Distillation of Quantum Steering

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We show—both theoretically and experimentally—that Einstein-Podolsky-Rosen steering can be distilled. We present a distillation protocol that outputs a perfectly correlated system—the singlet assemblage—in the asymptotic infinite-copy limit, even for inputs that are arbitrarily close to being unsteerable. As figures of merit for the protocol's performance, we introduce the assemblage fidelity and the singlet-assemblage fraction. These are potentially interesting quantities on their own beyond the current scope. Remarkably, the protocol works well also in the nonasymptotic regime of few copies, in the sense of increasing the singlet-assemblage fraction. We demonstrate the efficacy of the protocol using a hyper-entangled photon pair encoding two copies of a two-qubit state. This represents to our knowledge the first observation of deterministic steering concentration. Our findings are not only fundamentally important but may also be useful for semi-device-independent protocols in noisy quantum networks.

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Steering is a unique form of quantum nonlocality that appears in hybrid quantum networks with both trusted and untrusted components [1]. The first prototypes of the quantum internet may implement such a scenario, where only few members of the network (such as big servers) would have the resources to fully characterize, and therefore trust, their devices, while the remaining participants (such as the end users) would operate the untrusted devices. These scenarios are referred to as semi-device-independent (DI), in contrast to the fully DI context, where all apparatuses are untrusted, or the device-dependent one, with trusted components exclusively. A trusted device allows for full quantum control of the system it operates, e.g., through well-characterized quantum measurements on it. A device is untrusted if one can only control its classical settings (inputs), obtaining classical outcomes (outputs) of uncharacterized measurements from it, thus effectively working as a black-box device. Importantly, steering certifies the presence of entanglement in a semi-DI fashion. Because of this, apart from its fundamental relevance, it is important also from an applied point of view: Steering is known to be the key resource behind several information-processing tasks in the semi-DI scenario [2,3].

However, as experimental quantum networks grow ever more complex, the unavoidable noise and imperfections become increasingly significant. This can severely degrade the steering in the network, compromising the performance of the implemented task. Distillation protocols are ideal for these situations, as they concentrate the resource contained in multiple copies of a noisy system into a pure maximally resourceful system, which can then be directly used safely for the task in question. Interestingly, distillation protocols are known for the other two paradigmatic variants of quantum nonlocality—namely, entanglement in the device-dependent framework and Bell nonlocality in the fully DI one [4,5]—and also even for other important quantum resources [6–11]. Nevertheless, to our knowledge, almost nothing is known for the case of steering. The related phenomenon of steering superactivation [12–14] is known to exist, but requires complete quantum control on all parties. In particular, it is not known whether steering distillation exists (in the asymptotic regime of infinitely many copies of the noisy system) or even if steering can be partially purified in the finite-copy regime.

Here, we answer both questions in the affirmative. We theoretically prove that steering distillation exists, devising an explicit simple protocol for it. We show that such protocol not only distills pure *singlet assemblages* (i.e., the steering correlations generated by a maximally entangled singlet state under ideal von Neumann measurements) in the asymptotic infinite-copy regime but it also succeeds at concentrating steering in the finite-copy regime. Moreover, the proposed protocol distills quantum steering in a hybrid scenario, where only one device applies controlled operations, while the other just deals with its inputs and outputs. This is conceptually different from entanglement distillation, where fully quantum local operations and classical communication (LOCC) are applied on both sides. Remarkably, we demonstrated that an initial system with an arbitrarily small (constant) amount of pure steering can be distilled. To quantify the performance of the partial purification in the finite-copy regime, we introduce the *assemblage fidelity* as a measure of closeness between the steering correlations of two different systems. Finally, we experimentally demonstrate the efficacy of the protocol in an optical setup with two hyperentangled photons, encoding two copies of a 2-qubit state each. We observe a clear increase of the protocol's output's *singlet-assemblage* *fraction* (the assemblage fidelity with respect to the singlet assemblage).

Preliminary definitions.—We consider two parties, Alice and Bob, sharing initially a correlated system in a semi-DI scenario [Fig. 1(a)]. We assume that Alice has an untrusted black-box device, while Bob holds a fully characterized trusted quantum device. Alice's input is represented by a classical parameter $x \in [m]$, where $[n]$ is a shorthand notation for $\{0, 1, \ldots, n-1\}$ for any natural number n, and $m \geq 2$ is the number of possible settings. For a given choice x, a classical output $a \in [o]$ is obtained from the black box, where $o \geq 2$ is the number of possible outputs. Complete characterization of Bob's device allows him to reconstruct his quantum state.

The system is then completely described by the conditional distribution $\{P(a|x)\}_{a\in[o],x\in[m]}$ of obtaining an output a from the black box given the input choice x, and by the conditional quantum state $\rho_{a,x}$ on Bob's side, supported on a local Hilbert space $\mathcal{H}_B$ and possibly correlated to Alice's variables. Equivalently, both objects can be neatly encapsulated in the so-called *assemblage* [1], which is the list $\mathbf{\Sigma}_{A|X} \coloneqq \{\sigma_{a|x}\}_{a\in[o],x\in[m]}$ of subnormalized bounded operators $\sigma_{a|x}$ supported also on $\mathcal{H}_B$, such that $\mathrm{Tr}[\sigma_{a|x}] = P(a|x)$ and $\rho_{a,x} = \sigma_{a|x}/\mathrm{Tr}[\sigma_{a|x}]$.

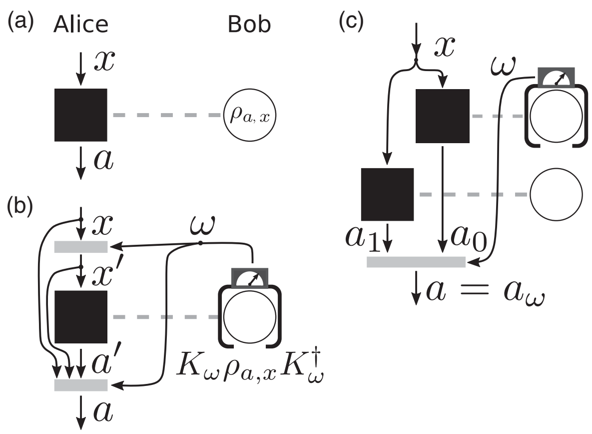

FIG. 1. (a) Bipartite semi-DI scenario. Alice can only perform uncharacterized measurements on her device, which is then effectively treated as a black box. Bob has full quantum control on his system, allowing complete knowledge of his quantum state. Together with Alice's probability distribution for her black box, the systems compose the assemblage $\mathbf{\Sigma}_{A|X}$. (b) Depiction of a generic 1W-LOCC. The assemblage can be manipulated locally by both parties and Bob is allowed to communicate any classical parameter to Alice. Alice's wirings, represented by gray boxes, allow creation of new random variables from previous ones by an arbitrary distribution. (c) Protocol 1 for two copies of the original assemblage. Bob applies a local filter on one of his qubits. A successful outcome results in a singlet assemblage shared between Alice and Bob, while failure produces an unsteerable assemblage. In both cases, communication of Bob's result to Alice is used and the appropriate subsystem is then chosen by both parties.

The assemblage is said to be *steerable* if it cannot be written in terms of a local-hidden-states (LHS) model. This means that there exists no hidden variable that turns Bob's state statistically independent from Alice's variables; i.e., an assemblage $\mathbf{\Sigma}^{\mathrm{LHS}}_{A|X} \coloneqq \{\sigma^{\mathrm{LHS}}_{a|x}\}_{a,x}$ admits a LHS model if there is a variable Λ admitting values $\lambda \in [L]$, such that

$$\sigma^{\mathrm{LHS}}_{a|x} = \sum_{\lambda} P_{\Lambda}(\lambda) P(a|x,\lambda)\rho_{\lambda}. \tag{1}$$

L is the number of possible configurations for Λ, and P_Λ is their probability distribution. Bob's local hidden states ρ_λ are independent of Alice's variables a and x in this case.

In the semi-DI scenario, operations are restricted due to the lack of characterization of Alice's device. Free operations—i.e., operations that do not create quantum steering out of LHS assemblages—are restricted to one-way local operations and classical communication (1W-LOCC) [15]. These correspond to local pre- and postprocessing operations of Alice's classical inputs and outputs, respectively, conditioned on the outputs of quantum operations on Bob's side. Examples of these are shown in Fig. 1.

Distillation of quantum steering.—The task of steering distillation consists of extracting from N copies of a weakly steerable assemblage a smaller number of an extremal assemblage with a purer form of steering, using free operations only. We consider as target here a singlet assemblage, i.e., an assemblage obtained from a singlet state by orthogonal rank-1 projective measurements on Alice's side. These are extremal in the sense that they cannot be obtained from other singlet assemblages via 1W-LOCC [15]. Furthermore, they are known to maximize important measures of quantum steering [3,16,17]. In particular, we consider the singlet assemblage $\mathbf{\Sigma}^{\Phi^+}_{A|X}$ obtained from the maximally entangled state $|\Phi^+\rangle \coloneqq (|00\rangle + |11\rangle)/\sqrt{2}$ when Alice's measurements correspond to the Pauli matrices Z and X. This assemblage is characterized by the components

$$\sigma^{\Phi^+}_{0|0} = \frac{1}{2}|0\rangle\langle 0|, \qquad \sigma^{\Phi^+}_{1|0} = \frac{1}{2}|1\rangle\langle 1|, \tag{2a}$$

$$\sigma^{\Phi^+}_{0|1} = \frac{1}{2}|+\rangle\langle +|, \qquad \sigma^{\Phi^+}_{1|1} = \frac{1}{2}|-\rangle\langle -|, \tag{2b}$$

where $|\pm\rangle \coloneqq (|0\rangle \pm |1\rangle)/\sqrt{2}$.

Then, we can now define the task of steering distillation (with respect to $\mathbf{\Sigma}^{\Phi^+}_{A|X}$ as target assemblage) as the following assemblage conversion:

$$(\mathbf{\Sigma}_{A|X})^{\otimes N} \xrightarrow{\mathrm{1W-LOCC}} (\mathbf{\Sigma}^{\Phi^+}_{A|X})^{\otimes rN}, \tag{3}$$

with unit probability as $N \to \infty$ and with $0 < r \leq 1$ the distillation rate of the protocol. The initial resource of the process is given by N independent copies of the assemblage $\mathbf{\Sigma}_{A|X}$, which is represented mathematically as $\mathbf{\Sigma}_{A|X}^{\otimes N} := \{\otimes_{i=1}^{N} \sigma_{a_i|x_i}\}_{a_1,x_1,\ldots,a_N,x_N}$.

In what follows, we will assume that Alice and Bob share initially $N \geq 2$ copies of the pure nonorthogonal assemblage $\mathbf{\Sigma}_{A|X}^{(\alpha)} := \{\sigma_{a|x}^{(\alpha)}\}_{a,x}$, obtained from the state

$$|\alpha\rangle := \alpha|00\rangle + \beta|11\rangle, \tag{4}$$

through Z and X Pauli measurements on Alice's side, where $0 < \beta < \alpha < 1$ and $\alpha^2 + \beta^2 = 1$. The assemblage is then characterized by the components

$$\sigma_{0|0}^{(\alpha)} = \alpha^2|0\rangle\langle 0|, \qquad \sigma_{1|0}^{(\alpha)} = \beta^2|1\rangle\langle 1|, \tag{5a}$$

$$\sigma_{0|1}^{(\alpha)} = \frac{1}{2}|\alpha_+\rangle\langle\alpha_+|, \qquad \sigma_{1|1}^{(\alpha)} = \frac{1}{2}|\alpha_-\rangle\langle\alpha_-|, \tag{5b}$$

where $|\alpha_\pm\rangle := \alpha|0\rangle \pm \beta|1\rangle$.

We also consider a dichotomic positive-operator-valued measure (POVM) $\boldsymbol{M} := \{M^{(0)}, M^{(1)}\}$ on Bob's subsystem, where $M^{(\omega)}$ are bounded operators satisfying $M^{(\omega)} \geq 0$ and $M^{(0)} + M^{(1)} = \mathbb{1}$. We say that $\boldsymbol{M}$ is applied on an assemblage $\mathbf{\Sigma}_{A|X}$ when Bob applies the corresponding POVM on his quantum state. When outcome ω is obtained, the assemblage's components are updated by [15,18]

$$\sigma'_{a|x,\omega} = \frac{\sqrt{M^{(\omega)}}\sigma_{a|x}\sqrt{M^{(\omega)}}^\dagger}{\mathrm{Tr}[M^{(\omega)}\rho_B]}, \tag{6}$$

where $\rho_B := \sum_a \sigma_{a|x}$ is Bob's reduced state (well defined by virtue of the no-signalling principle [19,20]). Introducing the notation $K^{(\omega)} := \sqrt{M^{(\omega)}}$, so that $M^{(\omega)} = K^{(\omega)\dagger}K^{(\omega)}$, we can now present our protocol:

Protocol 1: (Local filtering with one-sided quantum control.) Let Alice and Bob share $\mathbf{\Sigma}_{A|X}^{(\alpha)\otimes N}$, with $N \geq 2$, and let $\boldsymbol{M}$ be a dichotomic POVM of elements

$$K^{(0)} := \frac{\beta}{\alpha}|0\rangle\langle 0| + |1\rangle\langle 1|, \tag{7a}$$

$$K^{(1)} := \frac{\sqrt{\alpha^2 - \beta^2}}{\alpha}|0\rangle\langle 0|. \tag{7b}$$

Then, the following steps are executed:

(1) For $1 \leq i \leq N-1$, Bob measures $\boldsymbol{M}$ on each ith copy of $\mathbf{\Sigma}_{A|X}^{(\alpha)}$ and gets an outcome $\omega_i \in \{0, 1\}$.

(2) If $\omega_i = 1$ for all $1 \leq i \leq N-1$, he sets $\omega_N = 0$ without measuring the last copy; otherwise he sets $\omega_N = 1$. Then, he sends the string $\boldsymbol{\omega} := \omega_1, \ldots, \omega_N$ to Alice.

(3) Alice gets $\boldsymbol{\omega}$. Then, Alice and Bob discard every ith system for which $\omega_i = 1$, for all $1 \leq i \leq N$. The output of the protocol is given by the remaining assemblages.

The protocol is depicted in Fig. 1(c) for $N = 2$ copies of the initial assemblage.

Any steering distillation protocol must guarantee extraction of at least one such singlet assemblage in the regime of asymptotically many copies, $N \to \infty$. For a finite number of copies, however, perfect extraction may not be possible and only an approximation of $\mathbf{\Sigma}_{A|X}^{\Phi^+}$ is attainable. To quantify this notion of proximity and have a figure of merit for the protocol for finite N, we define the following quantities:

Definition 1.—(Assemblage fidelity.) Let $\mathbf{\Sigma}_{A|X} = \{\sigma_{a|x}\}_{a\in[o],x\in[m]}$ and $\mathbf{\Xi}_{A|X} = \{\xi_{a|x}\}_{a\in[o],x\in[m]}$ have the same number of inputs and outputs and act on the same Hilbert space $\mathcal{H}_B$. We define the *assemblage fidelity* between $\mathbf{\Sigma}_{A|X}$ and $\mathbf{\Xi}_{A|X}$ as

$$\mathcal{F}_A(\mathbf{\Sigma}_{A|X}, \mathbf{\Xi}_{A|X}) := \min_{x\in[m]} \sum_{a\in[o]} \mathcal{F}(\sigma_{a|x}, \xi_{a|x}), \tag{8}$$

with $\mathcal{F}(A, B) = \mathrm{Tr}[\sqrt{\sqrt{A}B\sqrt{A}}]$ the usual state fidelity between two density matrices A and B on $\mathcal{H}_B$.

The definition of assemblage fidelity retains many of the expected properties for a fidelitylike quantity from its dependence on the usual fidelity $\mathcal{F}$, see the Supplemental Material (SM) for demonstrations [21]. In particular, $\mathcal{F}_A$ is non-negative and $\mathcal{F}_A(\mathbf{\Sigma}_{A|X}, \mathbf{\Xi}_{A|X}) \leq 1$, with equality holding if and only if $\mathbf{\Sigma}_{A|X} = \mathbf{\Xi}_{A|X}$. It should be remarked that the minimization contained in definition Eq. (8) is used precisely to preserve these properties and should be understood as a way of better distinguishing assemblages that are in fact distinct. See the SM for details [21].

Assume now that the assemblage $\mathbf{\Sigma}_{A|X}$ is defined in the same space of $\mathbf{\Sigma}_{A|X}^{\Phi^+}$, defined in Eqs. (2). Then, we can now define our figure of merit for the distillation protocol's performance:

Definition 2.—(Singlet-assemblage fraction.) Let $\mathbf{\Sigma}_{A|X}^{\Phi^+}$ be the singlet assemblage defined in Eqs. (2). We define the *singlet-assemblage fraction* for an assemblage $\mathbf{\Sigma}_{A|X} = \{\sigma_{a|x}\}_{a\in[2],x\in[2]}$, with $\dim(\mathcal{H}_B) = 2$, as

$$\mathcal{F}_\Phi(\mathbf{\Sigma}_{A|X}) := \mathcal{F}_A(\mathbf{\Sigma}_{A|X}, \mathbf{\Sigma}_{A|X}^{\Phi^+}). \tag{9}$$

With this quantity, we may evaluate if a given protocol indeed allows extraction of an assemblage that is closer to the singlet assemblage than initially. Ideally, the singlet-assemblage fraction should be defined including an optimization over unitaries applied on Bob's side (or, more generally, over reversible 1W-LOCCs). This however enormously complicates its analytical computation even for the case considered in our results below, where we observe numerically that the values with and without this extra optimization coincide. We now present our main result, proven in Appendix B in the SM [21].

Theorem 1: (Distillation of Quantum Steering) Quantum steering can be distilled with the use of Protocol 1 with rate $r = 2\beta^2$ in the asymptotic regime of infinite copies of the initial assemblage $\mathbf{\Sigma}^{(\alpha)}_{A|X}$. Furthermore, in the regime of N copies, with N finite, an assemblage can be obtained on average which is closer to the singlet assemblage than $\mathbf{\Sigma}^{(\alpha)}_{A|X}$, attaining a singlet-assemblage fraction of $\sqrt{1-\frac{1}{2}(\alpha-\beta)^2(\alpha^2-\beta^2)^{N-1}}$.

Experimental realization.—We implemented the local filtering protocol experimentally using two copies of the original assemblage. A pair of hyperentangled photons in polarization and optical path, produced via spontaneous parametric down-conversion (SPDC), is used to encode the two copies, one in each degree of freedom (DOF). The setup is represented in Fig. 2. A 325-nm continuous-wave He-Cd laser pumps two type-I beta-barium borate (BBO) crystals in a cross-axis configuration [22], generating photon pairs centered at 650 nm. Wave plates H_0 (half) and Q_0 (quarter) are set to produce photons in a polarization state close to $|\alpha\rangle$ [Eq. (4)]. We use the encoding $|H\rangle \rightarrow |0\rangle$, $|V\rangle \rightarrow |1\rangle$, where $|H\rangle$ and $|V\rangle$ correspond to horizontal and vertical polarizations, respectively. Different values of α are realized by varying the angle on H_0. By keeping only two correlated directions produced in the SPDC we define the optical path qubits [Fig. 2(b)]. Path-dependent attenuators are used to make amplitudes match those of $|\alpha\rangle$. With this, we obtain another copy of the initial state between the parties.

The subsequent stage of the setup, with Alice's and Bob's devices, is illustrated in Fig. 2(c). On Alice's side, two black boxes are implemented with half-wave plates ($H_{A,p}$, $H_{A,s}$), quarter-wave plates ($Q_{A,p}$, $Q_{A,s}$), a beam displacer (BD_A) and a polarizing beam splitter (PBS_A). These components allow implementation of a fixed set of projections on both DOF utilized [23]. Inputs and outputs of the boxes are respectively given by the wave plates' angles and by the photon counts. Conditioning on Bob's state is implemented by coincidence detection.

On Bob's side, local filtering is implemented before photon detection. This is done with a variable mirror (VM) whose reflectance and transmittance depend on its position. The mirror acts only on the lower path on Bob's side, which is tailored to be more intense than the upper path. The VM is set so that transmission of the photon equalizes the amplitudes of both paths, thus implementing $K^{(0)}$ [Eq. (7a)]. Reflection, on the other hand, corresponds to $K^{(1)}$ and completely destroys steering in the path DOF—polarization is then used to prevent weakening of the final correlation.

Photon detectors after the VM register the outcomes, communication to Alice's side is done also through coincidence detection. Quantum state tomography of all assemblage components $\sigma_{a|x}$ is then realized and steering is analyzed on the reconstructed assemblage. Fully characterized components $H_{B,s}$, $Q_{B,s}$, $PBS_{B,s}$, BD_B allow path mode tomography on the transmitted path; wave plates $H_{B,p}$, $Q_{B,p}$, and $PBS_{B,p}$ on the reflected path allow for polarization tomography.

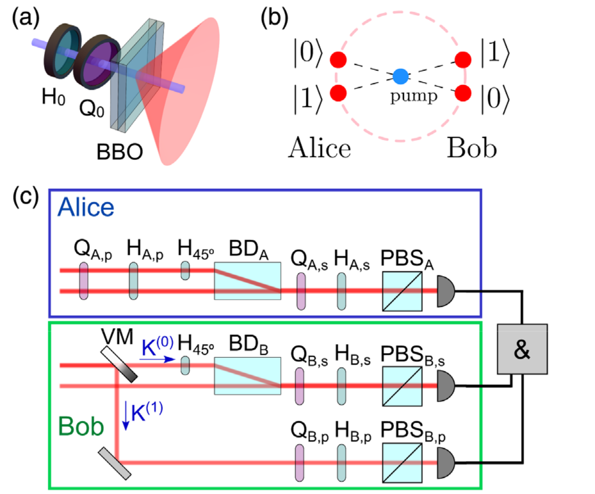

FIG. 2. (a),(b) Production of entangled photons via SPDC. The down-converted light is spectrally filtered to (650 ± 10) nm and collimated by a lens (not shown in figure), which converts the photons' momentum to spatial modes parallel to the pump beam. Only two pairs of correlated spatial modes are used in the remainder of the setup, as shown in (b), corresponding to two additional qubits besides polarization. (c) Setup for quantum steering distillation. Alice's interferometer allows for measurements both on polarization and on the spatial DOF and comprises Alice's two initial black boxes. On Bob's side, the amplitude filter is implemented by the variable reflectivity mirror (VM); reflectivity is tuned so that spatial DOF amplitudes become equalized when the photon is transmitted. Polarization is ignored and tomography of spatial mode DOF ensues in the upper branch of Bob's setup. Legend for the components: Q—Quarter-wave plate; H—Half-wave plate; BD—Beam displacer; PBS—Polarizing beam splitter; VM—Variable reflectivity mirror. Subindices indicate to which party (A—Alice, B—Bob) and to which type of measurement (p—polarization, s—spatial mode) the component pertains.

The results are shown in Fig. 3. Singlet-assemblage fractions for the distilled and original assemblages are shown as function of the population imbalance $\alpha^2 - \beta^2$ of Bob's reduced state. We also show the singlet-assemblage fraction for the case of postselection, where only the singlet assemblage is kept, given occurrence of a successful local filtering. Distillation with few copies is revealed, considering the average assemblage obtained by combining successful and unsuccessful runs of the protocol using experimentally obtained probabilities for each outcome ω.

As defined here, the singlet-assemblage fraction is not a steering monotone (meaning that it may increase under 1W-LOCCs). Hence, another comparison is made using the

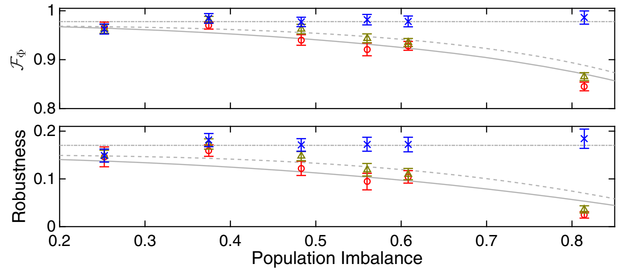

FIG. 3. Singlet-assemblage fraction (top) and LHS robustness of steering (bottom) as a function of Bob's reduced state amplitude imbalance $\alpha^2 - \beta^2$. Circles (red) correspond to the original assemblage $\mathbf{\Sigma}_{A|X}^{(\alpha)}$ shared as a base for the copies; crosses (blue) are postselected successfully distilled assemblages, obtained when Bob obtains the outcome $\omega = 0$ on his local filter; triangles (orange) correspond to the resulting average assemblages obtained by applying Protocol 1 to two copies of the original assemblage. Successful distillation can be observed as the values of both measures increase after the process, even for imbalances as high as 0.81.

steering LHS robustness, which is a proper steering monotone that measures the amount of unsteerable noise that a given assemblage tolerates before becoming itself unsteerable [3,15,16]. This is shown in the bottom part of Fig. 3. The same qualitative behavior can be observed for the robustness, with an increase observed for both the average assemblage and the postselected assemblage. Both observations then demonstrate the successful experimental distillation of an assemblage with stronger steering than initially.

Concluding remarks.—We have devised a steering distillation protocol inspired on the original local-filtering protocol for entanglement distillation [4], but exploiting quantum control only on one party. In other words, our protocol works with local operations assisted by classical communication from the untrusted part to the trusted one, a strict subclass of the LOCCs used in entanglement distillation that meets the natural restrictions of the semi-DI scenario. In contrast, we note that Ref. [14] also studies a steering-filtering protocol but to demonstrate steering superactivation, so it is not only device dependent (exploiting quantum control at both sides) but also probabilistic. Our protocol concentrates steering deterministically. Moreover, it works both in the limiting case of an infinite number of copies of the initial assemblage and in the nonasymptotic regime. In fact, we have experimentally demonstrated it for two copies of an input assemblage, each one encoded in a different degree of freedom (polarization or spatial) of the same twin photon pair. To the best of our knowledge, this is the first experimental demonstration of quantum steering distillation.

Our results offer a number of exciting open problems. For example, does the converse of steering distillation, i.e., dilution, exist? If so, are these two processes reversible? Or is there a notion of bound steering analogous to bound entanglement [24]? Finally, in our specific setup the quantum state underlying the assemblage also has its entanglement concentrated by the steering distillation protocol. Is this a generic feature, or can one distill steering without the underlying device-dependent process distilling entanglement?

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[18] The absence of Alice's parameters a and x in the denominator of the transformed assemblage can be understood as a consequence of the effect of the POVM also on Alice's black box: Given that outcome ω is obtained, the resulting assemblage is described by the components $\sigma'_{a|x} = P(a|x,\omega)\rho^{(\omega)}_{a,x}$, where $\rho^{(\omega)}_{a,x} = K^{(\omega)}\rho_{a,x}K^{(\omega)\dagger}/\mathrm{Tr}[K^{(\omega)}\rho_{a,x}K^{(\omega)\dagger}]$ and, from Bayes' rule, $P(a|\omega,x) = P(a|x)P(\omega|a,x)/P(\omega|x)$. Eq. (6) follows from noting that $P(\omega|a,x)=\mathrm{Tr}[K^{(\omega)}\rho_{a,x}K^{(\omega)\dagger}]$ and $P(\omega|x) = P(\omega) = \mathrm{Tr}[K^{(\omega)}\rho_B K^{(\omega)\dagger}]$.

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[21] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.124.120402 for a motivation for the assemblage fidelity and the demonstration of its properties. It also contains the proof of the main theorem.

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