Quantum Measurements of Time

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We propose a time-of-arrival operator in quantum mechanics by conditioning on a quantum clock. This allows us to bypass some of the problems of previous proposals, and to obtain a Hermitian time of arrival operator whose probability distribution arises from the Born rule and which has a clear physical interpretation. The same procedure can be employed to measure the "time at which some event happens" for arbitrary events (and not just specifically for the arrival time of a particle).

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Textbook quantum mechanics cannot describe measurements of time, since time is a parameter and not a quantum observable [1]. This is a clear shortcoming of the theory, since time measurements are routinely carried out in laboratories using quantum systems that act as clocks. Clever and creative tricks were devised to overcome this shortcoming, e.g., see reviews in Refs. [2–5]. However, many of these proposals give conflicting predictions and none of them provides a prescription that applies to generic time measurements: they all focus on specific measurements, e.g., the time of arrival, at a given position, of a particle subject to a specific potential, e.g., Refs. [6–10]. In this Letter we provide a general prescription for quantum measurements of the time at which an arbitrary event happens (the time of arrival being a specific instance). It entails quantizing the temporal reference frame, namely, employing a quantum system as clock [11–15]. Then, textbook quantum mechanics can be applied to describe time measurements through joint quantum observables of the system under analysis and the quantum clock. A simple Bayes conditioning of the Born rule probability of the joint state allows one to recover the full distribution of the time measurement.

It is not always recognized that, in the usual formulation of quantum mechanics, time is a conditioned quantity. The state $|\psi(t)\rangle$ is the state of the system *conditioned* on the time being t in the Schrödinger picture. (Analogously, in the Heisenberg picture the conditioning is on the observables.) This implies that the Born rule refers to conditional probabilities: the probability that the property $O=\sum_i o_i|o_i\rangle\langle o_i|$ has value o_i is $p(o_i|t)=|\langle\psi(t)|o_i\rangle|^2$, where o_i and $|o_i\rangle$ are eigenvalues and eigenvectors of O. It is a conditional probability, *conditioned* on the time being t. Because of this, time appears as a parameter and not as an observable in the usual formulation of quantum mechanics [1], and textbook quantum mechanics does not directly give a quantum description of time measurements, e.g., the arrival time of a particle at some position [2–10,16–23]. All the previous works considered the time of arrival as a property of the particle, and hence its corresponding observable as an operator acting on the particle's Hilbert space either a self-adjoint operator [6–8] or a positive operator valued measure (POVM) [16,17]]. Here, instead, we consider it as a joint property of the particle and of the clock that is used to measure time. It is an elegant way to tackle the conditioning described above. Indeed, it avoids many of the technicalities of previous proposals for time quantum observables (e.g., the distinctions between the interacting and the noninteracting case [9]). Here we will consider the time operator as the one obtained from the quantum clock. Different systems track time in different ways and laboratory-grade time measurements will employ the most accurate clocks which are typically macroscopic (classical) systems. Their energy spectrum well approximates a continuous unbounded spectrum, as is necessary to define a good time operator. Thus the approach used in conventional quantum mechanics of considering time as a conditioned classical parameter is well justified in practice [24]. However, a fully consistent theory must possess a prescription also for time measurements (not just as an approximation in the classical limit), and this is what we propose here.

In this Letter we use a quantum reference frame (a quantum clock) to describe time, the Page and Wootters formalism introduced in Refs. [11–15]. This allows us to obtain a description of the measurement of the time at which an event happens which bypasses most problems of previous proposals. Our proposal does not supersede previous ones, which are well suited when considering time as a property of the system itself, and not as a property of a reference (the clock). However, our proposal is simple, when compared to others: this allows us to extend it to situations beyond time of arrival that other proposals cannot treat and makes it appealing for experimental tests. While we employ a (slight) extension of textbook quantum theory, that theory can be recovered by conditioning the system state on what the clock shows [11–15].

For the sake of definiteness, we first focus on the description of the time of arrival of a one-dimensional particle at a position D (the position of the detector), and then show how the same mechanism can be easily extended to the measurement of the time of occurrence of any event. The particle at time t is described by its state $|\psi(t)\rangle$ and the time reference is a continuous quantum degree of freedom described by a Hilbert space $\mathcal{H}_c$ with delta-normalized basis states $|t\rangle$. We can write the global state of particle plus reference as [11–15]

$$|\Psi\rangle = \frac{1}{\sqrt{T}} \int_T dt |t\rangle |\psi(t)\rangle, \tag{1}$$

which is a state in the Hilbert space $\mathcal{H}_c \otimes \mathcal{H}_s$, with $\mathcal{H}_s$ the system's Hilbert space, and where the integral is performed on the time interval T (from $-(T/2)$ to $(T/2)$), a regularization parameter. One can think of T as a time interval much larger than all the other timescales: namely, the physical description of the system will be accurate for time intervals $\ll T$ and for energy intervals $\gg 2\pi\hbar/T$ (see the Supplemental Material [25] for a review of the Page and Wootters theory and for a justification of T). The $\sqrt{T}$ term is introduced in order to have a normalized $|\Psi\rangle$, given that $\langle\psi(t)|\psi(t)\rangle = 1$ for all t. The entanglement in Eq. (1) is not a result of any clock-system dynamics (they are isolated from each other), but arises solely from the fact that $|\Psi\rangle$ is an eigenstate of a constraint equation [11] (see the Supplemental Material [25]). The conventional formulation of quantum mechanics arises from conditioning the reference to time t [11–15,24]: indeed projecting the reference on the state $|t\rangle$ that indicates that time is t, one obtains the "state of the system given that time is t":

$$\langle t|\Psi\rangle \propto |\psi(t)\rangle, \tag{2}$$

and the corresponding wave function $\psi(x|t) = \langle x|\psi(t)\rangle$ can be obtained by projecting the system on the position eigenbasis $\{|x\rangle\}$. The physical meaning of the conditioning (2) is that, once the clock is read out, the system conditioned on the clock outcome is described by the state $|\psi(t)\rangle$, whereas all clock outcomes are equally likely, as expected from a (uniform) quantum clock that does not favor any particular time. We emphasize the conditioned nature of the wave function by using the Bayes notation for the conditional probability.

Since $|\Psi\rangle$ is normalized, it has a probabilistic interpretation that entails a (slight) extension of the Born rule: it uses the "history" state $|\Psi\rangle$ that contains the state of the system at all times, instead of using the state of the system $|\psi(t)\rangle$ at time t. Indeed one can construct a time of arrival POVM as

$$\forall\, t\colon\ \Pi_t \equiv |t\rangle\langle t| \otimes P_d; \qquad \Pi_{na} = \mathbb{1} - \int dt \Pi_t, \tag{3}$$

where $P_d = \int_D dx|x\rangle\langle x|$ is the projector of the system at the position D of the detector (D being the spatial interval occupied by it). This projective POVM returns the value t of the clock if the particle is in D or the value na (not arrived) if it is not. One can easily introduce an arrival observable from it, as

$$A = \int dt t |t\rangle\langle t| \otimes P_d + \mathbb{1}_c \otimes \lambda \int_{x\notin D} dx |x\rangle\langle x|, \tag{4}$$

where λ is an (arbitrary) eigenvalue that signals that the particle has not arrived (it will be dropped below by considering a vector-valued observable) and $\mathbb{1}_c$ is the identity on the clock Hilbert space. The above POVM (or the observable A) does not return the probability distribution of the time of arrival, because it also considers the case in which the particle has not arrived. Indeed, using it in the Born rule, one obtains the *joint* probability that the particle has arrived *and* that its time of arrival is t (i.e., the particle is at $x \in D$ and the clock shows t):

$$p(t, x \in D) = \mathrm{Tr}[|\Psi\rangle\langle\Psi|\Pi_t] = \frac{1}{T} \int_{x\in D} dx |\psi(x|t)|^2. \tag{5}$$

[The case of a pointlike detector is $p = |\psi(D|t)|^2/T$.] Then, the time of arrival distribution is recovered from the joint probability through the Bayes rule as

$$p(t|x \in D) = \int_{x\in D} dx |\psi(x|t)|^2 \Big/ \int_T dt \int_{x\in D} dx |\psi(x|t)|^2, \tag{6}$$

where the denominator (which is the dwell time [31]), divided by T, is the unconditioned probability that the particle is found in D at any time, and $p(t|x \in D)$ is normalized when integrated on the interval T. The dependence on the regularization parameter T disappears from the probability distribution (6) if the time integral converges for $T \to \infty$, typically if $\psi(x|t) \to 0$ sufficiently fast in this limit. In all other situations, the time distribution $p(t|x \in D)$ cannot be normalized over the whole time axis. While this might appear as a flaw of our proposal (as it does not satisfy Kijowski's normalization axiom [7]), it is actually a feature because it allows our distribution to treat situations where Kijowski's fails, e.g., the case in which the particle never arrives at D or the case in which the particle is stationary at the detector: in this case $p(t|x \in D)$ is a constant and can be normalized only on a finite interval T. One can dismiss this situation as uninteresting in the classical case, but due to the quantum superposition principle, one cannot ignore it in the quantum case, where most reasonable wave packets have a nonzero probability amplitude of being stationary at the detector position: thus, the experimental predictions of our proposal differ from the others in this case [32].

Consider some other special cases: (i) If the particle never reaches D the probability (6) is meaningless, as expected, since the numerator and denominator are null. (ii) If the particle crosses D multiple times, the distribution will have multiple peaks, as expected, corresponding to the "crossing times." (iii) If the particle is stationary at the detector position or if it reaches it after some finite time interval but remains there forever, then $\psi(x|t)$ is nonzero for large t and one cannot extend the time integration to infinity. In this case, $p(t|x \in D)$ explicitly depends on the interval T, since it is nonzero for arbitrarily large t as expected: the particle will be found at the detector at any sufficiently large time. (iv) A particle performing periodic evolution (e.g., a harmonic oscillator) is a combination of the previous two cases: whatever T interval yields a multipeak distribution, but again T cannot be increased to encompass all times t for which $\psi(x|t)$ is substantially nonzero. (v) In the simple case of a free nonrelativistic particle with Gaussian initial wave packet prepared far from the detector and negligible negative momentum components, we obtain [32] the same results of Refs. [6,7], as expected. (vi) If the particle is split into two wave packets approaching the detector from opposite directions, our probability distribution will display interference peaks due to the superposition principle, in contrast to other proposals [6,7] that do not [32]. This difference may be used to experimentally test our proposal.

All the above cases refer to single-shot measurements, where the time of arrival distribution refers to the probability of outcome of a *single* measurement, which is the usual interpretation of the Born-rule probability. Nonetheless, our mechanism can be extended to describe also multiple (successive) measurements on the same system. It takes into account measurement feedback and returns the multi-time correlations, a problem that was apparently not even ever considered in previous literature, see Ref. [15] and the Supplemental Material [25]. Equation (6) is our main result.

Discussion.—In previous literature, time observables are typically defined on the Hilbert space of the system (e.g., Refs. [6,7,10,17,18]). The only way the observables of these previous proposals can give rise to a time-of-arrival distribution through the Born rule $p(o_i|t) = |\langle\psi(t)|o_i\rangle|^2$ is by postulating that the observable is a constant of motion, such as Rovelli's evolving constants of motion [33,34]. This is a consequence of the Born rule's conditioned nature: it contains the state at one time only in the Schrödinger picture (or the observables at one time only in the Heisenberg picture). Namely, a time of arrival operator $\hat{t} = \int d\tau\tau|\tau\rangle\langle\tau|$ with eigenstates $|\tau\rangle$ in the particle's Hilbert space must return the same outcome τ at any time t through the Born rule: $p(\tau|t) = |\langle\psi(t)|\tau\rangle|^2 = |\langle\psi|\tau(t)\rangle|^2$ (in the Schrödinger and Heisenberg picture, respectively). This requirement leads to awkward statements such as "the time of arrival is τ at time t" and seems physically bizarre, since a constant of motion should give the same outcome *whenever* it is measured [10], but this is not the case for typical time-of-arrival experiments, which give an outcome at a well-defined time: the measurement cannot be performed before or after the particle has arrived. In contrast, our proposal does not suffer from this problem: our observable is not a constant of motion. It is a *joint* observable on the system and on the clock. Its time invariance is enforced not dynamically, but by the fact that the state $|\Psi\rangle$ of system plus clock is an eigenstate of the global Hamiltonian that defines the total energy constraint [11]. Moreover, the particle does not have to "stop" the clock nor interact with it [20,35], as there is no clock-system interaction in the global Hamiltonian that defines the constraint. The entanglement in Eq. (1) does not arise from a clock-system dynamics, but it is intrinsic in the zero-energy eigenstate of the global Hamiltonian (see Supplemental Material [25] and Refs. [11–15]). Indeed, a good clock should be *independent* from the system it is timing, and it is the experimentalist that notes the correlations between particle and clock. In other words, the conventional approach of considering the time as a property of the system, described by an operator acting on the system Hilbert space, is more appropriate if the system itself is used to measure time as in many traditional approaches (e.g., Refs. [4,10]): considering the system energy as the generator of time translations implies that *that* time operator refers to the system's evolution, but this is not what happens typically in a lab, where experiments are timed through an external clock. Indeed, the time operator $\int dtt|t\rangle\langle t|$ in Eq. (4) (where $|t\rangle$ is a state in the clock Hilbert space) is conjugated to the energy of the clock and not of the system [10,36]). This is the main advantage of our proposal, which also satisfies the desiderata for a time of arrival operator [6]: it obeys the superposition principle (trivially from linearity) and it originates from the Born rule. Finally, a definition of time through a quantum clock can describe real-life situations and experiments, once decoherence is accounted for [15], as discussed below. Elsewhere [15,36], we have shown how the usual objections against time quantization are overcome (see also the Supplemental Material [25]).

Environment and multiple clocks.—Our (idealized) description of Eq. (1) requires the system to be correlated with a single clock and the joint system-clock state to be in a pure state. While this is sufficient to give a fundamental prescription for time measurements in quantum mechanics, we need to show that this is compatible also with real world scenarios where multiple clocks are present and where there is an environment that may interact with the system and clocks. A less idealized description replaces Eq. (1) with

$$|\Phi\rangle = \frac{1}{\sqrt{T}}\int_T dt|t\rangle_{c_1}|t\rangle_{c_2}\sum_k \mu_k|\phi_k(t)\rangle_s|e_k(t)\rangle_e, \quad (7)$$

where c_1, c_2 indicate two different clock systems that are synchronized (they track each other because the joint measurement of both returns the same outcome t), $|\phi_k(t)\rangle_s$ indicates the system state at time t, which may be entangled with the orthonormal states $|e_k(t)\rangle_e$ of the environment with amplitude μ_k. The time entanglement of Eq. (7) is of the GHZ type, which is the one present in the branching states typically used in decoherence models [37]. Equation (7) is still idealized: good clocks should be sufficiently isolated from the environment (at least for the time interval in which it is considered a good clock) so that their evolution is unperturbed by it. This state contains correlations in time even when one considers only the reduced state $\rho_{c_1 s}$ of the system and of one of the clocks, say c_1. In this case, the reduced state is

$$\rho_{c_1 s} = \mathrm{Tr}_{c_2 e}[|\Phi\rangle\langle\Phi|] = \frac{1}{T}\int_T dt |t\rangle_{c_1}\langle t| \otimes \rho_s(t), \quad (8)$$

with $\rho_s(t)$ the reduced system state. One can obtain all previous results using the Born rule for $\rho_{c_1 s}$. Interestingly, even though this state has lost quantum coherence in the time correlations, retaining only classical correlations, the time of arrival distribution can still display intereference effects [32]. This loss of coherence translates into an effective superselection rule that prevents the creation and detection of superpositions of states of different times, as expected [38].

At first sight, the decohered state (8) seems inadequate to describe our perception of time: it describes a random time t, uniformly distributed in $[-(T/2),(T/2)]$ correlated to a state $|\phi(t)\rangle_s$ [39]. However, consider carefully our perception of time. We perceive a single instant (the present) and the past is contained into memory degrees of freedom, internal to some state $|\phi(t)\rangle_s$: St. Augustine's "the past is present memory." Would we be able to discriminate whether the "present" we perceive is a continuous succession of instants of time (as our naive intuition suggests) or as a random sampling of instants as described by Eq. (8)? No. There is no experimentally testable way to do that. Then such question is unscientific, and we can conclude that Eq. (8) is a good description of our perception of time even if it contains a random time. A more detailed discussion, with a review of previous literature, is in the Supplemental Material [25].

Time of arbitrary events.—Up to now we have considered the time of arrival, which is connected to the event "the particle is at position D." But the whole discussion can be easily extended to the "time at which any event happens." For example, if one considers a spin instead of a particle on a line, one can define the "time at which the spin is up," by substituting P_d with the spin-up projector $|\uparrow\rangle\langle\uparrow|$ in Eq. (3) and by replacing Eq. (6) with the conditional probability that time is t given that the spin was up: $p(t|\uparrow) = |\psi(\uparrow|t)|^2/\int_T dt|\psi(\uparrow|t)|^2$ with $\psi(\uparrow|t) \equiv \langle\uparrow|\psi(t)\rangle$ the probability amplitude of having spin up at time t. The general case of arbitrary events follows straightforwardly, by using a projector P that represents whatever value of a system property one wants to consider at time t, namely, P projects onto the eigenspace relative to some eigenvalue of a system observable. In this case the time distribution is

$$p(t|P) = \mathrm{Tr}[|\psi(t)\rangle\langle\psi(t)|P]/\int_T dt \mathrm{Tr}[|\psi(t)\rangle\langle\psi(t)|P], \quad (9)$$

where again the dependence on the regularization parameter T disappears if the integral can be taken on an interval containing all times when the integrand is substantially different from zero and *must* be retained otherwise. This captures the notion of "event" in quantum mechanics defined as "something that happens to a quantum system," where "something" means "a system observable property taking some value P."

Expectation values and uncertainty.—The presence of λ in the definition (4) of A implies that its expectation value $\langle A\rangle = \langle\Psi|A|\Psi\rangle$ is not the average time of arrival: the observable must account also for the case in which the particle does not arrive (or the event P does not happen). We can partially amend by considering a 2D vector-valued observable, where the first component of the vector contains the event time occurrence and the second component takes care of the cases in which the event does not occur:

$$\hat{T} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \int dt t |t\rangle\langle t| \otimes P + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbb{1}_c \otimes (\mathbb{1}_s - P). \quad (10)$$

The expectation value of the first component, which we denote by $\hat{T}_1$, is then proportional to the average event occurrence time

$$t_{\mathrm{ev}} \equiv \alpha\langle\hat{T}_1\rangle = \alpha\langle\Psi| \int dt t |t\rangle\langle t| \otimes P|\Psi\rangle. \quad (11)$$

The proportionality constant α, arising from the Bayes conditioning (9), is $\alpha = 1/\int_T dt\, \mathrm{Tr}[P|\Psi\rangle\langle\Psi|]$. Indeed, with this choice, we find the correct $t_{\mathrm{ev}} = \int dt t p(t|P)$. It is then clear that a null $\langle T_1\rangle$ may not lead to a null t_{ev} if the event never happens, as in this case $\alpha = \infty$. (We can be sure that there is a nonzero chance that the event happens if the second component's expectation value is $\langle\hat{T}_2\rangle \neq 1$.)

The presence of the constant α precludes the use of the Robertson [40] prescription to obtain a time-energy uncertainty relation for the event occurrence time t_{ev}, since its variance is $\Delta t_{\mathrm{ev}}^2 = \alpha\langle T_1^2\rangle - \alpha^2\langle T_1\rangle^2$.

Conclusions.—In conclusion, we have introduced a prescription for the time measurement of when arbitrary events happen, such as the time of arrival, by considering an observable acting on the extended Hilbert space of system plus time reference. It satisfies the desired properties for a time operator: it is a Hermitian operator (on the extended space), its probability distribution arises from the Born rule, it satisfies the superposition principle and it has the correct physical interpretation arising from a mathematical description of what happens in an actual experiment.

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