# Morphological Superfluid in a Nonmagnetic Spin-2 Bose-Einstein Condensate

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The two known mechanisms for superflow are the gradient of the U(1) phase and the spin-orbit-gauge symmetry. We find the third mechanism, namely a spatial variation of the order-parameter morphology protected by a hidden su(2) symmetry in a nonmagnetic spin-2 Bose-Einstein condensate. Possible experimental situations are also discussed.

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Superflow is usually generated by the gradient of the U(1) phase. In spinor Bose-Einstein condensates (BECs), the spin-gauge symmetry provides the second mechanism of superfluidity. For instance, in a ferromagnetic spin-1 BEC, superflow can be induced by spin textures via the spin-gauge symmetry [1,2], whereas a polar superfluid can only be carried by the gradient of the U(1) phase [2]. Similarly, in the superfluid $^3$He-A phase, superflow can be induced by textures of the $\boldsymbol{l}$ vector via the orbital-gauge symmetry [3]. Here we report our finding that for the case of a spin-2 BEC, spatial variation of the order-parameter shape can generate a supercurrent even in the nonmagnetic nematic and cyclic phases, offering the hitherto unexplored third mechanism of superfluidity. A full investigation of this possibility is the main theme of this Letter.

A spin-$F$ BEC can be described in the mean-field approximation by a $(2F+1)$-component order parameter $\boldsymbol{\psi} \equiv (\psi_{-F}, \psi_{-F+1}, \ldots, \psi_m, \ldots, \psi_F)^T$ [2,4], where $T$ denotes the transpose and $m = F, \ldots, -F$ is the magnetic sublevel. The superfluid velocity is defined in terms of the order parameter, the atomic mass $M$, and the local density $\rho = \sum_{m=-F}^{F} |\psi_m|^2$ as $\boldsymbol{v} \equiv (\hbar/2Mi\rho)[\psi_m^*(\nabla\psi_m) - (\nabla\psi_m^*)\psi_m]$. For a general spin-1 BEC, the order parameter can be expressed in the irreducible representation by six parameters (see Supplemental Material S1 D [5]):

$$\boldsymbol{\psi} = e^{i\varphi}\sqrt{\rho}R^{F=1}(\alpha,\beta,\gamma)\begin{pmatrix}\cos\vartheta \\ 0 \\ \sin\vartheta\end{pmatrix}, \tag{1}$$

where $\varphi$ is the U(1) phase, $\vartheta$ characterizes the relative amplitude between the $m = \pm 1$ states, and $R^F(\alpha,\beta,\gamma) = \exp(-\alpha F_z)\exp(-\beta F_y)\exp(-\gamma F_z)$ describes a Euler rotation in terms of the spin-$F$ matrices $F_\mu$'s ($\mu = x, y, z$) and the Euler angles $\alpha$, $\beta$, and $\gamma$. Equation (1) describes a ferromagnetic state at $\vartheta = n\pi/2$ ($n \in \mathbb{Z}$) and a polar state with $\vartheta = (2n+1)\pi/4$. The superfluid velocity for a spin-1 BEC can be expressed in terms of these parameters as [14]

$$\boldsymbol{v} = \frac{\hbar}{M}\{(\nabla\varphi) - [(\nabla\alpha)\cos\beta + (\nabla\gamma)]\cos 2\vartheta\}. \tag{2}$$

For the case of a nonmagnetic spin-1 BEC with $\vartheta = (2n+1)\pi/4$, Eq. (2) reduces to $\boldsymbol{v} = (\hbar/M)\nabla\varphi$ and hence superflow can only be generated from the U(1) phase.

However, a new situation arises for a nonmagnetic spin-2 BEC, where the order parameter can generally be described by seven parameters (Supplemental Material S1 E [5]):

$$\boldsymbol{\psi} = e^{i\varphi}\sqrt{\frac{\rho}{2}}R^{F=2}(\alpha,\beta,\gamma)\begin{pmatrix}e^{i\chi}\sin\eta \\ 0 \\ \sqrt{2}\cos\eta \\ 0 \\ e^{i\chi}\sin\eta\end{pmatrix}, \tag{3}$$

where $\eta$ and $\chi$ describe the relative amplitude and phase between the $m = \pm 2$ and $m = 0$ components. Here we note that the phase difference between the $m = 2$ and $m = -2$ components can be absorbed in the Euler angle $\gamma$. The symmetry of this order parameter can be described by the reciprocal spin representation [6–10] which employs the stereographic mapping of the four roots of the following algebraic equation:

$$\sum_{m=-2}^{2}\sqrt{\frac{24}{(2+m)!(2-m)!}}\xi_m^* w^{2+m} = 0, \tag{4}$$

where $\xi_m \equiv \psi_m/\sqrt{\rho}$ is the normalized order parameter. The four roots of Eq. (4) are stereographically mapped onto the Bloch sphere via $w = e^{i\phi}\tan(\theta/2)$ with $\phi$ and $\theta$ being the azimuth and polar angles, giving four vertices of a polyhedron. These vertices constitute a line segment, a rectangle, or a tetrahedron for the uniaxial nematic, biaxial nematic, and cyclic phases, respectively. The morphology of the order parameter of a spin-2 BEC depends crucially on $\chi$ and $\eta$ as illustrated in Fig. 1.

 

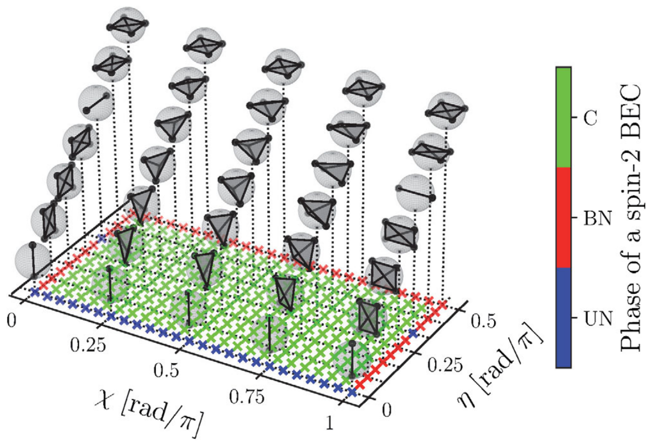


FIG. 1. Phase diagram and stereographically mapped polyhedrons plotted against $\chi$ and $\eta$ in Eq. (3). The blue, red, and green regions show the uniaxial nematic (UN), biaxial nematic (BN), and cyclic (C) phases, respectively. Each polyhedron shows the stereographic projection of the order parameter [i.e., the roots of Eq. (4)] on the Bloch sphere.

It follows from Eq. (3) that the superfluid velocity $\boldsymbol{v}$ is given by (see Supplemental Material S1 E [5])

$$\boldsymbol{v} = \frac{\hbar}{M}\left[(\nabla\varphi) + \frac{1}{2}(\nabla\chi)(1-\cos 2\eta)\right]. \tag{5}$$

The second term, which is absent in a nonmagnetic spin-1 BEC, implies that a supercurrent can be generated by a texture of the order parameter, the physical origin of which is a spatial variation of the morphology of the order parameter. The circulation of Eq. (5) along a two-dimensional closed loop $\mathcal{C}(x,y)$ gives

$$\oint_{\mathcal{C}(x,y)}\left[\frac{M}{\hbar}\boldsymbol{v}-(\nabla\varphi)\right]d\boldsymbol{l}=\frac{1}{2}\int_{\mathcal{S}'(2\eta(x,y),\chi(x,y))}d(1-\cos 2\eta)d\chi, \tag{6}$$

where $\mathcal{S}'[2\eta(x,y),\chi(x,y)]$ represents the surface of a unit sphere in spin space swept by the polar coordinates $(2\eta,\chi)$ when they are mapped from the region inside the loop $\mathcal{C}$ on the $x$-$y$ plane. The right-hand side of Eq. (6) may be interpreted as one half of the Berry phase swept by the unit vector $\hat{\boldsymbol{n}}$ with the azimuth angle $\chi$ and the polar angle $2\eta$, which is analogous to the circulation of a supercurrent in a fully polarized BEC [2] except for the factor of $1/2$.

To understand the correspondence between the superfluid circulations of a nonmagnetic spin-2 BEC and that of a ferromagnetic BEC, we let $\alpha=\beta=\gamma=0$ in Eq. (3) without loss of generality, since the Euler angles specify the direction of the order parameter but do not change its morphology. Then the nonvanishing components of the quardrupole, octupole, and hexadecapole moments are given by (see Supplemental Material S2 for details [5])

$$D_{xy} \equiv \sqrt{\frac{5}{21}}(F_x^2 - F_y^2), \tag{7}$$

$$Y_{\rm hyp} \equiv \frac{\sqrt{5}}{3\sqrt{7}}(-F_x^2 - F_y^2 + 2F_z^2), \tag{8}$$

$$T_{xyz} \equiv \frac{\sqrt{5}}{6\sqrt{3}}\overline{F_xF_yF_z}, \tag{9}$$

$$\Phi^s \equiv \frac{\sqrt{5}}{12}(F_x^4 + F_y^4 - \overline{F_x^2F_y^2}), \tag{10}$$

$$\Phi^a \equiv \frac{\sqrt{5}}{6\sqrt{7}}(F_x^4 - F_y^4 + \overline{F_y^2F_z^2} - \overline{F_z^2F_x^2}), \tag{11}$$

$$\Phi^z \equiv \frac{1}{12\sqrt{7}}(3F_x^4 + 3F_y^4 + 8F_z^4 + \overline{F_x^2F_y^2} - 4\overline{F_y^2F_z^2} - 4\overline{F_z^2F_x^2}), \tag{12}$$

where the coefficients are determined so as to make the $L-2$ norm of each matrix in Eqs. (7)–(12) equal to that of $F_\mu$'s (see Supplemental Material S2 [5]), and $\overline{F_{\mu_1}\cdots F_{\mu_n}}$ denotes the symmetrized product of $F_{\mu_i}$'s ($\mu_i = x,\ y,\ z$), that is, $\overline{F_{\mu_1}\cdots F_{\mu_n}} = \sum_{(\nu_1,\ldots,\nu_n)\in S(\{\mu_1,\ldots,\mu_n\})} F_{\nu_1}\cdots F_{\nu_n}$ where $S(\{\mu_1,\ldots,\mu_n\})$ is the permutation group of a given set $\{\mu_1,\ldots,\mu_n\}$ [11]. The physics behind the morphological supercurrent is a hidden su(2) symmetry whose generators can be constructed from Eqs. (7)–(12) as (see Supplemental Material S2 E for the derivation [5])

$$N_1 = \frac{2}{\sqrt{7}}D_{xy} - \sqrt{\frac{3}{7}}\Phi^a, \tag{13}$$

$$N_2 = -T_{xyz}, \tag{14}$$

$$N_3 = -\frac{2}{\sqrt{7}}Y_{\rm hyp} - \frac{1}{2}\Phi^s + \frac{\sqrt{5}}{2\sqrt{7}}\Phi^z. \tag{15}$$

Note that the structure factor of this algebra is $2\sqrt{5}$ which is to be distinguished from the usual spin su(2) subalgebra with the unit structure factor. Substituting Eq. (3) into Eqs. (7)–(12), we obtain the expectation values $\langle N_i\rangle$'s of $N_i$'s in Eqs. (13) and (14):

$$\langle N_1\rangle = \sqrt{5}\rho\cos\chi\sin 2\eta, \tag{16}$$

$$\langle N_2\rangle = \sqrt{5}\rho\sin\chi\sin 2\eta, \tag{17}$$

$$\langle N_3\rangle = \sqrt{5}\rho\cos 2\eta, \tag{18}$$

which form a vector $\langle\boldsymbol{N}\rangle \equiv (\langle N_1\rangle, \langle N_2\rangle, \langle N_3\rangle)^T$ pointing in the direction of $\hat{\boldsymbol{n}}$. In a spin-2 nonmagnetic BEC, $\hat{\boldsymbol{n}}$, which originates from the magnetic multipoles, plays the role of the spin vector in a fully polarized BEC.

A nonmagnetic superflow can also be induced between two weakly coupled BECs with different order-parameter symmetries, which we refer to as a morphological Josephson current. We assume that two nonmagnetic BECs are placed on the left and right of a potential wall in a well-localized manner. Then, the mean-field energy functional can be well approximated as

$$E_{\mathrm{tot}}[\boldsymbol{\psi}_L, \boldsymbol{\psi}_R] = E[\boldsymbol{\psi}_L] + E[\boldsymbol{\psi}_R] + K \sum_{m=-2}^{2} \int d\boldsymbol{r} (\psi_{Lm}^* \psi_{Rm} + \psi_{Rm}^* \psi_{Lm}), \tag{19}$$

where $\boldsymbol{\psi}_j$ $(j = L, R)$ represents the order parameters of the left $(L)$ and right $(R)$ BECs, $E[\boldsymbol{\psi}_j]$ indicates the energy functional of the BEC on side $j$, and $K$ denotes the coupling between them. When atoms interact via $s$-wave channels, the energy functional $E[\boldsymbol{\psi}_j]$ is given by [15,16]

$$E[\boldsymbol{\psi}_j] = \int d\boldsymbol{r} \left[ \frac{\hbar^2}{2M} \sum_{m=-2}^{2} |(\nabla \psi_{jm})|^2 + U(\boldsymbol{r}) \rho_j + \sum_{m=-2}^{2} q_j m^2 \psi_{jm}^* \psi_{jm} + \frac{1}{2} (c_0 \rho_j^2 + c_1 \boldsymbol{f}_j^2 + c_2 |A_j|^2) \right], \tag{20}$$

where $q_j$ denotes the quadratic Zeeman energy in each well and the coupling strengths $c_0$, $c_1$, and $c_2$ are given by $c_0 \equiv (4\hbar^2/M)(4a_2 + 3a_4)$, $c_1 \equiv (4\hbar^2/M)(a_4 - a_2)$, and $c_2 \equiv (4\hbar^2/M)(7a_0 - 10a_2 + 3a_4)$. Here, $a_{\mathcal{F}}$ represents the scattering length for binary collisions with the total hyperfine spin $\mathcal{F} = 0$, 2, and 4, and $\rho_j$, $\boldsymbol{f}_j \equiv \sum_{m,n=-2}^{2} (\boldsymbol{F})_{mn} \psi_{jm}^* \psi_{jn}$, and $A_j \equiv \sum_{m,n=-2}^{2} (A)_{mn} \psi_{jm} \psi_{jn}$ are the density, the magnetization vector, and the spin-singlet amplitude with a five-by-five antidiagonal matrix $(A)_{mn} \equiv \mathrm{adiag}(1, -1, 1, -1, 1)/\sqrt{5}$. The multicomponent Gross-Pitaevskii equation for $\boldsymbol{\psi}_j$ can be obtained from Eq. (19) as $i\hbar(d\psi_{jm}/dt) = \delta E_{\mathrm{tot}}/\delta\psi_{jm}^*$, from which we obtain $d\rho_L/dt = -d\rho_R/dt$ and

$$\frac{d\rho_L}{dt} = \frac{K}{i\hbar} \sum_{m=-2}^{2} (\psi_{Lm}^* \psi_{Rm} - \psi_{Rm}^* \psi_{Lm}). \tag{21}$$

To derive a general nonmagnetic current-phase relation, let us take the initial order parameter in Eq. (3) with $\alpha_j = \beta_j = \gamma_j = 0$ and assume that $\boldsymbol{\psi}_j$ is uniform in each well and decays exponentially on the other side of the potential barrier. Then, the populations of the $m = \pm 1$ components stay zero and those of the $m = \pm 2$ components remain equal to each other, since no population transfer occurs between $m = \pm 2, 0$ and $m = \pm 1$ and the energy functional in Eq. (19) is symmetric with respect to exchange of the $m = \pm 2$ states, from which we conclude that the order parameter can be expressed as in Eq. (3) with $\alpha_j = \beta_j = \gamma_j = 0$ during the time evolution. Then, Eq. (21) reduces to

$$\frac{d\rho_L}{dt} = \frac{2K}{\hbar} \sqrt{\rho_L \rho_R} [\sin \Delta\varphi (\cos \Delta\chi \sin \eta_L \sin \eta_R + \cos \eta_L \cos \eta_R) + \cos \Delta\varphi \sin \Delta\chi \sin \eta_L \sin \eta_R], \tag{22}$$

where $\Delta\varphi \equiv \varphi_R - \varphi_L$ and $\Delta\chi \equiv \chi_R - \chi_L$. When the BEC on the left side is in the biaxial nematic phase $(\varphi_L = 0, \chi_L = 0, \eta_L = \pi/2)$ and the BEC on the right side is in the cyclic phase $(\varphi_R = \Delta\varphi, \chi_R = \Delta\chi \neq 0, \eta_R = \pi/4)$, Eq. (22) gives

$$\frac{d\rho_L}{dt} = \frac{\sqrt{2}K}{\hbar} \sqrt{\rho_L \rho_R} \sin(\Delta\varphi + \Delta\chi), \tag{23}$$

which implies that the current flows depending on the difference of the parameter $\chi$ determining the shape of the order parameter and that of the U(1) gauge $\varphi$. Thus the supercurrent flows in a manner depending on the difference in the morphology of the order parameter between the left and right BECs. This is essentially different from the Josephson effect due to the Goldstone modes associated with symmetry breaking from $O(N)$ to $O(N-1)$ [17–20]. When the two BECs share the same morphology, Eq. (22) reduces to the familiar Josephson relation caused by the difference of the U(1) phase.

We now demonstrate the above general theory by numerical simulation. The nonmagnetic supercurrent given in Eq. (5) can be induced by a spatially varying quadratic Zeeman effect. To demonstrate this, we consider a cigar-shaped spin-2 BEC of $10^3$ $^{87}$Rb atoms, apply a spatially varying quadratic Zeeman field, and examine how the density profile of the BEC changes after the quadratic Zeeman field is switched off. We assume that the axial trapping frequency $\omega_x = 2\pi \times 10$ [Hz] in the $x$ direction is much smaller than those in the radial directions, i.e., $\omega_x \ll \omega_y$, $\omega_z = 2\pi \times 200$ [Hz] with the ratio $\gamma \equiv \sqrt{\omega_y \omega_z}/\omega_x = 20$. Then the mean-field dynamics of the spin-2 BEC can be described by the following multicomponent Gross-Pitaevskii equation [21]:

$$i\hbar \frac{\partial \psi_m}{\partial t} = \left[ -\frac{\hbar^2}{2M} \nabla^2 + U(x) + q(t, x) m^2 \right] \psi_m + \frac{\gamma}{2\pi} \sum_{n=-2}^{2} (\{c_0 \rho(t, x) \delta_{mn} + c_1 [\boldsymbol{f}(t, x) \cdot \boldsymbol{F}]_{mn}\} \psi_n + c_2 A(t, x) (A)_{mn} \psi_n^*), \tag{24}$$

where $M$ represents the mass of an $^{87}$Rb atom. The trapping potential $U(x)$ is assumed to be a box potential in the $x$ direction given by

$$U(x) = \begin{cases} \infty & (|x| > L/2); \\ 0 & (|x| \leq L/2), \end{cases} \tag{25}$$

where $L = 50 \times l_x$ with $l_x = \sqrt{\hbar/M\omega_x} \approx 3.41$ [$\mu$m]. In the numerical calculation, we set the height of the trapping potential to be $10^2 \times \gamma c_0/2\pi$. We vary the quadratic Zeeman field $q(t,x)$ as

$$q(t,\boldsymbol{r}) = \begin{cases} q' x^2 & (0 \leq t < T); \\ 0 & (t < 0 \quad \text{and} \quad t \geq T), \end{cases} \tag{26}$$

where $q' = 10h$ and $T = 0.1/\omega_x$. The scattering lengths $a_{\mathcal{F}}$'s for binary $s$-wave collisions with their total hyperfine

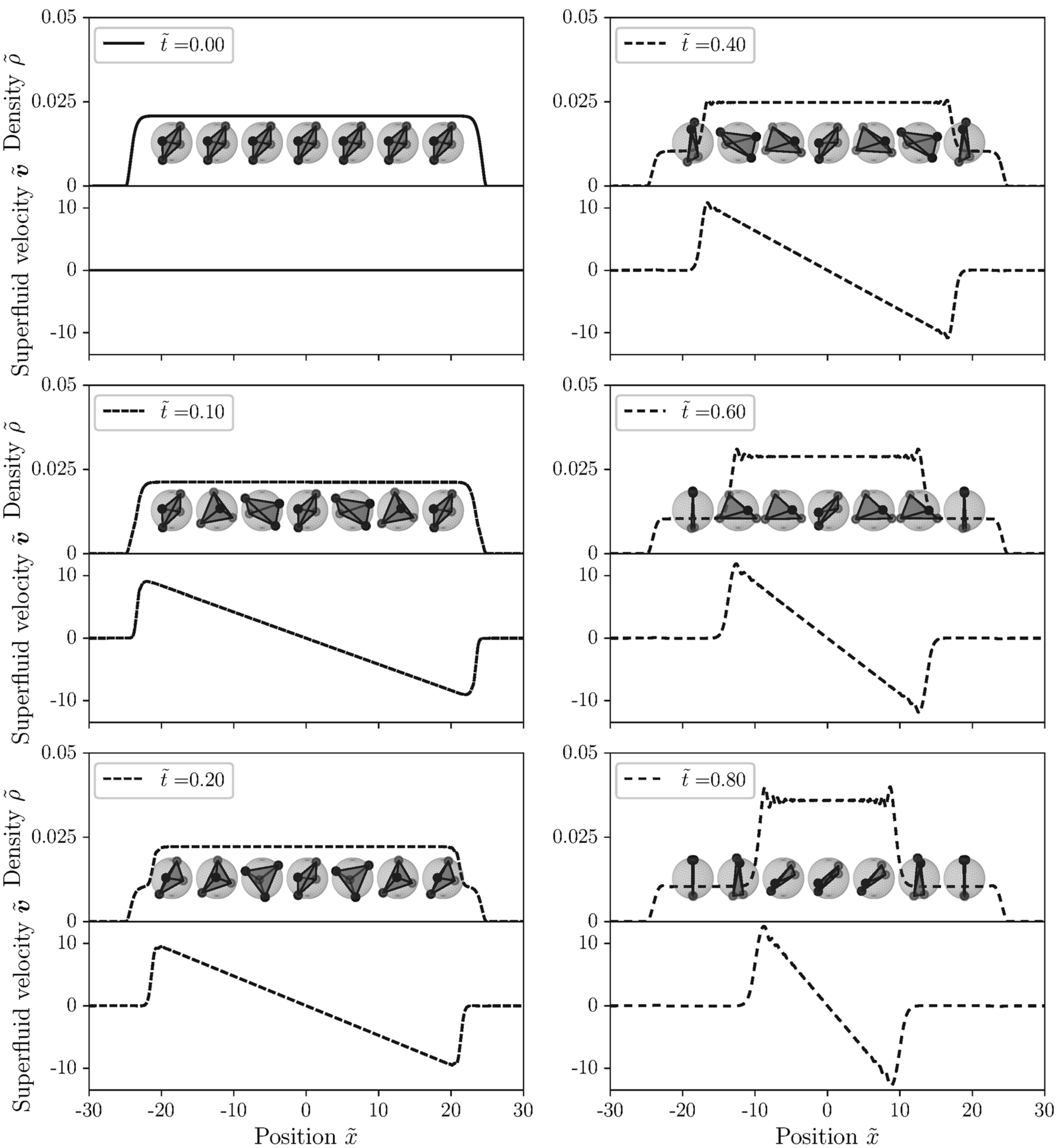


FIG. 2. Position dependences of the density profile and the morphology of the order parameter (upper panels) and the superfluid velocity (lower panels) of a spin-2 BEC at $\tilde{t} \equiv \omega_x t = 0$, 0.1, 0.2, 0.4, 0.6, and 0.8. The quadratic Zeeman field is switched on during $\tilde{t} = 0$–0.1. The coordinate, the density, and the superfluid velocity are normalized as $\tilde{x} \equiv x/l_x$, $\int d\tilde{x}\,\tilde{\rho}(\tilde{t},\tilde{x}) = 1$, and $\tilde{v} = v/l_x\omega_x$.

spins $\mathcal{F} = 0$, 2, and 4 are given by $a_0 = 89.4 a_B$, $a_2 = 94.5 a_B$, and $a_4 = 106 a_B$ with $a_B$ being the Bohr radius [16]. The density profile $\rho(0, x)$ of the initial order parameter is chosen to be the ground state of a scalar BEC with the same potential $U(x)$ and the interaction energy $c_0$ is chosen to be the same as that used in Eq. (24). The initial spin configuration is assumed to be spatially uniform and given by $\boldsymbol{\xi}(t = 0, x) = (1, 0, \sqrt{2}, 0, 1)^T/2$ corresponding to the biaxial nematic state (see Fig. 1). By numerically solving Eq. (24) with the Crank-Nicolson method, the multicomponent order parameter $\boldsymbol{\psi}$ can be obtained and the dynamics of the density profile $\rho(t, x)$, the superfluid velocity $\boldsymbol{v}(t, x)$, and the magnetization vector $\boldsymbol{f}(t, x)$ can be calculated from $\boldsymbol{\psi}$. The density profile and the superfluid velocity evolves in time as shown in Fig. 2, where $\boldsymbol{f}(t, x) = \mathbf{0}$, which implies that the BEC stays nonmagnetic throughout the time evolution. As shown in Fig. 2, the texture of the order-parameter morphology induces superflow without recourse to neither the U(1) gauge nor spin textures.

In summary, we have found the third mechanism of superflow originating from a spatial variation of the morphology of the order parameter in nonmagnetic spin-2 BECs. We have also discussed the morphological Josephson current due to different symmetries of the order parameter on the left and right sides of the potential barrier. We have numerically demonstrated that the morphological superflow can be generated by using a spatially varying quadratic Zeeman effect.

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