Efficiently Computable Bounds for Magic State Distillation

Xin Wang,[1,2,‡] Mark M. Wilde,[3,*] and Yuan Su[2,4,†]
[1]Institute for Quantum Computing, Baidu Research, Beijing 100193, China
[2]Joint Center for Quantum Information and Computer Science, University of Maryland, College Park, Maryland 20742, USA
[3]Hearne Institute for Theoretical Physics, Department of Physics and Astronomy, and Center for Computation and Technology, Louisiana State University, Baton Rouge, Louisiana 70803, USA
[4]Department of Computer Science and Institute for Advanced Computer Studies, University of Maryland, College Park, Maryland 20742, USA

(Received 17 January 2019; revised manuscript received 2 December 2019; accepted 7 January 2020; published 6 March 2020)

Magic-state distillation (or nonstabilizer state manipulation) is a crucial component in the leading approaches to realizing scalable, fault-tolerant, and universal quantum computation. Related to nonstabilizer state manipulation is the resource theory of nonstabilizer states, for which one of the goals is to characterize and quantify nonstabilizerness of a quantum state. In this Letter, we introduce the family of thauma measures to quantify the amount of nonstabilizerness in a quantum state, and we exploit this family of measures to address several open questions in the resource theory of nonstabilizer states. As a first application, we establish the hypothesis testing thauma as an efficiently computable benchmark for the one-shot distillable nonstabilizerness, which in turn leads to a variety of bounds on the rate at which nonstabilizerness can be distilled, as well as on the overhead of magic-state distillation. We then prove that the max-thauma can be used as an efficiently computable tool in benchmarking the efficiency of magic-state distillation, and that it can outperform previous approaches based on mana. Finally, we use the min-thauma to bound a quantity known in the literature as the "regularized relative entropy of magic." As a consequence of this bound, we find that two classes of states with maximal mana, a previously established nonstabilizerness measure, cannot be interconverted in the asymptotic regime at a rate equal to one. This result resolves a basic question in the resource theory of nonstabilizer states and reveals a difference between the resource theory of nonstabilizer states and other resource theories such as entanglement and coherence.

DOI: 10.1103/PhysRevLett.124.090505

Introduction.—Quantum computers hold the promise of a substantial speed-up over classical computers for certain algebraic problems [1–3] and the simulation of quantum systems [4,5]. One main obstacle to the physical realization of quantum computation is the decoherence that occurs during the execution of quantum algorithms. Fault-tolerant quantum computation [6,7] provides a framework to overcome this difficulty and allows reliable quantum computation when the physical error rate is below a certain threshold value.

According to the Gottesman-Knill theorem [8,9], a quantum circuit comprised of only Clifford gates confers no computational advantage because it can be simulated efficiently on a classical computer. However, the addition of a nonstabilizer state can lead to a universal gate set via a technique called state injection [10,11], thus, achieving universal quantum computation. The key of this resolution is to perform magic-state distillation [12] (see [13–19] for recent progress), wherein stabilizer operations are used to transform a large number of noisy nonstabilizer states into a small number of high quality nonstabilizer states. Therefore, a quantitative theory is highly desirable in order to fully exploit the power of nonstabilizer states in fault-tolerant quantum computation.

Quantum resource theories (QRTs) offer a powerful framework for studying different phenomena in quantum physics, and the seminal ideas of QRTs have recently been influencing diverse areas of physics [20]. In the context of the nonstabilizer-state model of universal quantum computation, the resource-theoretic approach reduces to the characterization and quantification of the usefulness of the resourceful nonstabilizer states [21,22]. In the framework of [21,22], the free operations are the stabilizer operations, those that possess a fault-tolerant implementation in the context of fault-tolerant quantum computation, and the free states are the stabilizer states (STAB). Stabilizer operations include preparation and measurement in the computational basis, and a restricted set of unitary operations, called the Clifford unitaries. The free states consist of all pure stabilizer states, which are eigenstates of the generalized Pauli operators, and their convex mixtures. The resource states, namely, the nonstabilizer states, are key resources that are required to achieve some desired computational tasks. For quantum computers acting on qudit registers with odd dimension d, the resource theory of nonstabilizer states (or equivalently, contextuality with respect to stabilizer measurements [23,24]) has been developed [21,22,25]. The resource theory of nonstabilizer states for multiqubit systems was recently developed in [26–28].

In this Letter, we solve some fundamental open questions in the resource theory of nonstabilizer states, and we develop a framework for one-shot magic state distillation. Our main tool for doing so is the thauma family of nonstabilizer monotones, which quantify the amount of nonstabilizerness in a given state by comparing it to a positive semidefinite operator with nonpositive mana (i.e., a subnormalized state with no nonstabilizerness). Our first contribution is to introduce the one-shot distillable nonstabilizerness of a quantum state and an upper bound for it named hypothesis testing thauma. This result leads to various applications for magic-state distillation, which can be interpreted as fundamental limits. The max-thauma is another member of the thauma family, and we prove that it is an efficiently computable nonstabilizerness monotone, which can, in turn, be used to evaluate the efficiency of magic-state distillation. Further, we provide an example to demonstrate that max-thauma outperforms mana in benchmarking the efficiency of magic-state distillation. We also prove that the min-thauma is an additive lower bound on the "regularized relative entropy of magic," the latter quantity defined in [22]. This bound then leads to the conclusion that two magic states with maximal negativity cannot be interconverted asymptotically at a rate equal to one.

Discrete Wigner function.—Now, we recall the definition of the discrete Wigner function [29–31], which is an essential tool in the analysis of the resource theory of nonstabilizer states. Throughout this Letter, a Hilbert space implicitly has an odd dimension, and if the dimension is not prime, it should be understood to be a tensor product of Hilbert spaces, each having an odd prime dimension.

Let $\mathcal{H}_d$ denote a Hilbert space of dimension d, and let $\{|j\rangle\}_{j=0,\ldots,d-1}$ denote the standard computational basis. For a prime number d, we define the respective shift and boost operators $X,\ Z \in \mathcal{L}(\mathcal{H}_d)$ as $X|j\rangle = |j \oplus 1\rangle$ and $Z|j\rangle = \omega^j |j\rangle$, with $\omega = e^{2\pi i/d}$. We define the Heisenberg-Weyl operators as $T_{\mathbf{u}} = \tau^{-a_1 a_2} Z^{a_1} X^{a_2}$, where $\tau = e^{(d+1)\pi i/d}$ and $\mathbf{u} = (a_1, a_2) \in \mathbb{Z}_d \times \mathbb{Z}_d$.

For each point $\mathbf{u}$ in the discrete phase space, there is a corresponding operator $A_{\mathbf{u}}$, and the value of the discrete Wigner representation of a quantum state ρ at this point is given by $W_\rho(\mathbf{u}) \coloneqq \mathrm{Tr} A_{\mathbf{u}} \rho / d$, where $\{A_{\mathbf{u}}\}_{\mathbf{u}}$ are the phase-space point operators: $A_{\mathbf{u}} \coloneqq T_{\mathbf{u}} A_0 T_{\mathbf{u}}^\dagger$, $A_0 \coloneqq (1/d) \sum_{\mathbf{u}} T_{\mathbf{u}}$. We give more details of this formalism in Appendix A [32].

Thauma.—It is well known that quantum computations are classically simulable if they consist of stabilizer operations acting on quantum states with a positive discrete Wigner function. Thus, such states are useless for magic-state distillation [21]. Let $\mathcal{W}_+$ denote the set of quantum states with a positive discrete Wigner function. States in $\mathcal{W}_+$ can be understood as being analogous to states with a positive partial transpose in entanglement distillation [42,43], in the sense that such states are undistillable.

To address open questions in the resource theory of nonstabilizer states, we are motivated by the idea of the Rains bound from entanglement theory [44], as well as its variants [45–47], which also have fruitful applications in quantum communication [48–51]. As developed in [44] and the later work [52], the Rains bound and its variants consider subnormalized states with nonpositive logarithmic negativity [53,54] as useless resources, and they use the divergence between the given state and such subnormalized states to evaluate the behavior of entanglement distillation.

Thus, inspired by the main idea behind the Rains bound, we introduce the set of subnormalized states with nonpositive mana: $\mathcal{W} \coloneqq \{\sigma : \mathcal{M}(\sigma) \le 0, \sigma \ge 0\}$, with the mana $\mathcal{M}(\rho)$ of a quantum state ρ defined as [22]

$$\mathcal{M}(\rho) \coloneqq \log_2 \|\rho\|_{W,1},$$

where the Wigner trace norm of an operator V is defined as $\|V\|_{W,1} \coloneqq \sum_{\mathbf{u}} |W_V(\mathbf{u})|$. It follows from definitions that $\mathrm{Tr}\sigma \le 1$ if $\sigma \in \mathcal{W}$. Note that the mana [22] is analogous to the logarithmic negativity [53,54]. Furthermore, the following strict inclusions hold: $\mathrm{STAB} \subsetneq \mathcal{W}_+ \subsetneq \mathcal{W}$.

Now, we define the thauma [55] of a state ρ as

$$\theta(\rho) \coloneqq \min_{\sigma \in \mathcal{W}} D(\rho \| \sigma),$$

where $D(\rho\|\sigma)$ is the quantum relative entropy [56], defined as $D(\rho\|\sigma) = \mathrm{Tr}\{\rho[\log_2 \rho - \log_2 \sigma]\}$ when the support of ρ is contained in the support of σ and equal to $+\infty$, otherwise. The thauma can be understood as the minimum relative entropy between a quantum state and the set of subnormalized states with nonpositive mana. The thauma is a nonstabilizerness measure that can be efficiently computed via convex optimization (see Appendix B [32]). Following from the definition of thauma above, we define the regularized thauma of a state ρ as $\theta^\infty(\rho) \coloneqq \lim_{n\to\infty} \theta(\rho^{\otimes n})/n$.

The definition of thauma given above can be generalized to a whole family of thauma measures of nonstabilizerness. Defining a generalized divergence $\mathbf{D}(\rho\|\sigma)$ to be any function of a quantum state ρ and a positive semidefinite operator σ that obeys data processing [57,58], i.e., $\mathbf{D}(\rho\|\sigma) \ge \mathbf{D}[\mathcal{N}(\rho)\|\mathcal{N}(\sigma)]$ where $\mathcal{N}$ is a quantum channel, we arrive at the generalized thauma of a quantum state ρ

$$\boldsymbol{\theta}(\rho) \coloneqq \inf_{\sigma \in \mathcal{W}} \mathbf{D}(\rho\|\sigma).$$

If the generalized divergence $\mathbf{D}$ is non-negative for a state ρ and a subnormalized state σ and equal to zero if $\rho = \sigma$, then it trivially follows that the generalized thauma $\boldsymbol{\theta}(\rho)$ is a nonstabilizerness monotone, meaning that it is nonincreasing under the free operations and equal to zero for stabilizer states. Examples of generalized divergences, in addition to the relative entropy, include the Petz-Rényi relative entropies [59] and the sandwiched Rényi relative entropies [60,61]. See Appendix C for further details [32].

Min- and max-thauma.—In what follows, we make use of the Petz-Rényi relative entropy of order zero [59] and the max-relative entropy [62] to define the min-thauma and the max-thauma, respectively. As we prove in what follows, these two members of the thauma family are efficiently computable by semidefinite programs (SDPs) [63] and are particularly useful for addressing open questions in the resource theory of nonstabilizer states.

The min-thauma of a state ρ is defined as

$$\theta_{\min}(\rho) \coloneqq \min_{\sigma\in\mathcal{W}} D_0(\rho\|\sigma) \coloneqq \min_{\sigma\in\mathcal{W}}[-\log_2 \mathrm{Tr} P_\rho \sigma],$$

where P_ρ denotes the projection onto the support of ρ. Note that $\theta_{\min}(\rho)$ is an SDP and the duality theory of SDPs [63] leads to the dual SDP

$$\theta_{\min}(\rho) = -\log_2 \min\{\|Q\|_{W,\infty} : Q \geq P_\rho\},$$

where $\|V\|_{W,\infty} \coloneqq d \max_{\mathbf{u}} |W_V(\mathbf{u})|$ denotes the Wigner spectral norm of an operator V acting on a space of dimension d. For any pure state $|\psi\rangle$,

$$\theta_{\min}(\psi) = -\log_2 \max_{\sigma\in\mathcal{W}} F(\psi,\sigma) \leq -\log_2 F_{\mathrm{Stab}}(\psi),$$

where $F_{\mathrm{STAB}}(\psi)$ is the stabilizer fidelity [64].

The max-thauma of a state ρ is defined as

$$\begin{aligned}\theta_{\max}(\rho) &\coloneqq \min_{\sigma\in\mathcal{W}} D_{\max}(\rho\|\sigma) \coloneqq \min_{\sigma\in\mathcal{W}}[\min\{\lambda : \rho \leq 2^\lambda \sigma\}] \\ &= \log_2 \min\{\|V\|_{W,1} : \rho \leq V\}.\end{aligned}$$

As the following proposition states, the min- and max-thauma are additive nonstabilizerness measures. Additionally, the min-thauma is a lower bound on the regularized thauma, and the max-thauma is an upper bound.

Proposition 1.—For states ρ and τ, it holds that

$$\begin{aligned}\theta_{\min}(\rho\otimes\tau) &= \theta_{\min}(\rho) + \theta_{\min}(\tau), \\ \theta_{\max}(\rho\otimes\tau) &= \theta_{\max}(\rho) + \theta_{\max}(\tau).\end{aligned}$$

Consequently, $\theta_{\min}(\rho) \leq \theta^{\infty}(\rho) \leq \theta_{\max}(\rho)$.

The proof of Proposition 1 relies on the Petz-Rényi relative entropy of order zero [62], the max-relative entropy [62], and the duality theory of SDPs [63] (see Appendix D for details [32]).

In Appendix E [32], we prove that the max-thauma possesses a stronger monotonicity property, in the sense that it does not increase on average under stabilizer operations.

Here, we note that an important consequence of the additivity of min-thauma is that the maximum overlap between $|\phi\rangle^{\otimes n}$ and the set $\mathcal{W}$ is $2^{-n\theta_{\min}(\phi)}$; i.e., for any $\tau\in\mathcal{W}$ (or STAB), we have that $\mathrm{Tr}[|\phi\rangle\langle\phi|^{\otimes n}\tau] \leq 2^{-n\theta_{\min}(\phi)}$.

Thauma for basic nonstabilizer states.—Proposition 2 below states that the min-, regularized, and max-thauma collapse to the same value for several interesting nonstabilizer states, including the Strange, Norrell, H, and T states.

The Strange and Norrell states are defined as [22]

$$|\mathbb{S}\rangle \coloneqq (|1\rangle - |2\rangle)/\sqrt{2}, \quad |\mathbb{N}\rangle \coloneqq (-|0\rangle + 2|1\rangle - |2\rangle)/\sqrt{6}.$$

The qutrit Hadamard gate is given by [65]

$$H = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \tag{1}$$

where we recall that $\omega = e^{2\pi i/3}$. The H gate has eigenvalues $+1$, -1, and i, and we label the three corresponding eigenstates as $|H_+\rangle$, $|H_-\rangle$, and $|H_i\rangle$. The $|H_+\rangle$ state is a nonstabilizer state that is typically considered in the context of magic-state distillation [12,66]. In what follows, we refer to it as the H_+ nonstabilizer state.

Another common choice for a non-Clifford gate is the T gate. The qutrit T gate is given by $T \coloneqq \mathrm{diag}(\xi, 1, \xi^{-1})$, where $\xi = e^{2\pi i/9}$ is a primitive ninth root of unity [65]. The nonstabilizer state corresponding to the qutrit T gate is $|T\rangle \coloneqq (1/\sqrt{3})(\xi|0\rangle + |1\rangle + \xi^{-1}|2\rangle)$, which is the state resulting from applying the T gate to the stabilizer state $(|0\rangle + |1\rangle + |2\rangle)/\sqrt{3}$.

In what follows, we employ the shorthand $\mathbb{S} \equiv |\mathbb{S}\rangle\langle\mathbb{S}|$, $\mathbb{N} \equiv |\mathbb{N}\rangle\langle\mathbb{N}|$, $H_+ \equiv |H_+\rangle\langle H_+|$, and $T \equiv |T\rangle\langle T|$. Before stating the theorem, let us recall the definition of the regularized relative entropy of magic and the relative entropy of magic [22]

$$R^{\infty}_{\mathcal{M}}(\rho) \coloneqq \lim_{n\to\infty}\frac{1}{n}R_{\mathcal{M}}(\rho^{\otimes n}), \qquad R_{\mathcal{M}}(\rho) \coloneqq \min_{\sigma\in\mathrm{STAB}} D(\rho\|\sigma).$$

Proposition 2.—The following equalities hold:

$$\begin{aligned}\theta_{\min}(\mathbb{S}) &= \theta^{\infty}(\mathbb{S}) = \theta_{\max}(\mathbb{S}) = \log_2(5/3), \\ \theta_{\min}(\mathbb{N}) &= \theta^{\infty}(\mathbb{N}) = \theta_{\max}(\mathbb{N}) = R^{\infty}_{\mathcal{M}}(\mathbb{N}) = \log_2(3/2), \\ \theta_{\min}(H_+) &= \theta^{\infty}(H_+) = \theta_{\max}(H_+) \\ &= R^{\infty}_{\mathcal{M}}(H_+) = \log_2(3-\sqrt{3}), \\ \theta_{\min}(T) &= \theta^{\infty}(T) = \theta_{\max}(T) = \log_2[1 + 2\sin(\pi/18)].\end{aligned}$$

Appendix F [32] contains a proof of Proposition 2. In the forthcoming sections, we provide applications of Propositions 1 and 2 to the resource theory of nonstabilizer states.

Fundamental limits for distilling nonstabilizer states.—The basic task of magic-state distillation [12] can be understood as follows. For any given quantum state ρ, we aim to transform this state to a collection of nonstabilizer states (e.g., $|T\rangle$) with high fidelity using stabilizer operations. The goal is to maximize the number of target states while keeping the transformation infidelity within some tolerance ε. After magic-state distillation, one can use a circuit gadget (which requires only stabilizer operations) to transform this nonstabilizer state into a non-Clifford gate [10,11]. Protocols for distillation in the qudit setting of quantum computing were recently developed in [66–69].

In the following, we study the fundamental limit of magic-state distillation of a pure target nonstabilizer state. We define the approximate one-shot distillable ϕ nonstabilizerness of a given state ρ as the maximum number of $|\phi\rangle\langle\phi|$ nonstabilizer states that can be obtained via stabilizer operations, while keeping the infidelity within a given tolerance. Formally, for any triplet $(\rho, \phi, \varepsilon)$ consisting of an initial state ρ, a target pure state ϕ, and an infidelity tolerance ε, the one-shot ε-error distillable ϕ nonstabilizerness of ρ is defined to be the maximum number of ϕ nonstabilizer states achievable via stabilizer operations, with an error tolerance of ε

$$\mathcal{M}^{\varepsilon}_{\phi}(\rho) = \sup\{k : \Lambda(\rho) \approx_{\varepsilon} |\phi\rangle\langle\phi|^{\otimes k}, \Lambda \in \mathrm{SO}\},$$

where $|\psi\rangle\langle\psi| \approx_{\varepsilon} \sigma$ is a shorthand for $\langle\psi|\sigma|\psi\rangle \geq 1-\varepsilon$ and SO for stabilizer operations.

In what follows, we focus on the one-shot distillable H_+ nonstabilizerness $\mathcal{M}^{\varepsilon}_{H_+}(\rho)$ and the one-shot distillable T nonstabilizerness $\mathcal{M}^{\varepsilon}_{T}(\rho)$.

First, we connect the task of magic-state distillation to quantum hypothesis testing between nonstabilizer states and operators in the set $\mathcal{W}$ (recall that $\mathrm{STAB} \subsetneq \mathcal{W}$), and we note, here, that such an approach was previously taken in entanglement theory [44,70]. Quantum hypothesis testing is the task of distinguishing two possible states ρ_0 and ρ_1 (null hypothesis ρ_0, alternative hypothesis ρ_1). We are allowed to perform a measurement characterized by the positive operator-valued measure $\{M, \mathbb{1}-M\}$ with respective outcomes 0 and 1. If the outcome is 0, we accept the null hypothesis. Otherwise, we accept the alternative one. The probabilities of type I and type II errors are given by $\mathrm{Tr}(\mathbb{1}-M)\rho_0$ and $\mathrm{Tr}M\rho_1$, respectively. The hypothesis testing relative entropy [71,72] quantifies the minimum type II error probability provided that the type I error probability is within a given tolerance: $D^{\varepsilon}_{H}(\rho_0\|\rho_1) := -\log_2 \min\{\mathrm{Tr}M\rho_1 | 0 \leq M \leq \mathbb{1}, 1-\mathrm{Tr}M\rho_0 \leq \varepsilon\}$.

Proposition 3.—Given a state ρ, the following holds:

$$\mathcal{M}^{\varepsilon}_{H_+}(\rho) \leq \frac{\min_{\sigma\in\mathcal{W}} D^{\varepsilon}_{H}(\rho\|\sigma)}{\log_2(3-\sqrt{3})}, \tag{2}$$

$$\mathcal{M}^{\varepsilon}_{T}(\rho) \leq \frac{\min_{\sigma\in\mathcal{W}} D^{\varepsilon}_{H}(\rho\|\sigma)}{\log_2[1+2\sin(\pi/18)]}. \tag{3}$$

A consequence of Proposition 3 is that the thauma of a quantum state is an upper bound on its distillable H_+ (or T) nonstabilizerness. Specifically, by applying the quantum Stein's lemma [73–76], we find the following.

Corollary 4.—The distillable nonstabilizerness of ρ satisfies

$$\mathcal{M}_{H_+}(\rho) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\mathcal{M}^{\varepsilon}_{H_+}(\rho^{\otimes n}) \leq \frac{\theta(\rho)}{\log_2(3-\sqrt{3})},$$
$$\mathcal{M}_{T}(\rho) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\mathcal{M}^{\varepsilon}_{T}(\rho^{\otimes n}) \leq \frac{\theta(\rho)}{\log_2[1+2\sin(\pi/18)]}.$$

Efficiency of magic-state distillation.—The efficiency of distilling a nonstabilizer state ξ from several independent copies of a resource state ρ is given by the minimum number of copies of ρ needed, on average, to produce ξ using stabilizer operations

$$N_{\mathrm{eff}}(\rho\to\xi) = \inf\{n/p : \Lambda(\rho^{\otimes n}) \to \xi \text{ with prob } p, \Lambda \in \mathrm{SO}\}.$$

Previously, the authors of [22] derived the following lower bound on the efficiency of magic-state distillation:

$$N_{\mathrm{eff}}(\rho\to\xi) \geq N_{\mathcal{M}}(\rho,\xi) := \mathcal{M}(\xi)/\mathcal{M}(\rho). \tag{4}$$

The lower bound in [22] was established by employing the mana of nonstabilizer states. Here, we utilize similar ideas and show that the max-thauma can also be applied to bound the efficiency of magic-state distillation.

Proposition 5.—The efficiency of distilling a nonstabilizer state ξ from resource states ρ is lower bounded by $N_{\theta_{\max}}(\rho,\xi) := \theta_{\max}(\xi)/\theta_{\max}(\rho)$.

Figure 1 demonstrates that the lower bound from Proposition 5 can be tighter than the lower bound in (4), thus, giving an improved sense of the efficiency.

On the overhead of magic-state distillation.—The overhead of magic-state distillation is defined as the ratio of the number of input to output states, under a target error rate [13,18]. Although our notion of error for magic-state distillation is different from that typically employed in the literature, here, we note that the inverse of the one-shot distillable ϕ nonstabilizerness (i.e., $[\mathcal{M}^{\varepsilon}_{\phi}(\rho)]^{-1}$) can be considered a reasonable way to measure the overhead of magic-state distillation. Then, our upper bounds in Proposition 3 and Corollary 4 become lower bounds on the overhead.

Inequivalence between nonstabilizer states with maximal mana.—A fundamental problem in any quantum resource theory is to determine the interconversion rate between

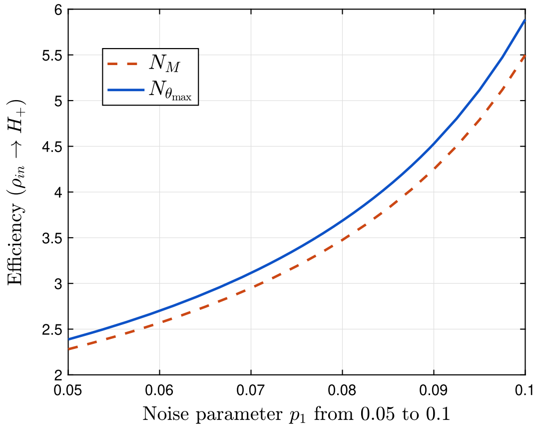

FIG. 1. Comparison between $N_{\theta_{\max}}(\rho_{\rm in} \to H_+)$ and $N_{\mathcal{M}}(\rho_{\rm in} \to H_+)$ for input $\rho_{\rm in} = (1 - p_1 - p_2)|H_+\rangle\langle H_+| + p_1\ |H_-\rangle\langle H_-| + p_2|H_i\rangle\langle H_i|$ with $p_2 = 1/10$.

different resource states [20], in particular, between given states and maximally resourceful states. This is rooted in the fact that, in any resource theory, maximally resourceful states play a unique role in quantifying the resourcefulness of other states and assessing the performance of resource manipulation. Considering entanglement theory (or coherence theory) as an example, the interconversion between a given state and maximally entangled (coherent) states leads to fundamental tasks such as entanglement (coherence) distillation and dilution [20,77,78]. Notably, any two maximally entangled (coherent) states under all resource measures in the same dimension are equivalent under free operations.

However, this is not the case in the resource theory of nonstabilizer states. Surprisingly, we find that, even though the Strange state and the Norrell state each have maximum mana and, thus, are the most costly resource to simulate on a classical computer using certain known algorithms [22,79], they are not equivalent even in the asymptotic regime. Note that the mana plays a significant role as a measure of nonclassical resources in quantum computation [25,79]. In particular, recall that mana is a nonstabilizerness measure analogous to logarithmic negativity in entanglement theory. In contrast, logarithmic negativity of a bipartite state is equal to its maximal value if and only if the state is maximally entangled.

To establish this result, we recall that the asymptotic conversion rate from ρ to ξ under asymptotically nonstabilizer-nongenerating transformations is given by the ratio of their regularized relative entropies of resource [80]. That is, $R(\rho \to \xi) = R^\infty_{\mathcal{M}}(\rho)/R^\infty_{\mathcal{M}}(\xi)$. Further, we recall that the Strange and Norrell states have maximum mana [22]: $\mathcal{M}(\mathbb{S}) = \mathcal{M}(\mathbb{N}) = \log_2(5/3)$. However, Proposition 2 and the fact that $R^\infty_{\mathcal{M}}(\mathbb{S}) \geq \theta^\infty(\mathbb{S})$ indicate that there is a gap between their regularized relative entropies of magic. As a consequence, we find that

Theorem 6.—For the Strange state $|\mathbb{S}\rangle$ and the Norrell state $|\mathbb{N}\rangle$, the following holds:

$$R(\mathbb{N} \to \mathbb{S}) = R^\infty_{\mathcal{M}}(\mathbb{N})/R^\infty_{\mathcal{M}}(\mathbb{S}) \leq \frac{\log_2(3/2)}{\log_2(5/3)} < 1.$$

Since stabilizer operations are included in the set of asymptotically nonstabilizer-nongenerating transformations, this result also establishes that the rate to obtain the Strange state from the Norrell state is smaller than one under stabilizer operations. Thus, the gap between $R^\infty_{\mathcal{M}}(\mathbb{N})$ and $R^\infty_{\mathcal{M}}(\mathbb{S})$, as established in Theorem 6, closes an open question from [[22] Section IV. D].

This result demonstrates a fundamental difference between the resource theory of nonstabilizer states and the resource theory of entanglement or coherence. Specifically, we show that the maximally resourceful nonstabilizer states under certain resource measure cannot be interconverted at a rate equal to one, even in the asymptotic regime, while the maximally resourceful states in entanglement theory or coherence theory can be interconverted equivalently in the one-copy setting. However, it remains open to determine whether the conversion rate from the Strange state to the Norrell state is strictly smaller than $\log_2(5/3)/\log_2(3/2)$. Such an inequality would imply the irreversibility of asymptotic magic state manipulation.

Conclusions.—We have introduced the thauma family of measures to quantify and characterize the nonstabilizerness resource possessed by quantum states that are needed for universal quantum computation. The min- and max-thauma are efficiently computable by semidefinite programming and lead to bounds on the rates at which one can interconvert nonstabilizer states. These bounds have helped to solve pressing open questions in the resource theory of nonstabilizer states. More generally, this Letter establishes fundamental limitations to the processing of quantum nonstabilizerness, opening new perspectives for its investigation and exploitation as a resource in quantum information processing and quantum technology. Along this line, we suspect that our results will have immediate impact on the quantum optics community working on the resource theory of non-Gaussianity [81–83] and continuous-variable quantum computing [84,85], because the main idea behind the thauma measure can be generalized to this setting.

We thank Ali Hamed Mosavian and Victor Veitch for discussions regarding [22]. X. W. acknowledges support from the Department of Defense. M. M. W. acknowledges support from NSF under Grant No. 1714215. Y. S. acknowledges support from the Army Research Office (MURI Grant No. W911NF-16-1-0349), the Canadian Institute for Advanced Research, the National Science Foundation (Grants No. 1526380 and No. 1813814), and the U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research, Quantum Algorithms Teams and Quantum Testbed Pathfinder programs.

*Corresponding author.
mwilde@lsu.edu
†buptsuyuan@gmail.com
‡wangxin73@baidu.com

[1] P. W. Shor, SIAM J. Sci. Stat. Comput. **26**, 1484 (1997).
[2] L. K. Grover, in *Proceedings of the Twenty-Eighth Annual ACM Symposium on Theory of Computing—STOC '96* (ACM Press, New York, 1996), pp. 212–219.
[3] A. M. Childs and W. van Dam, Rev. Mod. Phys. **82**, 1 (2010).
[4] S. Lloyd, Science **273**, 1073 (1996).
[5] A. M. Childs, D. Maslov, Y. Nam, N. J. Ross, and Y. Su, Proc. Natl. Acad. Sci. U.S.A. **115**, 9456 (2018).
[6] P. W. Shor, in *Proceedings of the 37th Conference on Foundations of Computer Science, Los Alamitos, CA, USA* (IEEE Comput. Soc. Press, 1996), pp. 56–65.
[7] E. T. Campbell, B. M. Terhal, and C. Vuillot, Nature (London) **549**, 172 (2017).
[8] D. Gottesman, Ph. D. thesis, California Institute of Technology, 1997, arXiv:quant-ph/9705052.
[9] S. Aaronson and D. Gottesman, Phys. Rev. A **70**, 052328 (2004).
[10] D. Gottesman and I. L. Chuang, Nature (London) **402**, 390 (1999).
[11] X. Zhou, D. W. Leung, and I. L. Chuang, Phys. Rev. A **62**, 052316 (2000).
[12] S. Bravyi and A. Kitaev, Phys. Rev. A **71**, 022316 (2005).
[13] S. Bravyi and J. Haah, Phys. Rev. A **86**, 052329 (2012).
[14] C. Jones, Phys. Rev. A **87**, 042305 (2013).
[15] J. Haah, M. B. Hastings, D. Poulin, and D. Wecker, Quantum **1**, 31 (2017).
[16] E. T. Campbell and M. Howard, Quantum **2**, 56 (2018).
[17] M. B. Hastings and J. Haah, Phys. Rev. Lett. **120**, 050504 (2018).
[18] A. Krishna and J.-P. Tillich, Phys. Rev. Lett. **123**, 070507 (2019).
[19] C. Chamberland and A. Cross, Quantum **3**, 143 (2019).
[20] E. Chitambar and G. Gour, Rev. Mod. Phys. **91**, 025001 (2019).
[21] V. Veitch, C. Ferrie, D. Gross, and J. Emerson, New J. Phys. **14**, 113011 (2012).
[22] V. Veitch, S. A. Hamed Mousavian, D. Gottesman, and J. Emerson, New J. Phys. **16**, 013009 (2014).
[23] M. Howard, J. J. Wallman, V. Veitch, and J. Emerson, Nature (London) **510**, 351 (2014).
[24] N. Delfosse, P. Allard Guerin, J. Bian, and R. Raussendorf, Phys. Rev. X **5**, 021003 (2015).
[25] A. Mari and J. Eisert, Phys. Rev. Lett. **109**, 230503 (2012).
[26] M. Howard and E. Campbell, Phys. Rev. Lett. **118**, 090501 (2017).
[27] S. Bravyi, G. Smith, and J. A. Smolin, Phys. Rev. X **6**, 021043 (2016).
[28] M. Heinrich and D. Gross, Quantum **3**, 132 (2019).
[29] W. K. Wootters, Ann. Phys. (N.Y.) **176**, 1 (1987).
[30] D. Gross, J. Math. Phys. **47**, 122107 (2006).
[31] D. Gross, Appl. Phys. B **86**, 367 (2007).
[32] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.124.090505 for detailed mathematical developments of the assertions in the main text, which includes Refs. [33–41].
[33] E. Hostens, J. Dehaene, and B. De Moor, Phys. Rev. A **71**, 042315 (2005).
[34] C. Ferrie, Rep. Prog. Phys. **74**, 116001 (2011).
[35] H. Fawzi and O. Fawzi, J. Phys. A **51**, 154003 (2018).
[36] M. Tomamichel, R. Colbeck, and R. Renner, IEEE Trans. Inf. Theory **55**, 5840 (2009).
[37] M. Mosonyi and F. Hiai, IEEE Trans. Inf. Theory **57**, 2474 (2011).
[38] A. Uhlmann, Rep. Math. Phys. **9**, 273 (1976).
[39] R. L. Frank and E. H. Lieb, J. Math. Phys. **54**, 122201 (2013).
[40] S. Beigi, J. Math. Phys. **54**, 122202 (2013).
[41] M. M. Wilde, J. Phys. A **51**, 374002 (2018).
[42] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[43] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
[44] E. M. Rains, IEEE Trans. Inf. Theory **47**, 2921 (2001).
[45] X. Wang and R. Duan, Phys. Rev. A **94**, 050301(R) (2016).
[46] X. Wang and R. Duan, Phys. Rev. A **95**, 062322 (2017).
[47] K. Fang, X. Wang, M. Tomamichel, and R. Duan, IEEE Trans. Inf. Theory **65**, 6454 (2019).
[48] M. Tomamichel, M. M. Wilde, and A. Winter, IEEE Trans. Inf. Theory **63**, 715 (2017).
[49] M. Tomamichel, M. Berta, and J. M. Renes, Nat. Commun. **7**, 11419 (2016).
[50] X. Wang and R. Duan, in *2016 IEEE International Symposium on Information Theory (ISIT), Barcelona, Spain* (IEEE, 2016), Vol. 2016, pp. 1690–1694.
[51] X. Wang, K. Fang, and R. Duan, IEEE Trans. Inf. Theory **65**, 2583 (2018).
[52] K. Audenaert, B. De Moor, K. G. H. Vollbrecht, and R. F. Werner, Phys. Rev. A **66**, 032310 (2002).
[53] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[54] M. B. Plenio, Phys. Rev. Lett. **95**, 090503 (2005).
[55] Greek for "wonder" or "marvel."
[56] H. Umegaki, Kodai Math. Sem. Rep. **14**, 59 (1962).
[57] Y. Polyanskiy and S. Verdu, in *48th Annual Allerton Conference on Communication, Control, and Computing (Allerton, Illinois), 2010* (IEEE, 2010), pp. 1327–1333.
[58] N. Sharma and N. A. Warsi, arXiv:1205.1712.
[59] D. Petz, Rep. Math. Phys. **23**, 57 (1986).
[60] M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr, and M. Tomamichel, J. Math. Phys. **54**, 122203 (2013).
[61] M. M. Wilde, A. Winter, and D. Yang, Commun. Math. Phys. **331**, 593 (2014).
[62] N. Datta, IEEE Trans. Inf. Theory **55**, 2816 (2009).
[63] S. Boyd and L. Vandenberghe, *Convex Optimization* (Cambridge University Press, Cambridge, England, 2004).
[64] S. Bravyi, D. Browne, P. Calpin, E. Campbell, D. Gosset, and M. Howard, Quantum **3**, 181 (2019).
[65] M. Howard and J. Vala, Phys. Rev. A **86**, 022316 (2012).
[66] H. Anwar, E. T. Campbell, and D. E. Browne, New J. Phys. **14**, 063006 (2012).
[67] E. T. Campbell, H. Anwar, and D. E. Browne, Phys. Rev. X **2**, 041021 (2012).
[68] E. T. Campbell, Phys. Rev. Lett. **113**, 230501 (2014).
[69] H. Dawkins and M. Howard, Phys. Rev. Lett. **115**, 030501 (2015).
[70] E. M. Rains, Phys. Rev. A **60**, 179 (1999).

[71] F. Buscemi and N. Datta, IEEE Trans. Inf. Theory **56**, 1447 (2010).
[72] L. Wang and R. Renner, Phys. Rev. Lett. **108**, 200501 (2012).
[73] F. Hiai and D. Petz, Commun. Math. Phys. **143**, 99 (1991).
[74] T. Ogawa and H. Nagaoka, IEEE Trans. Inf. Theory **46**, 2428 (2000).
[75] M. Tomamichel and M. Hayashi, IEEE Trans. Inf. Theory **59**, 7693 (2013).
[76] K. Li, Ann. Stat. **42**, 171 (2014).
[77] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[78] A. Streltsov, G. Adesso, and M. B. Plenio, Rev. Mod. Phys. **89**, 041003 (2017).
[79] H. Pashayan, J. J. Wallman, and S. D. Bartlett, Phys. Rev. Lett. **115**, 070501 (2015).
[80] F. G. S. L. Brandão and G. Gour, Phys. Rev. Lett. **115**, 070503 (2015).
[81] Q. Zhuang, P. W. Shor, and J. H. Shapiro, Phys. Rev. A **97**, 052317 (2018).
[82] R. Takagi and Q. Zhuang, Phys. Rev. A **97**, 062337 (2018).
[83] F. Albarelli, M. G. Genoni, M. G. A. Paris, and A. Ferraro, Phys. Rev. A **98**, 052350 (2018).
[84] S. Lloyd and S. L. Braunstein, Phys. Rev. Lett. **82**, 1784 (1999).
[85] D. Gottesman, A. Kitaev, and J. Preskill, Phys. Rev. A **64**, 012310 (2001).