle \delta\hat{\rho}(x,t)\delta\hat{\rho}(x',t)\rangle}{\rho_{\rm cl}(x)\rho_{\rm cl}(x')} = O(\hbar^2). \tag{16}$$

In Fig. 2 we numerically evaluate the Green's function and compare the QGHD prediction (15) with a time-dependent density-matrix renormalization group (tDMRG) [90,91] simulation of the microscopic model. The dimensionless Lieb parameter $\gamma = (\bar{g}/\hbar\rho)$ is chosen to be of order 1, so we are in the truly interacting regime of the 1D Bose gas, away from both the Gross-Pitaevski and the Tonks-Girardeau limits. The tDMRG simulation is performed for a lattice gas at very low density ($N \ll L$, where L is the number of lattice sites) [92,93], to be as close as possible to the continuum limit. The largest number of particles accessible with this method is of order of $N \sim 20$ [93], hence far from the thermodynamic limit. Consequently, finite-N effects are large in our data, which we display for $N = 10$, 20 (and $L = 800$). Still, the agreement between QGHD and numerics is good, and it improves as $\hbar$ decreases (i.e., $N \sim 1/\hbar$ increases). The tDMRG simulation becomes less accurate at large time; for this reason we stop the simulation at $t = 0.3\tau$. The limitations of tDMRG to small N and small t make the predictive power of QGHD even more apparent: QGHD does not suffer from those limitations as it works directly in the thermodynamic limit.

One interesting physical feature of Fig. 2 is the divergence of the density-density correlation, in the thermodynamic limit, at the points where a change in the number of Fermi points occurs. They come from the Jacobians in Eq. (15) and are genuine predictions of the theory, valid for large enough N. The presence of these peaks can be explicitly confirmed by direct computations in the Tonks-Girardeau limit, where they are superimposed to Friedel oscillations [94] (see also Ref. [65] about the equilibrium case in a trap, where these divergences appear near the edges of the system), but they are a general consequence of QGHD at any interaction strength. At the small value $N = 20$, the peaks' extent is smaller than that of Euler fluid cells, hence the peaks are washed away, as seen in the tDMRG result of Fig. 2.

Conclusion.—By focusing on the GHD description of the integrable 1D Bose gas in states of zero entropy, we showed that quantum effects which fall beyond the GHD description can be reconstructed by allowing quantum fluctuations of the Fermi contour. We have been partially inspired by linear fluctuating hydrodynamics [17,95], where fluctuations are accessed by phenomenologically adding thermal noise to the linear-response evolution of conserved fluid modes. We follow the general principles of this theory, but instead of adding thermal noise, we use ideas from quantum fluids (see, e.g., Ref. [17]) in order to access quantum fluctuations. To benchmark QGHD, we applied it to a zero-entropy quench in the 1D Bose gas, providing exact predictions for the equal-time density-density correlations, and checking that they are in good agreement with numerical tDMRG data obtainable for a small particle number and short times.

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