Work Fluctuations and Jarzynski Equality in Stochastic Resetting

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We consider the paradigm of an overdamped Brownian particle in a potential well, which is modulated through an external protocol, in the presence of stochastic resetting. Thus, in addition to the short range diffusive motion, the particle also experiences intermittent long jumps that reset the particle back at a preferred location. Due to the modulation of the trap, work is done on the system and we investigate the statistical properties of the work fluctuations. We find that the distribution function of the work typically, in asymptotic times, converges to a universal Gaussian form for any protocol as long as that is also renewed after each resetting event. When observed for a finite time, we show that the system does not generically obey the Jarzynski equality that connects the finite time work fluctuations to the difference in free energy. Nonetheless, we identify herein a restricted set of protocols which embraces the relation. In stark contrast, the Jarzynski equality is always fulfilled when the protocols continue to evolve without being reset. We present a set of exactly solvable models, demonstrate the validation of our theory and carry out numerical simulations to illustrate these findings. Finally, we have pointed out possible realistic implementations for resetting in experiments using the so-called engineered swift equilibration.

Introduction.—Stochastic thermodynamics is a cornerstone in nonequilibrium statistical physics [1–5]. Microscopic systems satisfy stochastic laws of motion governed by force fields and thermal fluctuations that arise due to the surrounding. The subject then teaches us that thermodynamic observables such as work, heat, entropy production etc. measured along the stochastic trajectories taken from ensembles of such dynamics will fluctuate too. Understanding the distribution and the statistical properties of these fluctuations is of great interest since they hold a treasure trove of information about microscopic systems and how they respond to external perturbations. Indeed there has been a myriad of studies to understand e.g., nonequilibrium dynamics of biopolymers [6,7], colloidal particles [8–14], efficiency of molecular biomotors [15,16] and microscopic engines [17], heat conduction [18,19], electronic transport in quantum systems [20], trapped-ion systems [21], and many more [22]. Although we observe such diverse small systems with no apparent similarity, it is remarkable to find that there exist some universal relations which are shared in common. One of the most celebrated ones is perhaps the Jarzynski equality (JE) that relates the nonequilibrium fluctuations of the work to the equilibrium free energy difference [23–25]. Universalities of such kind have always been considered as an important feature in physical sciences and in this Letter we seek out for thermodynamic invariant principles in stochastic resetting systems [26].

Dynamics with stochastic reset has drawn a lot of attention recently because of its rich nonequilibrium properties [26–41] and its broad applicability in first passage processes [41–51]. Nevertheless, thermodynamical perspective of resetting systems has been largely overlooked so far. It was only recently when first and second laws of thermodynamics were interpreted by identifying the contributions to the total entropy production [52], and furthermore it was shown to satisfy a universal integral fluctuation relation [53]. While these first studies focused exclusively on the entropy production, efforts are yet to be made to understand other response functions. Moreover, not much is known about the distribution of these observables. In particular, one important observable is the work function that is produced due to external perturbations to the system. Work statistics encode important features of an out-of-equilibrium thermodynamic process, but its computation is usually quite daunting. Here, we set out to characterize work fluctuations in a stochastic system that is subjected to resetting. Our detailed analysis to this account then reveals the emergence of robust universal pattern in work fluctuations: first, resetting renders work fluctuations Gaussian independent of the nature of the external perturbation that produces it. Second, work fluctuations are found to obey the JE under certain conditions which we identify through this comprehensive study.

General theory.—For the sake of generality, we put forward our results in the paradigmatic framework of

a one-dimensional overdamped Brownian particle in a potential $U[x, \lambda(t)]$, which is modulated externally through the protocol $\lambda(t)$. Motion of such a particle is governed by the Langevin equation of the form

$$\dot{x}(t) = -\gamma^{-1}\partial_x U[x, \lambda(t)] + \sqrt{2D}\eta(t), \qquad (1)$$

where γ and D are the friction and diffusion coefficients, respectively, that satisfy the fluctuation-dissipation relation, i.e., $D\gamma = k_B T$, with k_B being the Boltzmann constant and T being the temperature of the medium. We assume $\langle\eta(t)\rangle = 0$ and $\langle\eta(t)\eta(t')\rangle = \delta(t - t')$. Moreover, let us consider that the position of the particle at $t = 0$ is distributed according to the probability density function (PDF) $p_{\text{ini}}(x_0)$. At random times taken from an exponential distribution $f(t) = re^{-rt}$, the particle in motion is stopped and teleported to the initial configuration. A schematic of our model system is depicted in Fig. 1.

The external modulation of the potential performs work on the system, which can be defined as [23]

$$W = \frac{1}{k_B T}\int_0^t dt' \frac{\partial U[x(t'), \lambda(t')]}{\partial \lambda}\frac{d\lambda(t')}{dt'}, \qquad (2)$$

measured in units of $k_B T$. In what follows, we will set $k_B T = D = 1$ without any loss of generality.

Since the reset process is instantaneous, we will assume that no work was done during this course (a finite resetting work will be discussed later). In order to quantify the work fluctuations, it is convenient to first define the moment generating function (MGF) namely

$$H_r(k, t) \equiv \int_{-\infty}^{\infty} dW e^{-kW} P_r(W, t), \qquad (3)$$

where $P_r(W, t)$ is the PDF of the work at time t, averaged over the initial distribution $p_{\text{ini}}(x_0)$ and the underlying dynamics with stochastic resetting. To delve deeper, we make use of the renewal structure in resetting dynamics to construct a relation that connects the MGF for $r > 0$ to that of $r = 0$ for any initial and subsequent resetting positions

$$\tilde{H}_r(k, s) = \frac{\tilde{H}_0(k, s + r)}{1 - r\tilde{H}_0(k, s + r)}, \qquad (4)$$

where $\tilde{H}_r(k, s) = \int_0^{\infty} dt\, e^{-st} H_r(k, t)$ and the subscript 0 indicates the observables with $r = 0$. We have added a proof of Eq. (4) in the Supplemental Material [54], but it is imperative to stress the following points here. Note that Eq. (4) holds for any initial condition and naturally adheres to a fixed initial condition which was derived in [55,56], but in the absence of any protocol $\lambda(t)$. In the presence of protocol, one needs to be meticulous since the structure of this equation relies on the fact that $\lambda(t)$ *is also renewed* after each resetting. As we will see later, Eq. (4) *does not hold* when the protocol is *unaffected* under resetting [54].
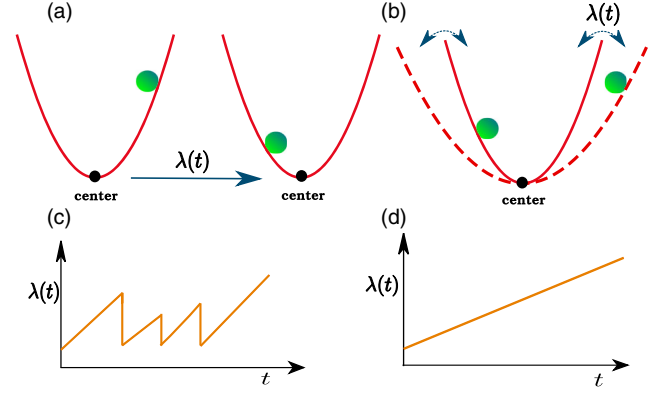

FIG. 1. Schematic of a Brownian particle confined in a harmonic trap $U[x, \lambda(t)] = \kappa(t)[x - y(t)]^2/2$, where $\lambda(t) = \{y(t), \kappa(t)\}$ represents the set of time-dependent protocols which are independently regulated. $\lambda = y(t)$ and $\lambda = \kappa(t)$ indicate the center of the trap (panel a) and the stiffness (panel b), respectively. The resetting mechanism acts both on the particle and the protocols as mentioned in the text. Here, we show the modulation of the protocol when it is renewed after each resetting (panel c) or remains unaltered (panel d).

The MGF, given by Eq. (4), can be inverted to obtain the full work statistics at a given time. But, we will show that it suffices to know the first and second moment to predict the universal behavior of the work fluctuations in the large time limit. To this end, we first note that the nth moment of W in Laplace space can be written as $\int_0^{\infty} dt\, e^{-st}\langle W^n(t)\rangle_r = \{\partial^n/[\partial(-k)^n]\}\tilde{H}_r(k, s)|_{k\to 0}$, which satisfies a recursive-renewal structure [54]

$$\tilde{W}_r^n(s) = \frac{s + r}{s}\left[\tilde{W}_0^n(s + r) + r\sum_{l=1}^{n}\binom{n}{l}\tilde{W}_r^{n-l}(s)\tilde{W}_0^l(s + r)\right],$$
$$(5)$$

where we have defined $\tilde{W}_r^n(s) \equiv \int_0^{\infty} dt e^{-st}\langle W^n(t)\rangle_r$. Equation (5) gives a simple recipe to compute all the moments of W recursively from the knowledge of the moments of the process without resetting.

Universal work fluctuations.—The infinite set of moments given by Eq. (5) contains the same information as that of the full distributions $P_r(W, t)$. However, physical intuition tells us that not all the moments contribute significantly for a long time. To see this, we consider a trajectory of time length t with multiple possible resetting events. The total work done along this long trajectory can then be decomposed into the sum of the partial works produced in each time interval between the resetting events. However, these intervals are statistically independent since the entire configuration of the system (comprising the particle and the trap) is renewed after each resetting event, and hence there are no correlations between the intervals. Therefore, for a long enough observation time t one would expect an average $\sim rt$ number of resetting events, and the

total work $W(t)$ can then be written as $W \approx W_1 + W_2 + W_3 + \cdots + W_{[rt]}$. Since the intervals are disjointed, the W_is are also independent and identically distributed. Moreover, if W_is are regular (with finite mean and variance), one would expect that the distribution of W, according to the central limit theorem, would converge to a Gaussian irrespective of the nature of the potential and choice of the external protocol

$$P_r(W, t) = \frac{1}{\sqrt{2\pi\sigma_W^2(t)}} \exp\left[-\frac{(W - \mu_t)^2}{2\sigma_W^2(t)}\right], \quad (6)$$

where the mean $\mu_t \equiv \langle W \rangle_r$ and the variance $\sigma_W^2(t) \equiv \langle W^2 \rangle_r - \langle W \rangle_r^2$ are computed from Eq. (5). In Fig. 2, we demonstrate Eq. (6) in the set up of a 1D Brownian particle confined in a harmonic trap $U[x, \lambda(t)] = \kappa(t)[x - y(t)]^2/2$, where $\kappa(t)$ and $y(t)$ represent the stiffness and center of the trap respectively (see details of the simulation in the Supplemental Material [54]).

Jarzynski equality—reset protocol.—The JE relates the finite time work fluctuations to the equilibrium free energy and here we ask whether such relations hold generically in resetting systems. We consider the same setup as before and assume that each resetting act renews both the particle and the protocol. We further assume that the initial condition is taken from an equilibrium distribution $p_{\text{ini}}(x_0) \propto \exp\{-U[x_0, \lambda(0)]\}$, which is an essential prerequisite for the JE. Employing Eq. (4) and substituting $k = 1$ there, we find a renewal expression for the average of the exponentiated work which connects to the same quantity with $r = 0$ in the Laplace space [54]
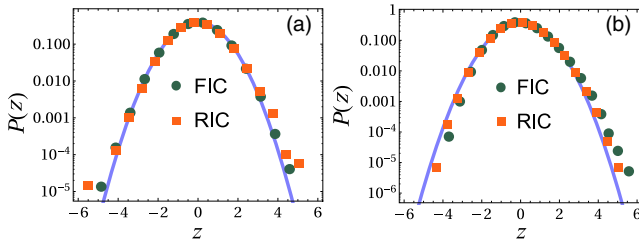

FIG. 2. Numerical computation of the PDF of the rescaled work $z = (W - \mu_t)/\sigma_W(t)$ performed on a Brownian particle in a harmonic trap for the linear modulation of the trap center, i.e., $y(t) = ut$ (panel a) and the stiffness, i.e., $\kappa(t) = \kappa_0 + vt$ (panel b), respectively. Simulations are performed for fixed initial condition (FIC): $p_{\text{ini}}(x_0) = \delta(x_0)$ (circle markers) and random initial condition (RIC): $p_{\text{ini}}(x_0) = p_{\text{eq}}(x_0) \propto \exp\{-U[x_0, \lambda(0)]\}$ (square markers), respectively, for each of the above cases. Parameters for panel (a): $\kappa_0 = 1.5, u = 0.2$ for FIC and $\kappa_0 = 0.5, u = 0.5$ for RIC, respectively, where $r = 0.5$ and $t = 10$ are set identical for both of these cases. Similarly, parameters for panel (b): $\kappa_0 = 0.5, v = 0.002, y = 0, r = 5$, and $t = 500$ for both FIC and RIC. Numerical simulations are corroborated with the theoretical prediction (solid line in both cases) given by $P(z) = e^{-z^2/2}/\sqrt{2\pi}$, and we see an excellent Gaussian collapse.

$$\mathcal{L}_{t \to s}[\langle e^{-W} \rangle_r] = \frac{\mathcal{L}_{t \to s+r}[\langle e^{-W} \rangle_0]}{1 - r\mathcal{L}_{t \to s+r}[\langle e^{-W} \rangle_0]}, \quad (7)$$

where $\mathcal{L}$ is the Laplace transform operator. Several comments are in order now. The exponential average on the rhs is along the trajectory without resetting and therefore must satisfy the JE, i.e., $\langle e^{-W_{[0,t]}} \rangle_0 = e^{-\{F_0[\lambda(t)] - F_0[\lambda(0)]\}}$, where $F_0[\lambda(t)]$ is the free energy of the underlying system (i.e., when the dynamics is not interrupted by resetting) corresponding to the value of λ evaluated at time t. However, it is evident that substituting this in Eq. (7) will not essentially lead to $e^{-\Delta F_0(t)}$ [where $\Delta F_0(t) = F_0[\lambda(t)] - F_0[\lambda(0)]$ is the free-energy difference] along the entire trajectory of length t in the presence of resetting, i.e., JE will not be obeyed generically for any arbitrary protocol. Nonetheless, we identify the protocols which will indeed satisfy this condition. This happens when the modulation of the protocol renders a linear change in the free energy, i.e., $\Delta F_0(t) = \alpha t$. The trivial scenario, i.e., $\Delta F_0(t) = 0$ is true under any external perturbation which is of the following form: $U[x, y(t)] = U[x - y(t)]$. This could happen, e.g., when we move the center of the trap $y(t)$ according to some specific schedule. On the other hand, the nontrivial linear change in $\Delta F_0(\neq 0)$ occurs when, e.g., the stiffness $\kappa(t)$ is varied exponentially as a function of time. Utilizing this condition in Eq. (7), we obtain $\langle e^{-W} \rangle_r = e^{-\Delta F_0(t)}$. In fact the linear time dependence in ΔF_0 is not only sufficient but also remains as a necessary condition [54].

We now briefly summarize the numerical setups that are used to verify these findings. We have simulated an overdamped Brownian particle in a harmonic trap $U[x, \lambda(t)] = \kappa(t)[x - y(t)]^2/2$ in the presence of resetting ($r = 0.5$), and measured e^{-W} till time $t = 5$. In Fig. 3(a), we have shown the convergence of the statistical average $\langle e^{-W} \rangle_r$ as a function of realizations N_R for the following protocol modulations (i) moving the center of the trap with $y(t) = 0.2t$, (ii) changing stiffness with a power law $\kappa(t) = \kappa_0(1 + 0.2t)^{-2}$, and (iii) an exponential law $\kappa(t) = \kappa_0 e^{-0.2t}$. As before, we have regulated one protocol at a time keeping the others fixed. The horizontal lines shown in the panel correspond to the theoretical prediction of $e^{-\Delta F_0(t)}$, which takes the values 1.0, 2.0, and ~1.65, respectively, for each of the modulations. The exact computation has been reserved to the Supplemental Material [54]. Note that modulations (i) leaves the free energy invariant, and (iii) imposes a linear change in ΔF_0 whereas modulation (ii) leads to a nonlinear evolution in ΔF_0 [54]. Thus, it is evident from Fig. 3(a) that the JE holds for modulations (i) and (iii), but not for modulation (ii).

Jarzynski equality is invariant under nonreset protocol.—The discussion so far focused on the case when we reset both the protocol and the particle. In the following, we relax this condition and assume that only the particle is reset while the protocol keeps evolving in time. Moreover,
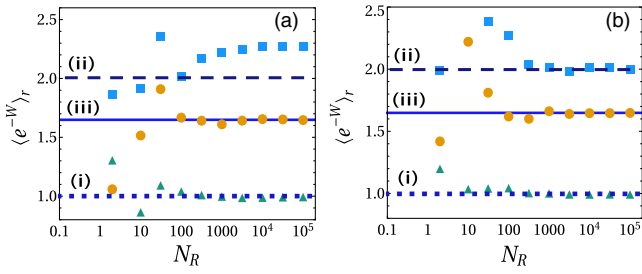
FIG. 3. Numerical verification of the JE: we have demonstrated convergence of $\langle e^{-W} \rangle_r$ as a function of the number of realizations N_R. We have used three different types of protocol modulations as mentioned in the main text. The analytical values of $e^{-\Delta F_0}$, shown by the horizontal lines [dotted for (i) moving trap, dashed and solid for the (ii) power law and (iii) exponential stiffness, respectively, in both panels], are plotted against numerical points for $\langle e^{-W} \rangle_r$ (shown by the triangles, squares, and circles, respectively). (a) Reset protocol. JE is seen to hold for protocols (i) and (iii) but not (ii). Parameters: $\kappa_0 = 1.5$ and $\kappa_0 = 0.5$ respectively for the center and stiffness modulation. (b) Nonreset protocol. JE holds for any protocols. Parameters: $\kappa_0 = 1$ and $\kappa_0 = 0.35$, respectively, for the center and stiffness modulation. In all the simulations, we have set $r = 0.5$ and $t = 5$.

we consider that after each resetting event, position of the particle is drawn from the equilibrium distribution $p_{\text{ini}}(x_0) \propto \exp\{-U[x_0, \lambda(t_i)]\}$ corresponding to λ measured at the times t_i of resetting. In this way, the particle is *effectively* equilibrated after each resetting event which is essential for the JE to hold. This construction correlates the intervals between resetting events: since the initial configuration of a given interval depends on the time spent in the previous one and hence renewal structure of Eq. (4) is lost [54]. However, notice that (i) the particle is prepared at the equilibrium state p_{ini} after each resetting event, and (ii) consequently, the equality is satisfied in any interval between two resetting events. Taking these two facts into account, one can show that the equality holds along the entire trajectory independent of the nature of the protocol [54]

$$\langle e^{-W} \rangle_r = e^{-\Delta F_0(t)}. \quad (8)$$

We numerically check Eq. (8) in Fig. 3(b) and show that indeed JE is invariant under *nonreset protocol* modulations.

Discussion.—Up to this point, we have strictly assumed that resetting is an instantaneous process and thus neglected any possible contributions coming from it to the total work done. However, in real world taking one particle from location A to B will require work and this contribution must be taken into account. Another essential aspect of the Letter is the imposed *equilibrated condition* upon each resetting which may appear artificial from an experimental point of view. To fill up these conceptual gaps, in this section, we put forward an explicit proposal for practical implementation of resetting which accounts for both these issues.

To proceed further, recall that the unhindered process, characterized by $p_0(x,t)$, satisfies the Fokker-Planck equation $\partial_t p_0(x,t) = \partial_x[(1/\gamma)U'[x, \lambda(t)]p_0(x,t)] + D\partial_x^2 p_0(x,t)$. Let us now focus on the first resetting event which occurs at a time t_r. Then the job of this engineered restart mechanism would be to take the current distribution $p_0(x, t_r)$ in the resetting time t_r and make a transformation to reach an arbitrary target distribution $p_f(x)$ (which in our case is the equilibrium density) in *a fixed time* $\Delta_r > 0$. There have been recent developments to design protocols, namely engineered swift equilibration [57–60], that short-cut the relaxation times between two target distributions whose properties can be controlled in time. We will now show how to choose the optimal protocol that renders the average irreversible work during the resetting mechanism minimum.

As a representative case, we will consider the Brownian particle diffusing in a harmonic trap whose stiffness is timely modulated, that is, $U(x, t) = \kappa(t)x^2/2$. At time zero, the particle is prepared in equilibrium with a zero-mean Gaussian and standard deviation $\sigma(0) \equiv \sigma_0 = \sqrt{\langle x^2(0) \rangle}$ with $\kappa(0) = \kappa_0$ which satisfies $\sigma_0^2 = k_B T/\kappa_0$ from the equipartition theorem. Due to the nature of the potential, position density remains to be zero-mean Gaussian at all times, i.e., $p_0(x,t) = \{[e^{-x^2/2\sigma^2(t)}]/[\sqrt{2\pi\sigma^2(t)}]\}$, where $\sigma(t)$ satisfies the following evolution equation: $\sigma(t)\dot{\sigma}(t) = -[\kappa(t)/\gamma]\sigma^2(t) + D$ [54]. Upon first restart at time t_r, the particle returns from a position which is distributed according to a Gaussian distribution with $\sigma(t_r)$ [corresponding to $\kappa(t_r)$]. The goal is then to *design an optimal protocol* $\kappa(t)$ $(t_r < t < t_r + \Delta_r)$, which drives the system from $\sigma(t_r)$ to σ_0 in time Δ_r within the most efficient energy consumption budget. The average work performed during this interval (with *reset protocol*) over many such trajectories is given by

$$\langle W_{rp} \rangle = \frac{1}{2} \int_{t_r}^{t_r + \Delta_r} dt\, \sigma^2(t)\dot{\kappa}(t) = W_1 + \Delta F_r + W^{\text{irr}}, \quad (9)$$

where $W_1 = \frac{1}{2}[\sigma_0^2 \kappa(0) - \sigma^2(t_r)\kappa(t_r)]$, and $\Delta F_r = -\frac{1}{2}k_B T \ln\{[\sigma_0^2]/[\sigma^2(t_r)]\}$ is the difference of free energy between the equilibrium states characterized by the initial and final variance [54]. Finally, the third term $W^{\text{irr}}[\dot{\sigma}(t)] = \gamma \int_{t_r}^{t_r + \Delta_r} dt\, \dot{\sigma}^2(t)$ can be identified as the irreversible work of the process which is always positive. Note that W_1 and ΔF_r are determined given the initial and target states leaving the dependence of the specific protocol and Δ_r only in W^{irr}. The optimal profile $\sigma_{\text{opt}}(t)$ that minimizes $W^{\text{irr}}[\dot{\sigma}(t)]$ can be immediately obtained using variational calculus and this reads [54]

$$\sigma_{\text{opt}}(t) = \sigma(t_r) + (t - t_r)\frac{\sigma_0 - \sigma(t_r)}{\Delta_r}. \quad (10)$$

Substituting this into the evolution equation for $\sigma(t)$, one immediately finds the corresponding profile for the optimal protocol $\kappa_{\text{opt}}(t)$. Interestingly, $\kappa_{\text{opt}}(t)$ develops finite discontinuities at the threshold times, i.e., $\kappa(t_r) \neq \lim_{t \to t_r} \kappa_{\text{opt}}(t)$ and $\kappa(t_r + \Delta_r) = \kappa(0) \neq \lim_{t \to t_r + \Delta_r} \kappa_{\text{opt}}(t)$, as was also found in other studies [59–61]. Implementing the protocol, we get the optimal irreversible work $W_{\text{opt}}^{\text{irr}} = \gamma\{[\sigma_0 - \sigma(t_r)]^2/\Delta_r\}$, which is exactly proportional to Δ_r^{-1} [54]. This ensures that a *perfect* instantaneous resetting (i.e., the $\Delta_r \to 0$ limit) is not physically viable (if a work indeed is accounted for the entire resetting mechanism) since the energetic cost for each resetting jump will be divergent. A similar analysis follows also for the nonreset protocol [54].

Conclusions and outlook.—In summary, this Letter discusses statistical properties of work fluctuations in a stochastic resetting system. We find that the introduction of resetting renders the work fluctuations Gaussian in the large time for the reset protocols. We infer that this is due to the renewal structure of the resetting process. Consequently, our approach also predicts emergence of Gaussian fluctuations for other thermodynamic observables such as dissipated heat, power flux or entropy production. A detailed analysis of this problem, however, remains to be seen. Furthermore, we note that only the typical fluctuations become Gaussian as a fallout of the central limit theorem. On the other hand, it is only rational to believe that such universality of the fluctuations will be lost while looking at the tail behavior (atypical fluctuations) of the work distribution. An outstanding challenge would be to extend our approach to capture such scenarios using large deviation theory [62].

Our research also presents an extensive study on JE in resetting systems and unravels different constraints on the temporal behavior of the protocols to preserve the JE. We have also put forward a neat feasible experimental pathway to implement real resets inspired by the recent developments on the edge between stochastic thermodynamics and control theory. We would like to stress the fact that the JE still prevails if the observable under study is the work W [Eq. (2)] performed only between the reset events, while the total work contribution namely $W + W_{rp}^{\text{tot}}$ (the latter averaged over all the resetting events) would lead to nongeneric results specific to resetting mechanism. Finally, we leave a note that our central results, although illustrated here under a one-dimensional framework, are also valid in higher dimensions. Furthermore, recent experiments allow us to explore one-dimensional systems with a high precision and accurate measurement (see, e.g., [13,17,57,63,65–67]). Naturally, our study opens up a new research avenue in stochastic thermodynamics of resetting with a great appeal to the experimental demonstration of the work fluctuation theorems in the resetting controlled biophysical [64] and single molecular systems using optical traps [57,63,65–67].

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