Negativity of Quasiprobability Distributions as a Measure of Nonclassicality

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We demonstrate that the negative volume of any s-parametrized quasiprobability, including the Glauber-Sudashan P function, can be consistently defined and forms a continuous hierarchy of nonclassicality measures that are linear optical monotones. These measures belong to an operational resource theory of nonclassicality based on linear optical operations. The negativity of the Glauber-Sudashan P function, in particular, can be shown to have an operational interpretation as the robustness of nonclassicality. We then introduce an approximate linear optical monotone, and we show that this nonclassicality quantifier is computable and is able to identify the nonclassicality of nearly all nonclassical states.

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Nonclassical states of light find useful applications in tasks such as quantum metrology [1,2], quantum teleportation [3], quantum cryptography [4], quantum communication [5], and quantum information processing [6]. Correspondingly, there has been great interest in the characterization, verification, and quantification of nonclassicality in quantum states [7–21]. Due to the linearity of quantum mechanics, measurement statistics from a nonclassical state may be reproduced by a classical state by appropriately modifying the measurement [22]. For this reason, nonclassicality in light is usually considered within the context of fixed measurement operations, such as homodyne measurements that measure quadrature variables of light. Such quadrature variables are related to quasiprobability distribution functions: the negativity of which is considered a nonclassicality indicator. The operational formalism of quasiprobability distribution functions has led to important results about the structure of quantum theory [23], quantum computation [24–26], and simulation of quantum optics [27,28].

It is typically considered that the most classical states of a bosonic field are the coherent states [29]. Defined as the eigenstates of the annihilation operator, $a|\alpha\rangle = \alpha|\alpha\rangle$, the dynamics of coherent states in a quadratic potential closely resemble that of a classical harmonic oscillator [30]. The seminal works of Glauber [29] and Sudarshan [31] showed that every quantum state of light may be written in the form of $\rho = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|$, where $P(\alpha)$ is referred to as the Glauber-Sudarshan P function. When $P(\alpha)$ corresponds to a proper probability density function, the quantum state is a statistical mixture of coherent states and is considered classical. More generally, $P(\alpha)$ may not correspond to any classical probability density function; in which case, the state is considered nonclassical. It is well known that the only classical pure states are the coherent states [32].

Because the P function is frequently highly singular [33], it is neither theoretically nor experimentally accessible in many instances. As such, previous efforts have largely focused on finding methods to quantify nonclassicality via other means. The Mandel Q parameter [7], for instance, measures the deviation from Poissonian statistics. The entanglement potential quantifies the maximum amount of entanglement that can be generated from a beam splitter [8]. The nonclassicality depth quantifies the amount of interaction with a thermal state in order to erase nonclassicality [9,10]. One may also count the number of superpositions of coherent states [11], the amount of coherent superposition between coherent states [12], the sensitivity of a quantum state to operator ordering [13], various geometric distances from the closest classical state [13–16], the negativity of the Wigner function [17], non-Gaussianity [18,19], or the amount of metrological advantage [20,21]. However, these nonclassicality measures are frequently computationally intractable, unable to detect every nonclassical state, or lack physical interpretations.

In this Letter, we propose a method to directly quantify the negativity of the P function and, more generally, any s-parameterized quasiprobability [34] in a consistent way. It is based on the nonclassicality filtering approach considered in Refs. [35,36]. We show that the negativity of the s-parameterized quasiprobabilities monotonically increases with s and that this approach leads to a continuous hierarchy of nonclassicality measures under the operational resource theory of nonclassicality proposed in Ref. [12]. In particular, the negativity of the P function is shown to have a direct operational interpretation in terms of the robustness of nonclassicality to statistical noise, as well as the cost of simulation. Finally, we propose an approximate nonclassicality monotone that is numerically computable for an arbitrary quantum state. For readability, detailed discussions of technical proofs are deferred to the Supplemental Material [37].

Preliminaries.—We first introduce the characteristic function of the Glauber-Sudarshan P function. A common convention is to define it as the integral $\int d^2\alpha P(\alpha) \times \exp[2i(\beta_i\alpha_r - \beta_r\alpha_i)]$, where α_r, β_r and α_i, β_i are the real and imaginary components of α and β, respectively. One observes that this is a multivariate Fourier transformation. For our purposes, we will adopt the following convention:

$$\chi(\beta) \coloneqq \int d^2\alpha P(\alpha) \exp[2\pi i(\beta_i\alpha_r + \beta_r\alpha_i)].$$

It should be clear that this definition essentially corresponds to a change in variables of types $\beta_i \to \pi\beta'_i$ and $\beta_r \to -\pi\beta'_r$, and so it does not alter the information content of the characteristic function. It also adheres more closely to the conventional definition of the Fourier transform in the ordinary frequency domain: $\mathcal{F}f(y) \coloneqq \int dx f(x) \times \exp(-2\pi ixy)$. The corresponding inverse Fourier transform is then $\mathcal{F}^{-1}f(y) \coloneqq \int dx f(x)\exp(2\pi ixy)$. This allows us to write $P(\alpha) = \mathcal{F}\chi(\alpha)$. All physical characteristic functions satisfy $|\chi(\beta)| \leq \exp(\pi^2|\beta|^2/2)$.

One major issue with the P function is that it is frequently highly singular. This complicates our ability to analyze and quantify the nonclassicality of quantum states via the P function alone, and it necessitates the use of other nonclassicality criteria.

We consider the filtered P functions proposed in Ref. [35]. Filtered P functions are based on the observation that $P(\alpha)$ is the (multivariate) Fourier transform of the characteristic function $\chi(\beta)$ such that $P(\alpha) = \mathcal{F}\chi(\alpha)$. This opens up the possibility of applying a filtering function $\Omega_w(\beta)$ prior to the Fourier transform. The non-negative parameter ω is to be interpreted as the width of the filter. The filtered function is then

$$P_{\Omega,w}(\alpha) \coloneqq \mathcal{F}\chi_{\Omega,w}(\alpha)$$

where $\chi_{\Omega,w}(\beta) \coloneqq \chi(\beta)\Omega_w(\beta)$. In general, the characteristic P and filtered P functions depend on the state ρ. When the state ρ is unambiguous, the characteristic function is denoted χ and $\chi(\alpha)$ is the function at point α. When ρ needs to be specified, the characteristic function is denoted $\chi(\rho)$, whereas $\chi(\alpha|\rho)$ is the function at α. Similar notations will also be used for the unfiltered and filtered P functions.

s-parameterized negativities.—The goal is to be able to consistently define the negativity of the P function, even when it is highly singular. For this purpose, we consider a carefully chosen nonclassicality filter Ω_w satisfying the following properties: (a) $\Omega_w(\beta)$ is factorizable such that $\Omega_w(\beta) = \Omega^1_w(\beta)\Omega^2_w(\beta)$ such that $\Omega^i_w(\beta)$ is square integrable for $i = 1,\ 2$. (b) $\Omega^1_w(\beta)e^{\pi^2|\beta|^2/2}$ is square integrable. (c) $\Omega_w(0) = 1$ and $\lim_{w\to\infty}\Omega_w(\beta) = 1$. (d) There exists $t > 0$ such that $\Omega_w(\beta) = \Omega_{w/|r|}(\beta)\Omega_t(\beta)$ for any $|r| < 1$, and some $t > 0$. (e) $\Omega_w(\beta) = \Omega_{kw}(k\beta)$ for any $k > 0$.

Note that these conditions are stronger than those proposed in Ref. [35]. There, the key requirement was for $\Omega_w(\beta)e^{\pi^2|\beta|^2/2}$ to be square integrable in order to ensure that its Fourier transform would also be square integrable due to Plancherel's theorem. Square integrability is, however, not sufficient to ensure that $P_{\Omega,w}(\alpha)$ is pointwise finite for every α. Our modified approach closes this gap by ensuring that $P_{\Omega,w}(\alpha)$ is always finite, which allows us to numerically determine whether there is negativity at a given point α.

Theorem 1: If Ω_w satisfies properties (a) and (b), then $P_{\Omega,w}(\alpha)$ contains no singularities and is finite for every α.

Theorem 1 thus allows us to assign definite positive or negative values at every point α of $P_{\Omega,w}(\alpha)$. This implies that we can determine unambiguously the positive and negative regions of $P_{\Omega,w}(\alpha)$. As such, for every w, the negative volume of $P_{\Omega,w}(\alpha)$ is well defined. Property (c) then guarantees that the filtered function is a proper quasiprobability function such that $\int d^2\alpha P_{\Omega,w}(\alpha) = 1$ and that, for sufficiently large w, $\mathcal{F}\Omega_w(\alpha) \approx \delta(\alpha)$; so, the original P function is retrieved.

It is well known that the characteristic function of P is related to the characteristic functions of other commonly studied quasiprobability distributions via the following relation:

$$\chi_s(\beta) \coloneqq \chi(\beta)e^{-(1-s)\pi^2|\beta|^2/2}.$$

This differs slightly from the usual convention due to the convention we employ for $\chi(\beta)$. For $s = 1$, we retrieve the characteristic function of the P function; for $s = 0$, the characteristic function leads to the Wigner function; whereas for $s = -1$, the characteristic function is related to the Husimi Q function. These form the set of s-parametrized quasiprobability distributions [34]. Just like for the P function, we can also apply a filter to other s-parametrized quasiprobabilities. This allows us to define the s-parametrized negativity in the following manner:

Definition 1 [s-parametrized negativity].—Let $P_s(\alpha) \coloneqq \mathcal{F}\chi_s(\alpha)$ be some s-parametrized quasiprobability; and let $P_{s,\Omega,w}(\alpha) \coloneqq \mathcal{F}\chi_{s,\Omega,w}(\alpha)$ be the filtered s-parametrized quasiprobability, where $\chi_{s,\Omega,w}(\beta) \coloneqq \chi_s(\beta)\Omega_w(\beta)$ for some filter Ω_w satisfying properties (a)–(c).

We can then write $P_{s,\Omega,w}(\alpha) = P^+_{s,\Omega,w}(\alpha) - P^-_{s,\Omega,w}(\alpha)$, where $P^\pm_{s,\Omega,w}(\alpha)$ are well defined non-negative functions.

The s-parametrized negativity is defined as

$$\mathcal{N}_s(\rho) \coloneqq \lim_{w\to\infty} \int d^2\alpha P^-_{s,\Omega,w}(\alpha).$$

In particular, when $s = 1$, $\mathcal{N}_1(\rho) \coloneqq \mathcal{N}(\rho)$ is the negativity of the P function.

Given the above definitions, one still needs to find an appropriate filter Ω_w. The astute reader may have noticed that properties (d) and (e) are not yet discussed. They will play an important role, which will be described in greater detail in a subsequent section. We first establish several properties of the s-parameterized negativities.

Negativity as a linear optical monotone.—In Ref. [12], a resource theoretical approach was proposed to quantify nonclassicality in radiation fields. There, it was argued that nonclassicality measures should be linear optical monotones; i.e., nonclassicality should be measured using quantities that do not increase under linear optical maps. Under this approach, nonclassicality may be considered as resources that overcome the limitations of linear optics.

Linear optical maps are formally defined to be any quantum map that can be written in the form

$$\Phi_L(\rho_A) \coloneqq \mathrm{Tr}_E[U_L(\rho_A \otimes \sigma_E)U_L^\dagger],$$

where σ_E is a classical state, and U_L is a linear optical unitary composed of any combination of beam splitters, phase shifters, and displacement operations. Such unitary transforms will always map an N mode bosonic creation operator

$$a_{\vec{\mu}}^\dagger \coloneqq \sum_{i=1}^{N} \mu_i a_i^\dagger$$

into $a_{\vec{\mu}'}^\dagger + \oplus_{i-1}^{N} \alpha_i \mathbb{1}_n i$, where $\vec{\mu}$, $\vec{\mu}'$, are N-dimensional complex vectors of unit length, and $\mathbb{1}_i$ is the identity operator on the ith mode.

One may also incorporate postselection into the definition by defining selective linear optical operations via a set of Kraus operators K_i for which there exist linear optical unitary U_L, classical ancilla $\sigma_{EE'}$, and a set of orthogonal vectors $\{|i\rangle_{E'}\}$ such that

$$\mathrm{Tr}_E[U_L(\rho_A \otimes \sigma_{EE'})U_L^\dagger] = \sum_i p_i \rho_A^i \otimes |i\rangle_{E'}\langle i|,$$

where $p_i \rho_A^i \coloneqq K_i \rho_A K_i^\dagger$ and $p_i \coloneqq \mathrm{Tr}(K_i \rho_A K_i^\dagger)$.

Based on this definition, the following theorem shows that the negativities $\mathcal{N}_s$ form a continuous hierarchy of linear optical monotones that belongs to the operational resource theory outlined in Ref. [12].

Theorem 2: The s-parametrized negativity $\mathcal{N}_s(\rho)$ is a nonclassicality measure satisfying the following properties: (1) $\mathcal{N}_s(\rho) = 0$ if ρ has a classical P function. (2a) (*Weak monotonicity*) $\mathcal{N}_s(\rho) \geq \mathcal{N}_s(\Phi_L(\rho))$. (2b) (*Strong monotonicity*)

$$\mathcal{N}_s(\rho) \geq \sum_i p_i \mathcal{N}_s(\rho_i),$$

where $p_i \coloneqq \mathrm{Tr}(K_i^\dagger K_i \rho)$, $\rho_i \coloneqq (K_i \rho K_i^\dagger)/p_i$, and

$$\Phi_L(\rho) = \sum_i K_i \rho K_i^\dagger$$

is a selective linear optical operation. (3) (*Convexity*), i.e.,

$$\mathcal{N}_s\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i \mathcal{N}_s(\rho_i).$$

In particular, if $s = 1$ and $\mathcal{N}_1(\rho) \coloneqq \mathcal{N}(\rho)$, then $\mathcal{N}(\rho) = 0$ if and only if ρ has a classical P function; i.e., $\mathcal{N}$ is a faithful measure identifying every nonclassical state.

Theorem 3: $\mathcal{N}_s(\rho)$ is a monotonically increasing function of $-1 \leq s \leq 1$ and is upper bounded by the negativity of the P function, i.e., $\mathcal{N}_s(\rho) \leq \mathcal{N}(\rho)$.

We can interpret the s-parametrized quasiprobability distributions as the P function with a Gaussian filter applied. In general, as s decreases, the width of the applied Gaussian filter increases, which decreases any observed negativity. This leads to correspondingly weaker measures. This decrease in the negativity may be interpreted as the effect of measurement inefficiencies [39]. For a double homodyne measurement with quantum efficiency $\eta \in [0, 1]$, the effectively observed P function is $P_s(\alpha)$, where $s = 2 - 1/\eta$. Ultimately, any negativity that is observed in the s-parametrized quasiprobabilities originates from the negativity of the Glauber-Sudarshan P function.

Operational interpretations of the negativity.—An operational measure that has been extensively studied in various quantum resource theories is the robustness [40,41]. It quantifies the minimum amount of mixing with noise that is necessary to make a given quantum state classical. It turns out that the negativity exactly quantifies the robustness of a given quantum state.

We can consider the following definition for the robustness of nonclassicality:

Definition 2 [Robustness of nonclassicality].—Let $\mathcal{P}$ be the set of all quantum states with classical P distributions.

The robustness of nonclassicality is defined as

$$\mathcal{R}(\rho) \coloneqq \min_{\sigma \in \mathcal{P}} \left\{ r | r \geq 0, \frac{\rho + r\sigma}{1 + r} \in \mathcal{P} \right\}.$$

Based on the above definition, one may show that the negativity and the robustness are, in fact, equivalent.

Theorem 4: The negativity and the robustness are equivalent measures of nonclassicality, i.e., $\mathcal{N}(\rho) = \mathcal{R}(\rho)$ for every quantum state ρ.

Theorem 4 therefore provides a direct operational interpretation for the negativity of the P function. The negativities of other s-parametrized quasiprobabilities can then be interpreted as lower bounds to the robustness. We also point out that $P_s(\alpha)$ may be interpreted as the P function after interaction with a thermal environment [2,9,10]. This means that, at $s < 1$, the negativity corresponds to the robustness of state under less than ideal environmental conditions; whereas at $s = 1$, it is the robustness under ideal conditions.

Another possible interpretation of the negativity is as the cost of simulating a nonclassical state in phase space. In particular, it is possible to show that the number of samples $s(\epsilon, \delta)$ required to classically simulate the measurement outcomes with a sampling error of less than ϵ and a success probability greater than $1 - \delta$ scales with [24,37]

$$s(\epsilon,\delta) \propto [1+2\mathcal{N}(\rho)]^2.$$

Because $\mathcal{N}(\rho)=0$ when the state is classical, the factor $[1+2\mathcal{N}(\rho)]^2$ describes the additional overhead required to simulate a nonclassical state. This suggests that states with greater negativities tend to be harder to simulate.

Approximate nonclassicality monotones.—The negativity of quasiprobabilities is well defined in Definition 1 but does not always lead to finite quantities. For instance, highly singular states such as squeezed states can possess infinite negativities. This can be verified numerically by applying an appropriate filter and computing the filtered negativities as $w\to\infty$. From Theorem 4, we know that this is because some states require an infinite amount of statistical mixing with classical states before their nonclassicality is erased. Nevertheless, $\mathcal{N}_s$ remains a linear optical monotone. For $s=1$, we retrieve the negativity $\mathcal{N}$ of the P function, which is able to unambiguously identify every nonclassical state. For $s<1$, $\mathcal{N}_s$ are weaker measures that may not be able to identify some nonclassical states. For instance, at $s=0$, $\mathcal{N}_s$ is the negativity of the Wigner function [17]. It is a well-known property of the Wigner function that its negativity cannot detect squeezed states.

It is natural to ask whether it is possible to avoid infinite values while simultaneously maximizing the number of identifiable nonclassical states. We show that this is possible via an appropriate choice of filters that satisfies the full suite of properties (a)–(e) (see Preliminaries section).

Theorem 5: If the filter Ω_w satisfies properties (a)–(e), with $\delta := \mathcal{N}(\mathcal{F}\Omega_{w=1})$, the filtered negativity $\mathcal{N}(P_{\Omega,w})$ is an approximate nonclassicality measure satisfying the following properties: (1) $\mathcal{N}(P_{\Omega,w})\le\delta$ if ρ is classical. (2) (*Approximate monotonicity*) For any linear optical map Φ_L,

$$(1+2\delta)\mathcal{N}[P_{\Omega,w}(\rho)]+\delta \ge \mathcal{N}\{P_{\Omega,w}[\Phi_L(\rho)]\}.$$

(3) (Convexity)

$$\mathcal{N}\left[P_{\Omega,w}\left(\sum_i p_i\rho_i\right)\right] \le \sum_i p_i \mathcal{N}[P_{\Omega,w}(\rho_i)].$$

Theorem 5 suggests that, given a filter that satisfies properties (a)–(e), the filtered negativity $\mathcal{N}_{\Omega,w}(\rho)$ is an approximate linear optical monotone when the negativity of $\mathcal{F}\Omega_w$ is small. Ideally, we would like the Fourier transform of the filter to be pointwise positive and still satisfy properties (a)–(e), which would imply that the filtered negativity is an exact linear optical monotone that can be computed for every $w>0$. It remains unclear whether this is possible, but we demonstrate that the negativity of the filter can at least be made arbitrarily small such that the filtered negativity is essentially a linear optical monotone to any required level of precision.

Proposition 1: Define $\Omega_{w,\epsilon}(\beta) := \exp(-|\beta/w|^{2+\epsilon})$, where $w>0$ is the width parameter, and $\epsilon>0$ is the error parameter.

Then, $\Omega_{w,\epsilon}$ is a filter that satisfies properties (a)–(e). Furthermore, $\mathcal{N}(\mathcal{F}\Omega_{w=1,\epsilon})\to 0$ as $\epsilon\to 0$.

Examples.—Here, we provide some numerical examples that illustrate our results for the negativity of the P function $\mathcal{N}$, the s-parametrized negativity $\mathcal{N}_s$ and the filtered negativity $\mathcal{N}_{\Omega,w} := \mathcal{N}(P_{\Omega,w})$ using several prominent nonclassical states. We will use the filter $\Omega_{w,\epsilon}$ from Proposition 1. The error parameter ϵ is chosen to be $\epsilon=0.21$ such that $2\delta = 2\mathcal{N}(\Omega_{w=1,\epsilon})\approx 0.05$. From Theorem 5, this means that the resulting filtered negativity $\mathcal{N}_{\Omega,w}$ is a linear optical monotone up to approximately a 5% error. Note that this choice is arbitrary because δ can be made as small as desired by decreasing ϵ.

For highly nonclassical states such as Fock and squeezed-vacuum states, $\mathcal{N}$ is infinitely large, which can be verified numerically via Definition 1. One example of a nonclassical state with finite $\mathcal{N}$ is the single-photon-added thermal (SPAT) state, which is defined by $\rho_{\mathrm{SPAT}} = a^\dagger e^{-\beta\hbar\omega a^\dagger a}a/\mathrm{Tr}(e^{-\beta\hbar\omega a^\dagger a}aa^\dagger)$. Its characteristic function is $\chi_{\mathrm{SPAT}}(\beta) = [1-\pi^2(1+\bar{n})|\beta|^2]e^{-\pi^2|\beta|^2/\bar{n}}$, and the corresponding P function is [42]

$$P_{\mathrm{SPAT}}(\alpha) = \frac{1+\bar{n}}{\pi\bar{n}^3}\left(|\alpha|^2-\frac{\bar{n}}{1+\bar{n}}\right)e^{-|\alpha|^2/\bar{n}}.$$

Figure 1(a), illustrates how the filtered negativity $\mathcal{N}_{\Omega,w}(\rho_{\mathrm{SPAT}})$ approaches $\mathcal{N}(\rho_{\mathrm{SPAT}})$ as $w\to\infty$, which comes directly from Definition 1. From Theorem 2, we know that the negativity $\mathcal{N}(\rho_{\mathrm{SPAT}})$ cannot be increased via linear optical processes.

From Theorem 3, we know that the s-parametrized negativity $\mathcal{N}_s$ is a monotonically decreasing function of s. We illustrate this using Fock states $|n\rangle$. Its s-parametrized characteristic function as given by $|n\rangle$ is $\chi_s(\beta) = e^{(s-1)\pi^2|\beta|/2}L_n(\pi^2|\beta|^2)$, with the corresponding s-parametrized quasiprobabilities given by [43]

$$P_s(\alpha) = \frac{2}{\pi(1+s)}\left(-\frac{1-s}{1+s}\right)^n \exp\left(-\frac{2|\alpha|^2}{1+s}\right)L_n\left(\frac{4|\alpha|^2}{1-s^2}\right).$$

Plotting $\mathcal{N}_s$, Fig. 1(b) illustrates its monotonic dependence on s for $n=1$, 2, and 3. Also note how, for every s, $\mathcal{N}_s(|n\rangle)$ increases with n. Theorem 2 says that $\mathcal{N}_s(|n\rangle)$ for $s<1$ are also valid, albeit weaker, nonclassicality measures according to the resource theory of Refs. [12,20].

The s-parametrized negativities can be infinite in general. One example is the squeezed vacuum state $|r\rangle = e^{r(a^{\dagger 2}-a^2)/2}|0\rangle$. Its characteristic function is

$$\chi_s(\beta = x+iy) = \exp\left\{\frac{\pi^2}{2}[(s-e^{2r})x^2+(s-e^{-2r})y^2]\right\}$$

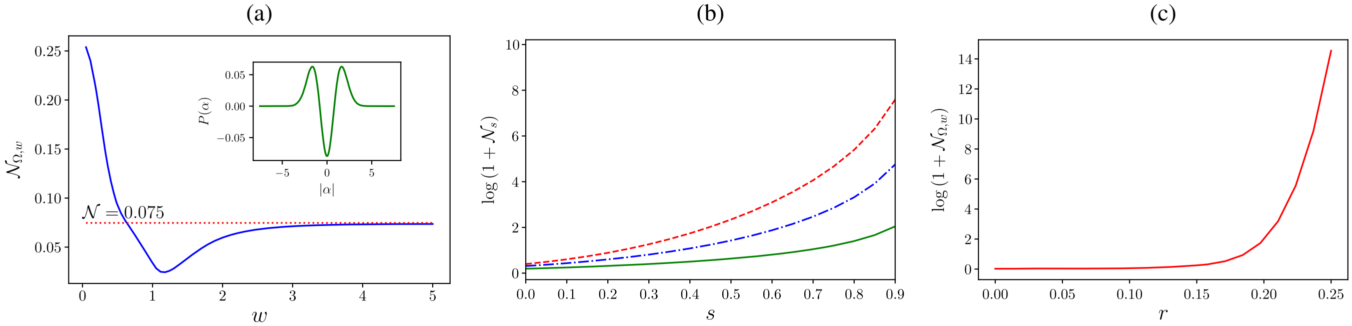

FIG. 1. (a) Convergence of filtered negativity (solid line) $\mathcal{N}_{\Omega,w}$ to negativity(dotted line) $\mathcal{N}$ for single photon added thermal state ρ_{SPAT} with $\bar{n} = 2$. (b) The (logged) s-parametrized negativity $\log(1+\mathcal{N}_s)$ of Fock state $|n\rangle$ for $n = 1$ (solid line), 2 (dotted-dashed line), and 3 (dashed line). (c) The (logged) filtered negativity $\log(1+\mathcal{N}_{\Omega,w})$ for squeezed vacuum $|r\rangle$.

for $r > 0$. If $s \leq e^{-2r}$, then the s-parametrized quasiprobability of $|r\rangle$ is Gaussian, and so it does not show any negative value. However, if $s > e^{-2r}$, then its quasiprobability distribution shows extremely singular behavior; and one can numerically verify that $\mathcal{N}_s$ is infinite. In such cases, $\mathcal{N}_s$ is useful to identify the nonclassicality of the state, but it is unable to capture the increase in nonclassicality that one gets from additional squeezing. This can be circumvented by considering the filtered negativity $\mathcal{N}_{\Omega,w}$.

Figure 1(c) illustrates the filtered negativities $\mathcal{N}_{\Omega,w}$ of the squeezed vacuum states $|r\rangle$ with squeezing parameter r. We see that the filtered negativity captures the increase in nonclassicality due to the increase in squeezing r. As the filter $\Omega_{w,\epsilon}$ has nonzero negativity, $\mathcal{N}_{\Omega,w}$ is only an approximate monotone (see Theorem 5), but this error can be made arbitrarily small by decreasing the parameter ϵ. This may, however, require increased numerical precision, and hence incur additional computational costs.

Conclusion.—We introduced a method to define the negativity of the s-parametrized quasiprobabilities. Our method is based on a modified version of the filtered P function in Ref. [35]. Based on this definition, it is possible to show that the negativity of the set of s-parametrized quasiprobabilities are all linear optical monotones and form a continuous hierarchy of increasingly weaker nonclassicality measures that all belong to the operational resource theory of nonclassicality considered in Refs. [12,20].

In general, the s-parametrized negativities may have infinite values. In order to circumvent this, we introduce an approximate linear optical monotone that is finite computable and able to identify nearly every nonclassical state. A key advantage of this approach is that the set of unidentifiable nonclassical states can be made to converge to nil by increasing the parameter w. The error can also be controlled via a single parameter ϵ.

We also demonstrate in Theorem 4 that the negativity of the P function has a direct operational interpretation as the robustness. Because $\mathcal{N}(\rho)$ is not always finite, this means that there are some states for which the nonclassicality cannot be erased by simple statistical mixing with classical noise. This is a characteristic it shares with quantum coherence, where simple mixing with an incoherent state cannot make the state classical in general [41].

Finally, we comment that our proposed measures are practical under realistic settings. In order to compute the proposed measures, one only requires the characteristic function of the quantum state, with no limitations on whether the state is mixed or pure. The characteristic function may be sampled directly in the laboratory using only homodyne measurements [44]. More generally, the reconstruction of any of the s-parametrized quasiprobabilities [45] allows you to infer the characteristic function, and hence compute our proposed measures.

We hope our work will spur continued interest in the study of nonclassicality in light fields.

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