Multipartite Generalization of Quantum Discord

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A generalization of quantum discord to multipartite systems is proposed. A key feature of our formulation is its consistency with the conventional definition of discord in bipartite systems. It is by construction zero only for systems with classically correlated subsystems and is a non-negative quantity, giving a measure of the total nonclassical correlations in the multipartite system with respect to a fixed measurement ordering. For the tripartite case, we show that the discord can be decomposed into contributions resulting from changes induced by nonclassical correlation breaking measurements in the conditional mutual information and tripartite mutual information. The former gives a measure of the bipartite nonclassical correlations and is a non-negative quantity, while the latter is related to the monogamy of the nonclassical correlations.

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Introduction.—One of the foremost aims of quantum information theory is to understand and quantify the various forms of quantum correlations. Quantum correlations are ubiquitous in many areas of modern physics, ranging from condensed matter physics, quantum optics, high-energy physics, to quantum chemistry. They can be regarded as the most fundamental type of nonclassical correlation which includes entanglement, EPR (Einstein-Podolsky-Rosen) steerable states, and nonlocal correlations [1,2]. Much work has been done towards constructing resource theories [3–8] as well as understanding the operational relevance of information theoretic quantities [9–13].

For bipartite systems, the best-known measure of the nonclassical correlations is quantum discord (or discord for short) [14,15]. This is defined as the minimized difference between the quantum mutual information with and without a von Neumann projective measurement applied on one of the subsystems. The role of the projective measurement is to break the quantum correlations (for simplicity, we henceforth use this term interchangeably with "nonclassical correlations") between the subsystems, which results in a classically correlated state [16,17]. The intuition is that by comparing the mutual information before and after the breaking of quantum correlations, one can quantify the amount of quantum correlations in the original state with respect to the measured subsystem. Quantum generalization of such entropies has focused on applications in quantum state distribution [18–20], optimal source coding [21], quantum information processing [3,22], and simulation of classical channels with quantum side information [23,24]. Quantum discord has shown to be a powerful characterization tool for complex quantum states, such as in quantum many-body systems [25,26].

For tripartite and larger systems, several generalizations of discord have been proposed. In Ref. [16] a symmetric multipartite discord was defined based on relative entropy and local measurements. Another definition of multipartite discord was provided in Ref. [27], as the sum of bipartite discords after making successive measurements. An approach using relative entropy was defined in Ref. [28] to define genuine quantum and classical correlations in multipartite systems. Reference [29] introduced the notion of quantum dissension defined as the difference between tripartite mutual information after a single measurement. A distance-based approach was formulated in Ref. [30], including a multipartite measure of quantum correlations [31–33].

Surprisingly, a definition of quantum discord to multipartite systems that is consistent with the original bipartite definition of Refs. [14,15] does not seem to exist. A reasonable set of properties [34] possessed by such a measure include (i) zero if and only if the state is a classically correlated state; (ii) a non-negative quantity; and (iii) reduction to the standard definition of discord for bipartitelike correlated subsystems. In the original definition of quantum discord [14,31,32], a classically correlated state is one such that the "classical information is locally accessible, and can be obtained without perturbing the state of the system." This means that given a classically correlated state, there exists a measurement that can be performed such that the classical correlations can be recovered, without altering the density matrix. For example, consider the state $(|00\rangle\langle 00| + |1+\rangle\langle 1+|) \otimes |0\rangle\langle 0|/2$. As a tensor product of a bipartite zero discord state with a single qubit state, one expects such a state to have zero tripartite discord, taking the first qubit to be the measured qubit. Past works based on multipartite mutual information [16,27,28] give nonzero values for this state, either because of the type of measurements performed, or a symmetric definition. Meanwhile, quantum dissension [29] allows negative values which are not present with discord. For distance-based definitions [30], one would not expect to obtain completely equivalent results due to the different measure used. However, we note that Ref. [30] also uses a different notion of a classically correlated state than that of Ref. [14], and takes a nonzero value even for the bipartite component of the above state.

In this Letter, a natural generalization of discord—as originally defined in Refs. [14,15]—is made for multipartite systems. Our definition satisfies all of the postulates of a multipartite discord (i)–(iii), thanks to the concept of conditional measurements which we introduce here. We further examine the entropy change to various mutual information quantities as a result of projective measurements, which leads to a method of decomposing the multipartite discord into various contributions. This leads us to propose two more quantities based on mutual information, which measure the purely bipartite quantum correlations, and satisfy the properties (i)–(iii) as well as the monogamy of the quantum correlations in the tripartite system.

Multipartite measurements.—Let us first start by reviewing the original definition of discord, which is defined as [14,15]

$$D_{A;B}(\rho) = \min_{\Pi^A}[S_{B|\Pi^A}(\rho) - S_{B|A}(\rho)] \tag{1}$$

where the conditional entropy without measurement is defined $S_{B|A}(\rho) = S_{AB}(\rho) - S_A(\rho)$ [14,35,36], where $S_n(\rho) = -\mathrm{Tr}\rho_n \log \rho_n$ is the von Neumann entropy for the (reduced) density matrix on the system labeled by n. The subsystem labels on the discord follow the notation such that a measurement is made on the label preceding the semicolon. The conditional entropy with measurement is defined [14]

$$S_{B|\Pi^A}(\rho) = \sum_j p_j^A S_{AB}(\Pi_j^A \rho \Pi_j^A / p_j^A), \tag{2}$$

where Π_j^A is a one-dimensional von Neumann projection operator on subsystem A and $p_j^A = \mathrm{Tr}(\Pi_j^A \rho \Pi_j^A)$ is its probability. The discord is zero if and only if there is a measurement such that $\rho = \sum_j \Pi_j^A \rho \Pi_j^A$. The fact that one can measure one system and yet leave the state unchanged is a signal that there are no quantum correlations taking A to be the measured subsystem.

In the above formulation, only one of the subsystems is measured. For bipartite systems, this is sufficient since the correlations are only between two subsystems. We first generalize the bipartite discord to the case where both subsystems are measured. Although redundant for the bipartite case, understanding this will prove useful when generalizing discord to multipartite systems. In order to keep a consistent definition of discord, we seek a measurement for zero discord states such that $\rho = \sum_{jk} \Pi_{jk}^{AB} \rho \Pi_{jk}^{AB}$. Such a measurement can always be constructed according to the form [37]

$$\Pi_{jk}^{AB} = \Pi_j^A \otimes \Pi_{k|j}^B \tag{3}$$

where $\Pi_{k|j}^B$ is a projector on subsystem B that is conditional on the measurement outcome of A [14]. The projectors satisfy $\sum_k \Pi_{k|j}^B = \mathbb{1}^B$, $\sum_j \Pi_j^A = \mathbb{1}^A$. As mentioned in the original work of Ref. [14], this would physically correspond to some classical communication from A to B being exchanged to modify the measurement on B.

Using this form of a measurement, we can then write an equivalent expression for the discord [Eq. (1)], where measurements are made on both systems [37]

$$D_{A;B}(\rho) = \min_{\Pi^{AB}}[S_{B|A}(\rho_{\Pi^{AB}}) - S_{B|A}(\rho)], \tag{4}$$

where $\rho_{\Pi^{AB}} = \sum_{jk} \Pi_{jk}^{AB} \rho \Pi_{jk}^{AB}$ is the state after measurement. Here the optimization is performed over projective measurements of the type given in Eq. (3). For example, the zero discord state $(|00\rangle\langle 00| + |1+\rangle\langle 1+|)/2$ has an optimal basis $\Pi^{AB} \in \{|00\rangle\langle 00|, |01\rangle\langle 01|, |1+\rangle\langle 1+|, |1-\rangle\langle 1-|\}$, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Without conditional measurements, it would be impossible to obtain consistent results with the conventional definition of discord because the states $|0\rangle$, $|+\rangle$ are not orthogonal by themselves.

For multipartite systems with N subsystems, in general $N-1$ local measurements will be necessary in order to break all the quantum correlations [16,27]. In an analogous way to Eq. (4) it is possible to equally make N measurements, but this is unnecessary and adds an extra overhead to the optimization, hence we consider $N-1$ measurements henceforth. For multipartite systems, each successive measurement is conditionally related to the previous measurement. The $N-1$-partite measurement is written

$$\Pi_{j_1\ldots j_{N-1}}^{A_1\ldots A_{N-1}} = \Pi_{j_1}^{A_1} \otimes \Pi_{j_2|j_1}^{A_2} \ldots \otimes \Pi_{j_{N-1}|j_1\ldots j_{N-2}}^{A_{N-1}}, \tag{5}$$

where the N subsystems are labeled as A_i. Here the measurements take place in the order $A_1 \to A_2 \to \ldots A_{N-1}$.

Multipartite quantum discord.—We now show that the relevant quantity to be minimized in Eq. (1) can be deduced by a simple procedure, which always ensures that the discord takes a zero value for measured states. Evaluating the entropy of the measured system $S_{AB}(\rho_{\Pi^A})$, we observe that this can always be decomposed as

$$S_{AB}(\rho_{\Pi^A}) - S_A(\rho_{\Pi^A}) = S_{B|\Pi^A}(\rho), \quad (6)$$

where $\rho_{\Pi^A} = \sum_j \Pi_j^A \rho \Pi_j^A$. The left-hand side takes the form of conditional entropy $S_{B|A}(\rho_{\Pi^A})$ and all terms involve the measured subsystem A. The right-hand side takes the form of Eq. (2), and is the average entropy of the unmeasured system B. If ρ_{Π^A} is replaced by a more general state ρ, the equality does not hold. The comparison of the left- and right-hand side for a general state is then related to the degree of quantum correlations for the measurement performed on A.

We can follow the same strategy to obtain a multipartite generalization of discord. Examining tripartite systems first, the total entropy of $S_{ABC}(\rho_{\Pi^{AB}})$ can be decomposed to give [37]

$$S_{ABC}(\rho_{\Pi^{AB}}) - S_A(\rho_{\Pi^{AB}}) - S_{B|\Pi^A}(\rho_{\Pi^{AB}}) = S_{C|\Pi^{AB}}(\rho) \quad (7)$$

where we have defined $S_{C|\Pi^{AB}}(\rho) = \sum_{jk} p_{jk}^{AB} \times S_{ABC}(\Pi_{jk}^{AB}\rho\Pi_{jk}^{AB}/p_{jk}^{AB})$ and $p_{jk}^{AB} = \mathrm{Tr}(\Pi_{jk}^{AB}\rho\Pi_{jk}^{AB})$. Here, the left-hand side contains terms which involve the entropy of the subsystems AB that are measured, and the right-hand side is the average entropy of the unmeasured system C. We thus define

$$D_{A;B;C}(\rho) = \min_{\Pi^{AB}}[-S_{BC|A}(\rho) + S_{B|\Pi^A}(\rho) + S_{C|\Pi^{AB}}(\rho)] \quad (8)$$

as a tripartite generalization of discord, for the measurement ordering $A \to B$. This is a non-negative quantity, and by construction is zero for any postmeasured state; e.g., for the state $(|00\rangle\langle 00| + |1+\rangle\langle 1+| \otimes |0\rangle\langle 0|)/2$ we have $D_{A;B;C}(\rho) = 0$. Importantly, it is also true in the reverse direction, that $D_{A;B;C}(\rho) = 0$ implies that the state is of the form $\rho_{\Pi^{AB}}$ [37]. The tripartite discord has the attractive property that it reduces to the standard bipartite discord when only bipartite quantum correlations are present: $D_{A;B;C}(\rho^{AB} \otimes \rho^C) = D_{A;B}(\rho^{AB})$, $D_{A;B;C}(\rho^{BC} \otimes \rho^A) = D_{B;C}(\rho^{BC})$, $D_{A;B;C}(\rho^{AC} \otimes \rho^B) = D_{A;C}(\rho^{AC})$ [37].

In Fig. 1 we show several examples of the tripartite discord for various states. For the Werner states, we see that the tripartite discord generally follows a similar relation to bipartite discord, only diminishing to zero when $\mu = 0$, showing a similar behavior for the Greenberger-Horne-Zeilinger (GHZ) state and W states. For a GHZ state, it is known that entanglement is present only for $\mu > 1/5$ [41–43], showing quantum correlations can be present even when entanglement is zero. The optimal measurement [Eq. (3)] on the A subsystem is found to not necessarily coincide with the optimization for the bipartite discord

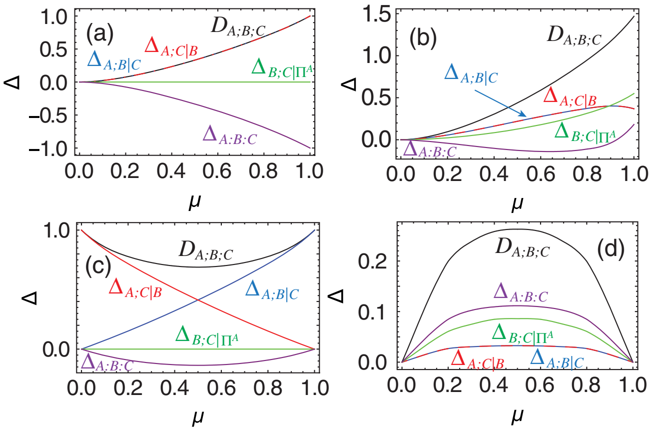

FIG. 1. The tripartite quantum discord and its decompositions. Definitions of quantities are given in Eqs. (8), (12), (13), and (15). The states are (a) Werner-GHZ states $\rho_W = \mu|\psi\rangle\langle\psi| + (1-\mu)\mathbb{1}/8$, where $|\psi\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$; (b) Werner-W states ρ_W defined the same as (a), but with $|\psi\rangle = (|001\rangle + |010\rangle + |001\rangle)/\sqrt{3}$; (c) mixed Bell states $\rho = \mu|\Phi_{AB}^+\rangle\langle\Phi_{AB}^+| + (1-\mu)|\Phi_{AC}^+\rangle\langle\Phi_{AC}^+|$, where $|\Phi_{AB}^+\rangle = (|000\rangle + |110\rangle)/\sqrt{2}$, $|\Phi_{AC}^+\rangle = (|000\rangle + |101\rangle)/\sqrt{2}$; (d) tripartite quantum correlated states $\rho = \mu|000\rangle\langle 000| + (1-\mu)|{+}{+}{+}\rangle\langle{+}{+}{+}|$. The optimization is performed by minimizing the expression Eq. (8) over all projection measurements $\Pi_j \in \{\cos\theta|0\rangle + e^{i\phi}\sin\theta|1\rangle, \sin\theta|0\rangle - e^{i\phi}\cos\theta|1\rangle\}$ for the form Eq. (3), giving six parameters to optimize.

between the A and BC subsystems. This is because the expression Eq. (8) contains contributions from other subdivisions. For Bell states the tripartite discord reduces to the bipartite values [Fig. 1(c)]. The tripartite separable state shows quantum correlations as expected for any state that is not a product state [Fig. 1(d)]. This shows the nonconvexity of the tripartite discord—a property also shared by bipartite discord—where a mixture of zero discord states can give a nonzero discord.

The multipartite generalization can be performed by following the same logic. Evaluating the entropy of an N-partite system measured using the conditional measurements [Eq. (5)] we have the N-partite discord

$$D_{A_1;A_2;\ldots;A_N}(\rho) = \min_{\Pi^{A_1\ldots A_{N-1}}}[-S_{A_2\ldots A_N|A_1}(\rho) + S_{A_2|\Pi^{A_1}}(\rho)\ldots + S_{A_N|\Pi^{A_1\ldots A_{N-1}}}(\rho)] \quad (9)$$

for the measurement ordering $A_1 \to A_2 \to \ldots A_{N-1}$. Here we have defined $S_{A_k|\Pi^{A_1\ldots A_{k-1}}}(\rho) = \sum_{j_1\ldots j_{k-1}} p_j^{(k-1)} \times S_{A_1\ldots A_k}(\Pi_j^{(k-1)}\rho\Pi_j^{(k-1)}/p_j^{(k-1)})$ with $\Pi_j^{(k)} \equiv \Pi_{j_1\ldots j_k}^{A_1\ldots A_k}$, $p_j^{(k)} = \mathrm{Tr}(\Pi_j^{(k)}\rho\Pi_j^{(k)})$. This is again a non-negative quantity, and reduces to lower order discords for states that have classically correlated subdivisions [44]. For a single m-dimensional system, the number of parameters to specify a projector is $m(m-1)$ [45]. For N qubits, there are a total of

$\sum_{n=1}^{N-1} m^{n-1}$ local projectors in Eq. (5), giving a total of $m^N - m$ parameters to optimize in the discord, Eq. (9).

Quantum discord as an entropy flux.—The multipartite generalization of discord gives a quantification of the total quantum correlations in the system with respect to a particular measurement ordering. In a multipartite system, it is desirable to identify exactly where the quantum correlations exist in the system, to see the contributions between subsystems. Before examining the multipartite case, it is interesting to revisit the bipartite case first. The quantum correlation breaking measurement causes a pattern of entropy flux through the system. The entropy contributions before and after the measurement can be written as given in Figs. 2(a) and 2(b), where the same definitions of the entropies are used throughout except the state changes from ρ to ρ_{Π^A} [37]. The entropy change for the three contributions are shown in Fig. 2(c). We see that the measurement causes the mutual information to decrease by an amount equal to the discord, and the conditional entropies increase by the same amount. The conditional entropy for the measured system A also increases by a local contribution $\delta S_{\Pi^A}(\rho)$, since a measurement is applied on this subsystem. This has the interpretation that the entropy corresponding to the quantum correlations are redistributed into subsystems A and B separately, since the measurement destroys this for the mutual information.

For tripartite systems, a similar redistribution of entropies occur. The measurement [Eq. (3)] can be performed in two steps, first performing a measurement on A, then conditionally performing another measurement on B. The initial distribution is shown in Fig. 2(d), which changes to Fig. 2(e) after the first measurement. We define the measured version of the conditional mutual information $I_{A:B|C}$ and tripartite mutual information $I_{A:B:C}$ according to [16,27,29]

$$J_{A:B|C}(\rho) = I_{A:B|C}(\rho_{\Pi^A}) \tag{10}$$

$$K_{A:B|C}(\rho) = I_{A:B|C}(\rho_{\Pi^{AB}}) \tag{11}$$

and similarly for the remaining quantities (mutual information is denoted with a colon). For a classically correlated state the above definition ensures $I_{A:B|C} = J_{A:B|C} = K_{A:B|C}$, but more generally these quantities are not equal. This naturally leads us to define various contributions to the entropy change as a result of the measurement. After one measurement, the conditional mutual information changes by an amount

$$\begin{aligned} \Delta_{A;B|C}(\rho) &\equiv I_{A:B|C}(\rho) - J_{A:B|C}(\rho) \\ &= d_{A;BC}(\rho) - d_{A;C}(\rho), \end{aligned} \tag{12}$$

which we call the *conditional tripartite discord*, and can be interpreted as the bipartite like quantum correlations in the system. We may similarly define $\Delta_{A;C|B} \equiv I_{A:C|B}(\rho) - J_{A:C|B}(\rho)$, where $d_{A;C}(\rho) = S_{C|\Pi^A}(\rho) - S_{C|A}(\rho)$ is the argument to be minimized for the bipartite discord. This is a non-negative quantity $\Delta_{A;B|C}(\rho), \Delta_{A;C|B}(\rho) \geq 0$, and reduces to the bipartite discord without the minimization: $\Delta_{A;B|C}(\rho^{AB} \otimes \rho^C) = d_{A;B}(\rho^{AB})$ [37].

Similarly for the tripartite mutual information we define

$$\begin{aligned} \Delta_{A:B:C}(\rho) &\equiv I_{A:B:C}(\rho) - J_{A:B:C}(\rho) \\ &= d_{A;B}(\rho) + d_{A;C}(\rho) - d_{A;BC}(\rho). \end{aligned} \tag{13}$$

This can take positive or negative values [29]. The fact that this can be negative is not entirely surprising from the point of view that even classically, the tripartite mutual information can be negative. From the decomposition into discords, it is evident that this is a monogamy quantity, giving a negative value for monogamous and positive value for polygamous quantum correlations [46].

Figure 2(e) shows the changes in the entropy after a measurement on A [37]. We see that the entropy changes follow an analogous structure to the bipartite case [Fig. 2(c)]. The three contributions to the entropy $\Delta_{A;B|C}, \Delta_{A;C|B}, \Delta_{A:B:C}$ are "extruded" to the unmeasured parts of the system. The total of the three parts is equal to the conventional bipartite discord

$$d_{A;BC}(\rho) = \Delta_{A;B|C}(\rho) + \Delta_{A;C|B}(\rho) + \Delta_{A:B:C}(\rho) \tag{14}$$

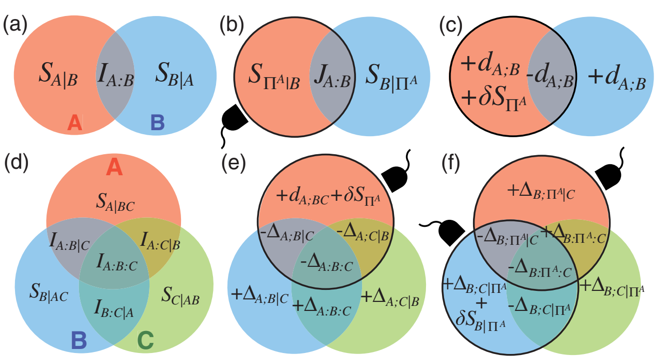

FIG. 2. Distribution of various entropies during a measurement in bipartite and tripartite systems. (a),(d) The initial state before the measurement. (b) The final state after a measurement on subsystem A. The thick outline indicates the measured system. (c),(e) The change in entropy after a measurement on A. (f) The change in entropy after measuring both A and B. Bipartite entropy contributions are defined according to $I_{A:B}(\rho) = S_A(\rho) + S_B(\rho) - S_{AB}(\rho)$, $J_{A:B}(\rho) = S_B(\rho) - S_{B|\Pi^A}(\rho) = I_{A:B}(\rho_{\Pi^A})$, $S_{\Pi^A|B}(\rho) \equiv S_{AB}(\rho_{\Pi^A}) - S_B(\rho_{\Pi^A})$, $\delta S_{\Pi^A}(\rho) = S_A(\rho_{\Pi^A}) - S_A(\rho)$, $d_{A;B}(\rho) = S_{B|\Pi^A}(\rho) - S_{B|A}(\rho)$. Tripartite entropy contributions are defined by $I_{A:B|C}(\rho) = S_{A|C}(\rho) - S_{A|BC}(\rho)$, $I_{A:B:C}(\rho) = I_{A:C}(\rho) - I_{A:C|B}(\rho)$, $\delta S_{B|\Pi^A}(\rho) = S_{B|A}(\rho_{\Pi^{AB}}) - S_{B|A}(\rho_{\Pi^A})$, $\Delta_{B;\Pi^A|C}(\rho) = J_{A:B|C}(\rho) - K_{A:B|C}(\rho)$, $\Delta_{B;\Pi^A;C}(\rho) = J_{A:B:C}(\rho) - K_{A:B:C}(\rho)$.

which, combined with a local entropy increase δS_{Π^A}, is also the increase in the conditional entropy of A.

After an additional measurement on B, a similar pattern emerges, except that the entropy shifts are in the direction of CA and CB instead of AB and AC as before. Changes in the conditional mutual and tripartite mutual information are defined similarly to Eqs. (12) and (13). The most interesting of these terms is

$$\Delta_{B;C|\Pi^A}(\rho) \equiv J_{B:C|A}(\rho) - K_{B:C|A}(\rho) = d_{B;\Pi^A C}(\rho) - d_{B;\Pi^A}(\rho), \quad (15)$$

which is conditional discord after the measurement of A, and is also non-negative: $\Delta_{B;C|\Pi^A}(\rho) \geq 0$ [37]. In addition to the similar pattern of entropy changes, there are again local entropy contributions on subsystem B.

The above definitions allow us to write the generalized discord [Eq. (8)] in an equivalent form:

$$D_{A;B;C}(\rho) = \min_{\Pi^{AB}}[\Delta_{A;B|C}(\rho) + \Delta_{A;C|B}(\rho) + \Delta_{B;C|\Pi^A}(\rho) + \Delta_{A:B:C}(\rho)]. \quad (16)$$

The tripartite discord can thus be equivalently viewed as the sum of all conditional discords and the change in the tripartite mutual information.

This decomposition allows us to attribute various contributions of the total discord to various parts of the system. Figure 1 shows the decompositions of the multipartite discord into various components. For the Werner-GHZ state we see that the conditional discords between AB and AC take the values $\Delta_{A;B|C} = \Delta_{A;C|B} = D_{A;B;C}$, showing that bipartite quantum correlations exist within the GHZ state, prior to a measurement on A. Meanwhile, the remaining conditional discord is $\Delta_{B;C|\Pi^A} = 0$ due to all quantum correlations (and hence entanglement) collapsing to zero after the measurement on A is made. The monogamous nature of the GHZ state is verified with the change in the triparite mutual information, giving a negative value $\Delta_{A:B:C} = -D_{A;B;C}$. For the Werner-$W$ states, the conditional discord for all three pairings take nonzero values, since the measurement on A does not completely break the quantum correlations between BC. It is well known that the tripartite quantum systems can be divided into these two classes, which are not related to each other via local operations and classical communication [47]. Interestingly, the monogamy swaps sign from polygamous to monogamous behavior at lower purities. A similar effect was also found using a different measure in Ref. [46]. For the bipartite states in Fig. 1(c), the conditional discords reduce to the bipartite discords at $\mu = 0$, 1. Finally, for the tripartite correlated state all quantities are positive [Fig. 1(d)].

Conclusions.—We have introduced a generalization of discord for tripartite [Eq. (8)] and multipartite [Eq. (9)] states. One of the main features of our approach is the use of conditional measurements. The conditioning is essential to take into account all of the classical correlations that may exist between subsystems. Viewing the measurements as an operation to break the quantum correlations, optimizing over all such measurements allows one to recover the purely quantum contribution. We note that there is an obvious asymmetry due the fixed ordering of the measurements, which is also present in the original definition of the bipartite discord. While symmetric definitions of discord exist such that there is no dependence upon the choice (and order) of measured subsystems [16,48], here we take the point of view that we wish to have a definition consistent with the most commonly used definition of bipartite discord as defined in Refs. [14,15]. This asymmetry has similarities with quantum steering which also considers a measurement on part of a system [49]. The aims are somewhat different in that for discord, it is to minimize the disturbance due to measurement rather than compare to a local hidden state theory [50]. Some applications, such as one-way quantum computing [51], have a definite ordering of measurements which makes our multipartite discord naturally compatible. By identifying the various contributions to the terms which make up our definition of tripartite discord in terms of conditional entropies, we provide an exact decomposition [Eq. (16)]. The contributions give a definition of a conditional discord which characterizes the bipartite correlations $\Delta_{A;B|C}, \Delta_{A;C|B}, \Delta_{B;C|\Pi^A}$ in a tripartite system, as well as a quantity related to the monogamy of quantum correlations $\Delta_{A:B:C}$. Similar decompositions can be made for the multipartite system, which we leave as future work.

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