Quadrature Coherence Scale Driven Fast Decoherence of Bosonic Quantum Field States

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We introduce, for each state of a bosonic quantum field, its quadrature coherence scale (QCS), a measure of the range of its quadrature coherences. Under coupling to a thermal bath, the purity and QCS are shown to decrease on a timescale inversely proportional to the QCS squared. The states most fragile to decoherence are therefore those with quadrature coherences far from the diagonal. We further show a large QCS is difficult to measure since it induces small scale variations in the state's Wigner function. These two observations imply a large QCS constitutes a mark of "macroscopic coherence." Finally, we link the QCS to optical classicality: optical classical states have a small QCS and a large QCS implies strong optical nonclassicality.

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Introduction.—Both in order to obtain an ever better understanding of quantum physics and to solve problems in quantum information theory, there is continued interest in the exploration of the classical-quantum boundary and the identification of those quantum states exhibiting specifically quantum features, such as coherence and interference that cannot be explained with classical mechanics and/or classical probability theory. It has been shown on example states in model systems [1–6] that fast decoherence results from the interaction of the system with its environment when the system is suitably "macroscopic." It is therefore much harder to generate, maintain, and detect coherence on a macroscopic scale than on a microscopic one. These results contribute to clarifying why the observation of coherent superpositions is not part of our every day experience and why building large-scale quantum computers is a major challenge.

To render the previous observations quantitative and general, several different characterizations have been proposed of the "coherence" [7,8], "large-scale quantum coherence" [9–11], "macroscopic coherence" [10,12], "quantum macroscopicity" [10,13,14], "macroscopic quantumness" [10,15–18] and "macroscopic distinctness" [13,18,19] of quantum states. Resource theories for those closely related properties of states have also been developed [7,8,10,12,20]. An important and to the best of our knowledge unaddressed question concerning these quantities is to evaluate the rate at which they decrease when the system is coupled to its environment, i.e., to evaluate their sensitivity to environmental decoherence.

We will address the latter question for the states of a bosonic quantum field for which we introduce the quadrature coherence scale (QCS), defined as a measure of the scale on which the coherences of its quadratures are appreciable [see (4)]. As we will show, a small QCS means the coherences *for all* quadratures are small far from the diagonal. We call such states *quadrature quasi incoherent*. A large QCS means on the contrary that, given any pair of conjugate quadratures, at least one has appreciable coherences far from the diagonal. In addition, it implies the state is strongly optically nonclassical.

We show the QCS of a state, if initially large, decreases fast when the system is coupled to an environment. The corresponding characteristic timescale is inversely proportional to the (square of the) QCS itself. Purity loss takes place on a similar timescale. Therefore, the states very sensitive to environmental decoherence are precisely those with a large QCS. This result generalizes known results on the decoherence of optical cat states [2–6] to all mixed or pure states of the field mode. We further establish that states with a large QCS are hard to observe since their Wigner functions have small scale structures. The QCS thus furnishes a physical parameter that measures the "coherence size" of the state and that is directly related to the decoherence rate.

Coherence, interference and coherence scale.—To reveal the quantum nature of a state ρ, one may proceed as follows. Consider two noncommuting observables A and B that we suppose for simplicity to have associated orthonormal eigenbases $|a_i\rangle$, $|b_m\rangle$. Let $p_A(a_i) = \langle a_i|\rho|a_i\rangle$ and $p_B(b_m) = \langle b_m|\rho|b_m\rangle$. Then

$$p_B(b_m) = p_B^{\mathrm{diag}}(b_m) + \sum_{i\neq j} \langle b_m|a_i\rangle\langle a_j|b_m\rangle\langle a_i|\rho|a_j\rangle. \qquad (1)$$

Here, $p_B^{\mathrm{diag}}(b_m) = \sum_i |\langle b_m|a_i\rangle|^2 p_A(a_i)$ is of "classical" nature in the sense that it is the term expected from an application of classical probability theory. The second term—the interference term—is of a typical quantum nature. It is absent when ρ has no off-diagonal matrix elements $\langle a_i|\rho|a_j\rangle$, the so-called "coherences." For an overview of measures and monotones of coherence for observables with discrete spectrum, we refer to [8], and references therein. If ρ is diagonal in the A basis (ρ is then said to be A incoherent) or if its coherences in this basis are small, the quantum nature of ρ is not revealed in this manner and one can then say that, in this restricted sense, the state "behaves classically." The state's quantum nature may, however, still be revealed by another choice of observables. In this view, there is no such thing as "the" nonclassical nature of a state, but rather the degree to which various measurements can reveal its quantum nature, notably through interference terms. The relation of this analysis to the independent notion of optical (non)classicality of the states of a bosonic quantum field mode will be discussed below.

As will become clear below, for our purposes, it is not so much the size of the coherences as their location which is important. To evaluate how far from the diagonal the coherences occur, we write $\mathcal{P} = \mathrm{Tr}\rho^2$ for the purity, and consider the probability density on the (a, a') plane:

$$\mu(a, a') = \sum_{i,j} \frac{|\langle a_i|\rho|a_j\rangle|^2}{\mathcal{P}} \delta(a - a_i)\delta(a' - a_j).$$

It describes the spatial repartition of the matrix elements of ρ and in particular of its coherences. We define the A-coherence scale $\mathcal{C}_A(\rho)$ of ρ via

$$\begin{aligned}\mathcal{C}_A^2(\rho) &= \sum_{i,j} (a_i - a_j)^2 \frac{|\langle a_i|\rho|a_j\rangle|^2}{\mathcal{P}} \\ &= \int (a - a')^2 \mu(a, a') da da',\end{aligned}$$

which is the variance of the eigenvalue spacings of A so that, when $\mathcal{C}_A(\rho)$ is large, there are coherences far from the diagonal: $\mathcal{C}_A(\rho)$ determines the scale on which the coherences of ρ live. It is easy to check that for pure states $\mathcal{C}_A^2(\rho) = 2(\Delta A)^2$. A simple calculation shows, furthermore, $\mathcal{C}_A^2(\rho) = \mathcal{P}^{-1}\mathrm{Tr}[\rho, A][A, \rho]$, an expression valid also when A has continuous spectrum. We stress that $\mathcal{C}_A$ is *not* a measure of the A coherence of the state; it does not establish "how much" coherence there is, but "where" it is. For example, two states $|a_i\rangle + |a_j\rangle$ and $|a_i\rangle + |a_k\rangle$ have off-diagonal matrix elements of the same size, and in this sense the same "amount" of coherence, but their A-coherence scale is proportional to $|a_i - a_j|$, respectively $|a_i - a_k|$, and can therefore strongly differ [10].

Quadrature coherence scale-quadrature quasicoherence.— We consider a state ρ of a single-mode field, characterized by an annihilation-creation operator pair a, $a^\dagger$. We are interested in the coherence scale of its quadratures and define, in analogy with what precedes, its quadrature coherence scale (QCS) $\mathcal{C}(\rho)$ through

$$\mathcal{C}^2(\rho) = \frac{1}{2\mathcal{P}}(\mathrm{Tr}[\rho, X][X, \rho] + \mathrm{Tr}[\rho, P][P, \rho]), \tag{2}$$

where $X = [(a^\dagger + a)/\sqrt{2}], P = [i(a^\dagger - a)/\sqrt{2}]$. With $X_\theta = \cos\theta X + \sin\theta P$, $P_\theta = -\sin\theta X + \cos\theta P$, one has also

$$\mathcal{C}^2(\rho) = \frac{1}{2\mathcal{P}}(\mathrm{Tr}[\rho, X_\theta][X_\theta, \rho] + \mathrm{Tr}[\rho, P_\theta][P_\theta, \rho]), \tag{3}$$

so that $\mathcal{C}^2(\rho)$ is the average coherence scale (squared) of any pair of conjugate quadratures. Equation (2) implies

$$\begin{aligned}\mathcal{C}^2(\rho) = \frac{1}{2\mathcal{P}}\bigg(&\int (x - x')^2 |\rho(x, x')|^2 dx dx' \\ &+ \int (p - p')^2 |\rho(p, p')|^2 dp dp'\bigg).\end{aligned} \tag{4}$$

Here, $\rho(x, x')$ [respectively, $\rho(p, p')$] is the operator kernel of ρ in the X representation (respectively, P representation). It follows from (3) and (4) that a large $\mathcal{C}(\rho)$ implies that for every pair (X_θ, P_θ) of conjugate quadratures, at least one has a large coherence scale. Conversely, a small $\mathcal{C}(\rho)$ implies that the off-diagonal coherences of all quadratures must be small away from the diagonal. We stress that no state ρ can be X_θ or P_θ incoherent in the sense that ρ cannot be diagonal in the corresponding representation [21]. Mixed states can nevertheless have an arbitrarily small QCS, as we will see below. For pure states, (2) implies $\mathcal{C}^2(\rho) = (\Delta X)^2 + (\Delta P)^2$, the so-called total noise of ρ [27]. It follows that, on pure states, the QCS is larger than 1; it reaches its minimal value of 1 only on the coherent states $|\alpha\rangle = D(\alpha)|0\rangle$, where $|0\rangle$ is the vacuum state and $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$. For optical cat states $|\psi_\alpha\rangle \sim (|\alpha\rangle + |-\alpha\rangle)$, a simple computation [21] yields $\mathcal{C}_\alpha \simeq |\alpha|$ $(|\alpha| \gg 1)$: "large" cats have a large QCS in agreement with the observation that $\rho_\alpha(x, x') = \langle x|\psi_\alpha\rangle\langle\psi_\alpha|x'\rangle$ has large off-diagonal elements in the neighborhood of $x = -x' = \pm\alpha$ if α is real. We will refer to states for which $\mathcal{C}(\rho) \leq 1$ as quadrature quasi-incoherent states. In fact, we will see below that $\mathcal{C}^2$ has the particular feature of providing a measure of optical nonclassicality. It follows from [28,29] that the right hand side of (2) can be expressed in terms of the Wigner function $W(\alpha)$ or the characteristic function $\chi(\xi)$ [30,31] as follows:

$$\mathcal{C}^2(\rho) = \frac{\||\xi|\chi\|_2^2}{\|\chi\|_2^2} = \frac{1}{4}\frac{\|\nabla W\|_2^2}{\|W\|_2^2}. \tag{5}$$

Here, with $\xi, \alpha \in C$, $\|\cdot\|_2$ stands for the L^2 norm, meaning, for example, $\|W\|_2^2 := \int |W|^2(\alpha) d^2\alpha$ and

$$\begin{aligned}\chi(\xi) &= \mathrm{Tr}\rho D(\xi), \\ W(\alpha) &= \frac{1}{\pi^2}\int \chi(\xi) \exp(\xi^*\alpha - \xi\alpha^*) d^2\xi.\end{aligned}$$

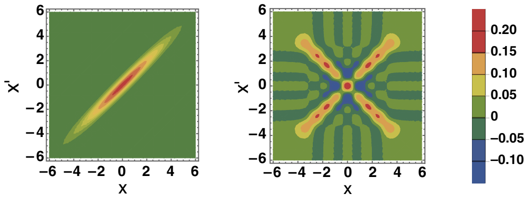


FIG. 1. Plots of $\rho(x, x')$. Left panel: thermal state with $\bar{n} = 5$. Right panel: even state ρ_M, with $M = 4$, $\bar{n} = 5$.

The definition (2) and expression (5) carry over to multimode systems by summing over a complete set of conjugate quadratures.

For a centered Gaussian state ρ_G with covariance matrix $V = \begin{pmatrix} 2\mathrm{Tr}\rho X^2 & \mathrm{Tr}\rho(XP+PX) \\ \mathrm{Tr}\rho(XP+PX) & 2\mathrm{Tr}\rho P^2 \end{pmatrix}$, one finds [21]:

$$\mathcal{C}_G^2 = \mathcal{C}^2(\rho_\mathrm{G}) = [(\Delta X)^2 + (\Delta P)^2]\mathcal{P}^2 = \frac{1}{2}\mathrm{Tr}V^{-1}. \qquad (6)$$

It follows that Gaussian mixed states can have an arbitrarily small QCS. This can happen even if the total noise is very large. One notes for example in Fig. 1 that the coherences of the thermal state with mean photon number $\bar{n} = 5$ are concentrated along the diagonal. This reflects the fact that for thermal states $\mathcal{C}(\rho_\mathrm{th}) = (1 + 2\bar{n})^{-1/2}$, which follows from (6). We note that for Gaussian states $4\mathcal{C}_G^2$ coincides with the sum of the quantum Fisher information of two conjugate quadratures [21], which is known to provide a useful lower bound for proposed measures and monotones of quantum macroscopicity [10] and nonclassicality [32]. On non-Gaussian states, however, the two quantities can differ greatly (for an example, see [29]).

As an example of non-Gaussian states we consider the family of even states, with M a positive integer:

$$\rho_M = \frac{1}{M}\sum_{k=1}^{M} |2k\rangle\langle 2k|.$$

One has $\mathcal{C}(\rho_M) = \sqrt{2M+3}$ [29] and Fig. 1 shows that, indeed, the coherences have a large off-diagonal branch that can be checked to grow as $\sqrt{2M}$, as expected. Since $\mathcal{P}_M = M^{-1}$, this shows that very strongly mixed states can have a very large QCS. Other examples of this phenomenon are the strongly squeezed thermal states for which a very small purity can be compensated by a very large total noise [see (6) and [21]].

Environment induced quadrature coherence scale loss.—We consider a field weakly coupled to a thermal bath through the standard master equation in Lindblad form [1,3,6,33,34]

$$\frac{d}{dt}\rho(t) = -i\omega[a^\dagger a, \rho(t)] + \frac{1}{2}\gamma\{[a\rho(t), a^\dagger] + [a, \rho(t)a^\dagger]\} + \frac{1}{2}\delta\{[a^\dagger\rho(t), a] + [a^\dagger, \rho(t)a]\},$$

where $\gamma > \delta \geq 0$. This dynamics converges to a thermal state with mean photon number $\bar{n}_\infty = \delta t_R$, where $t_R = (\gamma - \delta)^{-1}$ is the relaxation time. Purity evolution is determined by $\dot{\mathcal{P}}(t) = (1/t_R)[1 - (2\bar{n}_\infty + 1)\mathcal{C}^2(t)]\mathcal{P}(t)$. Using the affine approximation to $\mathcal{P}(t)$ at small t shows the purity half time

$$\tau_\mathcal{P} \approx \frac{1}{2}\frac{1}{(2\bar{n}_\infty + 1)\mathcal{C}_0^2 - 1}t_R,$$

provided $\mathcal{C}_0^2 = \mathcal{C}^2(0) > 1$: the purity half-life decreases as $\mathcal{C}_0^{-2}$ when the QCS is large. This approximation gives the right order of magnitude [21] and reduces to the known result for pure states [2,3,5]. Simultaneously with the purity loss, there is QCS loss. Indeed, the time evolution of the QCS, and in particular its sharp initial drop (Fig. 2), can be explained by analyzing the differential equation for $\mathcal{C}(t)$ [21]:

$$\dot{\mathcal{C}}(t) = \frac{1}{2t_R}[1 - \kappa(t)(2\bar{n}_\infty + 1)\mathcal{C}^2(t)]\mathcal{C}(t), \qquad (7)$$

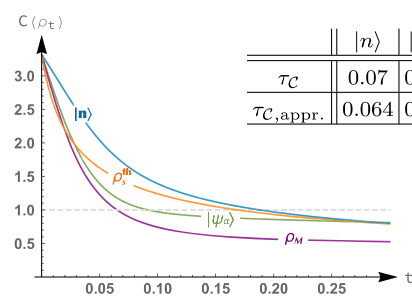
with

$$\kappa(t) = [(\langle\!\langle \xi^4 \rangle\!\rangle_t / \langle\!\langle \xi^2 \rangle\!\rangle_t^2) - 1],$$
$$\langle\!\langle \xi^{2k} \rangle\!\rangle = \int |\xi|^{2k}(|\chi(\xi)|^2/\|\chi\|_2^2)dx$$

Hence, the half-life $\tau_\mathcal{C}$ of the QCS is given approximately by (Fig. 2)

$$\tau_\mathcal{C} \approx -\frac{1}{2}\frac{\mathcal{C}(0)}{\dot{\mathcal{C}}(0)} = \frac{1}{\kappa_0(2\bar{n}_\infty + 1)\mathcal{C}_0^2 - 1}t_R. \qquad (8)$$

where $\kappa_0 = \kappa(0)$. For Gaussian states, more precise estimates can be obtained from a more detailed computation [21]:



	$\lvert n\rangle$	$\lvert\psi_\alpha\rangle$	ρ_M	ρ_s^{th}
$\tau_\mathcal{C}$	0.07	0.038	0.033	0.047
$\tau_{\mathcal{C},\mathrm{appr.}}$	0.064	0.032	0.026	0.048

FIG. 2. Evolution of $\mathcal{C}(\rho_t)$ under the dynamics (7) of an initial Fock state $|n\rangle$ ($n = 5$), a squeezed thermal state [$V = 1.8\begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix}$, $r = \cosh^{-1}(19.8)/2 \approx 1.84$], an optical cat state $|\psi_\alpha\rangle \sim (|\alpha\rangle + |-\alpha\rangle)$ ($\alpha \approx 2.24$), and an even state ρ_M ($M = 4$). $\mathcal{C}(0) = \sqrt{11}$ for all states shown. $\bar{n}_\infty = 1$. $t_R = 1$. The table show the numerically exact value of the half-life $\tau_\mathcal{C}$ of the QCS and its approximation obtained by Eq. (8) for the first three columns and by Eq. (9) for the Gaussian state.

$$\tau_{\mathcal{P},G} \approx \frac{2^{\kappa_0}-1}{[(2\bar{n}_\infty+1)\mathcal{C}_0^2-2^{\kappa_0}]\kappa_0} t_R,$$
$$\tau_{\mathcal{C},G} \approx \frac{3}{\kappa_0(2\bar{n}_\infty+1)\mathcal{C}_0^2-4} t_R. \tag{9}$$

Comparing (8) to (9), one sees that for the same value of $\mathcal{C}_0 \gg 1$ and κ_0, a Gaussian state is less sensitive to decoherence than the non-Gaussian states considered above. A further calculation [21] permits us to determine the time $\tau_{1,G}$ at which the state becomes quasi-incoherent, i.e., $\mathcal{C}(\tau_{1,G}) = 1$. It is, remarkably, to leading order in $\mathcal{C}_0^{-2}$, independent of the QCS:

$$\tau_{1,G} \approx \left[\ln\left(\frac{\kappa_0(2\bar{n}_\infty+1)}{\kappa_0(2\bar{n}_\infty+1)-1}\right) - \frac{1}{\mathcal{C}_0^2\kappa_0(2\bar{n}_\infty+1)}\right] t_R.$$

These results show in all generality that the purity loss and the destruction of the large-scale quadrature coherences of any initial state are determined by the temperature of the environment and by two parameters characteristic of the initial state: the QCS $\mathcal{C}_0$ and κ_0. They generalize the known results for optical cat states [2–5] to all pure and mixed states.

Effects of a large QCS.—The expressions in (5) show that a large value of the QCS corresponds to a large spread of the characteristic function and to the existence of small scale structures in the Wigner function [28,29]. Indeed, $\||\xi|\chi\|_2^2/\|\chi\|_2^2$ is the mean of $|\xi|^2$ with respect to the probability density $|\chi(\xi)|^2/\|\chi\|_2^2$. Hence, a large value of the QCS corresponds to a characteristic function with a wide spread in at least some directions in the ξ plane, a manifestation of the well known link between the characteristic function and the coherences [35]:

$$\chi\left(-\frac{\mu\sin\theta}{\sqrt{2}}, \frac{\mu\cos\theta}{\sqrt{2}}\right) = \mathrm{Tr}\exp(i\mu X_\theta)$$
$$= \int \rho(p_\theta, p_\theta+\mu)dp_\theta.$$

On the other hand, a large QCS implies the gradient of W is large, which means the graph of W must have steep slopes, at least in some places of the phase plane, a signature either of oscillations or of sharp peaks [15,29]. For Gaussian states, this phenomenon manifests itself in that the variance of the probability distribution of one of the quadratures is of order $\mathcal{C}^{-2}$ [21]. A faithful reconstruction of the Wigner function through quantum tomography therefore requires great accuracy when $\mathcal{C} \gg 1$. States with a large QCS are therefore hard to observe. That it is generally difficult to measure optical cat states and analogous states in other systems, when their components have a "macroscopic" separation, was proven in [19]. We have here established the same result for all mixed or pure states of a bosonic quantum field with a large QCS.

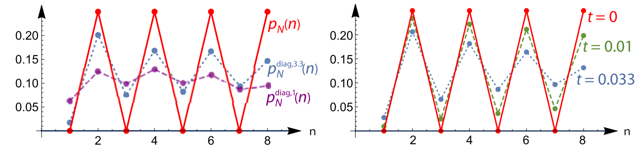


FIG. 3. Full red line: values of $p_N(n) = 1/M$ for the even state ρ_M with $M = 4$ at $t = 0$. Left panel: values of $p_N^{\mathrm{diag},3.3}$ (dotted blue line) and $p_N^{\mathrm{diag},1}$ (dashed purple line) as defined in (10), both at $t = 0$. Right panel: values of $p_N(n)$ at $t = 0.01$ (dashed green line) and $t = 0.033 = \tau_{\mathcal{C}}$ (dotted blue line).

To see how a large coherence scale can lead to strong interference effects, we consider the states ρ_M (Fig. 1) and choose $A = X$ and $B = N = a^\dagger a$ and write

$$p_N(n) = p_N^{\mathrm{diag},\ell}(n) + \int_{|x'-x|\geq\ell} \langle x'|n\rangle\langle n|x\rangle\rho(x,x')dxdx',$$

in analogy with (1). Here,

$$p_N^{\mathrm{diag},\ell}(n) = \int_{|x'-x|\leq\ell} \langle x'|n\rangle\langle n|x\rangle\rho(x,x')dxdx'. \tag{10}$$

Contrary to when A has a discrete spectrum, as in (1), one cannot sharply isolate the diagonal part of the state. Nevertheless, as the left panel of Fig. 3 illustrates, it is the contribution of the coherences far from the diagonal that generate the sharp oscillations or fringes in $p_N(n)$. In fact, it is clear (see Fig. 3) that the term $p_N^{\mathrm{diag},\ell}(n)$ shows a mildly oscillating behavior for $\ell = 1$, which is, as ℓ grows, enhanced by the interference terms to yield $p_N(2k) = 1/M$ (constructive interference), $p_N(2k+1) = 0$ (destructive interference). That the dynamical loss of large-scale coherences leads to a sharp decrease of this interference effect is illustrated in the right panel of Fig. 3: at the QCS half life $\tau_{\mathcal{C}} = 0.033$ of the state, the interferences are already considerably suppressed.

Quadrature coherence scale and optical (non) classicality.—Let C_{cl} be the set of optical classical states, i.e., all mixtures of coherent states [36]. A number of witnesses, measures, and monotones of optical nonclassicality have been designed [29,32,36–59] to identify non-optical classical states and to quantify their degree of nonoptical classicality. Those quantities are often hard to compute, to measure, or to give a clear physical meaning. It is, in particular, not evident how they relate to standard manifestations of specifically quantum behavior such as coherence and interference, nor how they evolve when coupled to a thermal bath. We show here a quantitative link between optical (non)classicality, the presence of coherences, and (fast) decoherence. Our analysis is based on the optical nonclassicality distance $d(\rho, C_{\mathrm{cl}})$ defined in [29] using a quantity denoted $S_o(\rho)$ which measures the sensitivity of the state to operator ordering. One of its expression is $S_o(\rho) = \frac{1}{4}(\|\nabla W\|_2^2/\|W\|_2^2)$. In view of (5), this means $\mathcal{C}^2(\rho) = S_o(\rho)$. In other words, the QCS provides a new physical interpretation of the ordering sensitivity in terms of quadrature coherences, and the associated physical phenomena described above. In view of the bound

$$\mathcal{C}(\rho) - 1 \leq d(\rho, C_{\mathrm{cl}}) \leq \mathcal{C}(\rho) \tag{11}$$

proven in [29], $\mathcal{C}(\rho)$ is a good estimate of the distance between ρ and the optical classical states when $\mathcal{C}(\rho) \gg 1$. Hence, the states far from the optical classical states are those with quadrature coherences far from the diagonal. In view of what precedes, they are the most fragile to decoherence. Conversely, when $\rho \in C_{\mathrm{cl}}$, $d(\rho, C_{\mathrm{cl}}) = 0$ and it follows from (11) that $\mathcal{C}(\rho) \leq 1$: optical classical states are quadrature quasi-incoherent. Finally, the smaller the QCS of ρ, the closer it is to the optical classical states. This link between coherence and optical nonclassicality is specific to the quadrature coherences. It is, for example, not present in the $a^\dagger a$ coherence of the state.

Conclusion.—We introduced, for any bosonic state, its QCS, a measure of how far from the diagonal its quadrature coherences lie. We established that the states with a large QCS are strongly optically nonclassical, hard to observe, and very sensitive to environmental decoherence. These results generalize the known fast decoherence of large optical cat states [2–6] to all pure or mixed states with a large QCS.

One may thus legitimately argue that the QCS provides a measure of quantum macroscopicity. Indeed, when the QCS is large, the state is "strongly nonclassical" in the sense that it is far from the optical classical states and its far off-diagonal coherences can be understood as a form of "macroscopicity." Also, when the QCS is small, the states are close to the optical classical states and in this sense have a low degree of "quanticity." Our results thus strongly support a suggestion in [32], were it is surmised that there may be a link between optical nonclassicality and macroscopic quantum effects.

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