Stellar Representation of Non-Gaussian Quantum States

Ulysse Chabaud,[1,*] Damian Markham,[1] and Frédéric Grosshans[2,1]
[1]*Laboratoire d'Informatique de Paris 6, CNRS, Sorbonne Université, 4 place Jussieu, 75005 Paris, France*
[2]*Laboratoire Aimé Cotton, CNRS, Université Paris-Sud, ENS Cachan, Université Paris-Saclay, 91405 Orsay Cedex*

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The so-called stellar formalism allows us to represent the non-Gaussian properties of single-mode quantum states by the distribution of the zeros of their Husimi Q function in phase space. We use this representation in order to derive an infinite hierarchy of single-mode states based on the number of zeros of the Husimi Q function: the stellar hierarchy. We give an operational characterization of the states in this hierarchy with the minimal number of single-photon additions needed to engineer them, and derive equivalence classes under Gaussian unitary operations. We study in detail the topological properties of this hierarchy with respect to the trace norm, and discuss implications for non-Gaussian state engineering, and continuous variable quantum computing.

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Introduction.—Quantum information processing takes advantage of nonclassical phenomena, such as superposition and entanglement, to provide applications beyond what classical information processing may offer [1,2]. Quantum information may be encoded in physical systems using either discrete variables, e.g., the polarization of a photon, or continuous variables, e.g., quadratures of the electromagnetic field. Continuous variable quantum information processing [3] represents a powerful alternative to its discrete variable counterpart, as deterministic generation of highly entangled states [4,5] and high efficiency measurement are readily available with current technologies.

In continuous variable quantum information, quantum states are described mathematically by vectors in a separable Hilbert space of infinite dimension. Alternatively, phase-space formalism describes quantum states conveniently using generalized quasiprobability distributions [6], among which are the Husimi Q function, the Wigner W function, and the Glauber-Sudarshan P function, which is always singular. The states that have a Gaussian Wigner or Husimi function are called Gaussian states, while all the other states are called non-Gaussian. By extension, the operations mapping Gaussian states to Gaussian states are called Gaussian operations, and measurements projecting onto Gaussian states are called Gaussian measurements. Gaussian states and processes feature an elegant mathematical description with the symplectic formalism, and are useful for a wide variety of quantum information protocols [7–10]. However, Gaussian computations, composed of input Gaussian states, Gaussian operations, and Gaussian measurements, are easy to simulate classically [11]. On the other hand, non-Gaussian states are needed, and are actually useful for achieving universal qubit quantum computing with continuous variables [12,13], and they are crucial for many other quantum information tasks [14–22]. Characterizing and understanding the properties of these states is thus of major importance [23–26].

Hudson [27] has notably shown that a single-mode pure quantum state is non-Gaussian if and only if its Wigner function has negative values, and this result has been generalized to multimode states by Soto and Claverie [28]. This characterization is an interesting starting point for studying non-Gaussian states. From this result, one can introduce measures of a state being non-Gaussian using Wigner negativity, e.g., the negative volume [29], that are invariant under Gaussian operations. However, computing these quantities from experimental data is complicated in practice. Other measures and witnesses for non-Gaussian states have been derived [30–33], which make it possible to discriminate non-Gaussian states from mixtures of Gaussian states from experimental data.

The Husimi function, which is a smoothed version of the Wigner function, also characterizes non-Gaussian states: for pure states, the Husimi function having zeros is actually equivalent to the Wigner function having negative values, as shown by Lütkenhaus and Barnett [34]. Informally,

Theorem 1.—A pure quantum state is non-Gaussian if and only if its Husimi Q function has zeros.

An interesting point is that for single-mode states, the zeros of the Husimi Q function form a discrete set, as we will show later on. The non-Gaussian properties of single-mode states may thus be described by the distribution of these zeros in phase space.

Based on this result, we introduce in this Letter an infinite hierarchy of states, which we call stellar hierarchy, which allows us to characterize single-mode continuous variable pure quantum states with respect to their non-Gaussian properties. We make use of the so-called stellar representation, or Segal-Bargmann formalism [35,36], in order to derive this hierarchy. We give a brief introduction to this formalism in what follows, and we review and prove additional relevant properties. We then define the stellar rank of a state, which induces the stellar hierarchy, and we characterize the set of states of each rank. In particular, we show that each rank is left invariant under Gaussian operations. At rank zero lie Gaussian states, while non-Gaussian states populate all higher ranks. We show that the stellar rank of a state is equivalent to the minimal number of photon additions necessary to engineer the state. We then use this hierarchy to study analytically Gaussian convertibility of states, and we derive equivalence classes under this relation. We study the topology of the stellar hierarchy, with respect to the trace norm, and show that it is robust. We show that the stellar hierarchy matches the hierarchy of genuine n-photon quantum non-Gaussian light introduced in Ref. [37], and we discuss implications of our results for non-Gaussian quantum state engineering, and continuous variable quantum computing.

The stellar function.—The so-called stellar representation, or Segal-Bargmann representation [35,36], has been used to study quantum chaos [38–41], and the completeness of sequences of coherent states [42–44]. We give hereafter an introduction to this formalism. Further details may be found, e.g., in Ref. [45].

Let $\mathcal{H}_\infty$ be the infinite-dimensional Hilbert space of single-mode pure quantum states. In the following, we consider normalized states, and we denote by $\{|n\rangle\}_{n\in\mathbb{N}}$ the Fock basis of $\mathcal{H}_\infty$. We introduce below the stellar function. This function has been recently studied, in the context of non-Gaussian quantum state engineering [46], in order to simplify calculations related to photon-subtracted Gaussian states.

Definition 1.—Let $|\psi\rangle = \sum_{n\geq 0}\psi_n|n\rangle \in \mathcal{H}_\infty$ be a normalized state. The stellar function of the state $|\psi\rangle$ is defined as

$$F^\star_\psi(\alpha) = e^{|\alpha|^2/2}\langle\alpha^*|\psi\rangle = \sum_{n\geq 0}\psi_n\frac{\alpha^n}{\sqrt{n!}}, \qquad (1)$$

for all $\alpha\in\mathbb{C}$, where $|\alpha\rangle = e^{-(1/2)|\alpha|^2}\sum_{n\geq 0}(\alpha^n/\sqrt{n!})|n\rangle \in \mathcal{H}_\infty$ is the coherent state of amplitude α.

The stellar function is a holomorphic function over the complex plane, which provides an analytic representation of a quantum state. For any state $|\psi\rangle\in\mathcal{H}_\infty$, we may write

$$|\psi\rangle = \sum_{n\geq 0}\psi_n|n\rangle = F^\star_\psi(\hat{a}^\dagger)|0\rangle, \qquad (2)$$

using the definition of the stellar function. An important result is that the stellar representation is unique, up to a global phase:

Lemma 1.—Let $|\phi\rangle$ and $|\psi\rangle$ be normalized single-mode pure states such that $F^\star_\phi = F^\star_\psi$, up to a phase. Then $|\phi\rangle = |\psi\rangle$. Moreover, let $|\chi\rangle = f(\hat{a}^\dagger)|0\rangle$ be a normalized single-mode pure state, where f is analytical. Then $f = F^\star_\chi$ up to a phase.

These results follow directly from Eq. (2), as detailed in the Supplemental Material [47].

The stellar function of a state $|\psi\rangle\in\mathcal{H}_\infty$ is related to its Husimi Q function, a smoothed version of the Wigner function [6], given by

$$Q_\psi(\alpha) = \frac{1}{\pi}|\langle\alpha|\psi\rangle|^2 = \frac{e^{-|\alpha|^2}}{\pi}|F^\star_\psi(\alpha^*)|^2, \qquad (3)$$

for all $\alpha\in\mathbb{C}$. The zeros of the Husimi Q function are the complex conjugates of the zeros of $F^\star_\psi$. Hence, by Theorem 1, a single-mode pure quantum state is non-Gaussian if and only if its stellar function has zeros. These zeros form a discrete set, as the stellar function is a nonzero analytical function. The non-Gaussian properties of a single-mode pure state are then described by the distribution of the zeros over the complex plane. Using antistereographic projection [48], this amounts to describing the non-Gaussian properties of a pure state with a set of points on the sphere, hence the name stellar representation, where the points on the sphere looked at from the center of the sphere are seen as stars on the celestial vault [40,49].

The stellar rank.—The Hilbert space $\mathcal{H}_\infty$ is naturally partitioned into classes of states having the same number of zeros. We introduce the following related definition:

Definition 2.—The stellar rank $r^\star(\psi)$ of a normalized single-mode pure quantum state $|\psi\rangle\in\mathcal{H}_\infty$ is defined as the number of zeros of its stellar function $F^\star_\psi$, counted with multiplicity.

We introduce hereafter the notation $\bar{\mathbb{N}} = \mathbb{N}\cup\{+\infty\}$, so that $r^\star(\psi)\in\bar{\mathbb{N}}$. For $N\in\bar{\mathbb{N}}$, we define

$$R_N = \{|\psi\rangle\in\mathcal{H}_\infty, r^\star(\psi) = N\} \qquad (4)$$

as the set of states with stellar rank equal to N. The stellar hierarchy is the hierarchy of states induced by the stellar rank. By Lemma 1, if $M\neq N$ then $R_M\cap R_N = \varnothing$, for all $M, N\in\bar{\mathbb{N}}$, so all the ranks in the stellar hierarchy are disjoint. We have $\mathcal{H}_\infty = \bigcup_{N\in\bar{\mathbb{N}}}R_N$, i.e., the stellar hierarchy covers the whole space of normalized states, and the set of states of finite stellar rank is given by $\bigcup_{N\in\mathbb{N}}R_N$. By Theorem 1, the rank zero of the stellar hierarchy R_0 is the set of normalized single-mode pure Gaussian states. For all $N\in\mathbb{N}$ the photon number state $|N\rangle$ is of stellar rank N, since $F^\star_{|N\rangle}(\alpha) = \alpha^N/\sqrt{N!}$, while the cat state $|\mathrm{cat}\rangle \propto (|ix\rangle - |-ix\rangle)$ is of infinite stellar rank, since $F^\star_{|\mathrm{cat}\rangle}(\alpha)\propto \sin(\alpha x)$, so all ranks are nonempty.

By analogy with the Schmidt rank in entanglement theory [50], we define the stellar rank of a mixed state ρ as $r^\star(\rho) = \inf_{p_i,\psi_i}\sup r^\star(\psi_i)$, where the infimum is over the statistical ensembles such that $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$.

In the following, we investigate further the properties of the stellar hierarchy. We prove a first general decomposition result for pure states of finite stellar rank:

Theorem 2.—Let $|\psi\rangle \in \bigcup_{N\in\mathbb{N}} R_N$ be a pure state of finite stellar rank. Let $\{\beta_1, \ldots, \beta_{r^\star(\psi)}\}$ be the roots of the Husimi Q function of $|\psi\rangle$, counted with multiplicity. Then,

$$|\psi\rangle = \frac{1}{\mathcal{N}}\left[\prod_{n=1}^{r^\star(\psi)} \hat{D}(\beta_n)\hat{a}^\dagger \hat{D}^\dagger(\beta_n)\right]|G_\psi\rangle, \qquad (5)$$

where $\hat{D}(\beta)$ is a displacement operator, $|G_\psi\rangle$ is a Gaussian state, and $\mathcal{N}$ is a normalization constant. Moreover, this decomposition is unique up to reordering of the roots.

The proof of this statement, which combines Eq. (2) with the Hadamard-Weierstrass factorization theorem [51], is detailed in the Supplemental Material [47].

This decomposition implies that any state of finite stellar rank may be obtained from a Gaussian state by successive applications of the creation operator at different locations in phase space, given by the zeros of the Husimi Q function. Experimentally, this corresponds to the probabilistic non-Gaussian operation of single-photon addition [52–54]. Using this decomposition, we obtain the following property:

Theorem 3.—A unitary operation is Gaussian if and only if it leaves the stellar rank invariant.

This result follows directly from Theorem 2, and we refer to the Supplemental Material [47] for a formal proof.

An interesting consequence is that the number of single-photon additions in the decomposition of Theorem 2 is minimal. Indeed, if a quantum state is obtained from the vacuum by successive applications of Gaussian operations and single-photon additions, then its stellar rank is exactly the number of photon additions, because each single-photon addition increases by one its stellar rank (it adds a zero to the stellar function at zero), while each Gaussian operation leaves the stellar rank invariant by Theorem 3. Hence, the stellar rank is a measure of the non-Gaussian properties of a quantum state that may be interpreted as a minimal non-Gaussian operational cost, in terms of single-photon additions, for engineering the state from the vacuum.

Gaussian convertibility.—Now that the first properties of the stellar hierarchy are laid out, we consider as an application the convertibility of quantum states using Gaussian unitary operations:

Definition 3.—Two states $|\phi\rangle$ and $|\psi\rangle$ are Gaussian-convertible if there exists a Gaussian unitary operation $\hat{G}$ such that $|\psi\rangle = \hat{G}|\phi\rangle$.

Note that this notion is different from the more restrictive notion of Gaussian conversion introduced in Ref. [55], which denotes the conversion of Gaussian states with passive linear optics, and a subclass of Gaussian measurements and feed forward.

Gaussian convertibility defines an equivalence relation in $\mathcal{H}_\infty$. By Theorem 3, having the same stellar rank is a necessary condition for Gaussian convertibility. However, this condition is not sufficient. In order to derive the equivalence classes for Gaussian convertibility, we introduce the following definition:

Definition 4.—Core states are defined as the normalized single-mode pure quantum states which have a polynomial stellar function.

By Eq. (2) and Lemma 1, core states are the states with a bounded support over the Fock basis, i.e., finite superpositions of Fock states. These correspond to the minimal non-Gaussian core states introduced in Ref. [56], in the context of non-Gaussian state engineering. With this definition, we can state our result on Gaussian convertibility of states of finite stellar rank:

Theorem 4.—Let $|\psi\rangle \in \bigcup_{N\in\mathbb{N}} R_N$ be a state of finite stellar rank. Then, there exists a unique core state $|C_\psi\rangle$ such that $|\psi\rangle$ and $|C_\psi\rangle$ are Gaussian-convertible.

Moreover, by Theorem 2, $|\psi\rangle = P_\psi(\hat{a}^\dagger)|G_\psi\rangle$, where P_ψ is a polynomial of degree $r^\star(\psi)$ and $|G_\psi\rangle = \hat{S}(\xi)\hat{D}(\beta)|0\rangle$ is a Gaussian state, where $\hat{D}(\beta) = e^{\beta\hat{a}^\dagger - \beta^*\hat{a}}$ is a displacement operator, and $\hat{S}(\xi) = e^{1/2(\xi\hat{a}^2 - \xi^*\hat{a}^{\dagger 2})}$ is a squeezing operator, with $\xi = re^{i\theta}$. Then,

$$|\psi\rangle = \hat{S}(\xi)\hat{D}(\beta)|C_\psi\rangle = \hat{S}(\xi)\hat{D}(\beta)F^\star_{C_\psi}(\hat{a}^\dagger)|0\rangle, \qquad (6)$$

where the (polynomial) stellar function of $|C_\psi\rangle$ is given by

$$F^\star_{C_\psi}(\alpha) = P_\psi(c_r\alpha - s_r e^{i\theta}\partial_\alpha + c_r\beta^* - s_r e^{i\theta}\beta)\cdot 1, \qquad (7)$$

for all $\alpha \in \mathbb{C}$.

The proof of this result follows from combining Theorem 2 together with Lemma 1, and is detailed in the Supplemental Material [47].

This result has several important consequences. First, it implies a second general decomposition result, in addition to Theorem 2: by Eq. (6), any state of finite stellar rank can be uniquely decomposed as a finite superposition of equally displaced and equally squeezed number states. This shows that the stellar hierarchy matches the genuine n-photon hierarchy introduced in Ref. [37]: a pure state exhibits genuine n-photon quantum non-Gaussianity if and only if it has a stellar rank greater or equal to n. Formally, for all $N \in \mathbb{N}$, the set R_N of states of stellar rank equal to N is obtained by the free action of the group of single-mode Gaussian unitary operations $\mathcal{G}$ on the set of core states of stellar rank N, which is isomorphic to the set of normalized complex polynomials of degree N. Second, it also gives an analytical way to check if two states of finite stellar rank are Gaussian-convertible, given their stellar functions, by checking with Eq. (7) if they share the same core state. A simple example is given in the Supplemental Material [47], where it is shown using this criterion that single photon states and single photon-subtracted squeezed vacuum states are Gaussian-convertible. Third, it shows that two different core states are never Gaussian-convertible, while any state of finite stellar rank is always Gaussian-convertible to a unique core state. This implies that equivalence classes for Gaussian convertibility for states of finite stellar rank correspond to the orbits of core states under Gaussian operations.

Stellar robustness.—Having characterized the states of finite stellar rank, we study in the following the topology of the stellar hierarchy, with respect to the trace norm. In order to discuss the robustness of this hierarchy up to small deviation in trace distance, we introduce the following definition:

Definition 5.—Let $|\psi\rangle \in \mathcal{H}_\infty$. The stellar robustness of the state $|\psi\rangle$ is defined as

$$R^\star(\psi) = \inf_{r^\star(\phi)<r^\star(\psi)} D_1(\phi, \psi), \tag{8}$$

where D_1 denotes the trace distance, and where the infimum is over all states $|\phi\rangle \in \mathcal{H}_\infty$ such that $r^\star(\phi) < r^\star(\psi)$ (with the convention $N < +\infty \Leftrightarrow N \in \mathbb{N}$).

The stellar robustness quantifies how much one has to deviate from a quantum state in trace distance to find another quantum state of lower stellar rank. A similar notion is the quantum non-Gaussian depth [57], which quantifies the maximum attenuation applicable on a quantum state, after which quantum non-Gaussianity can still be witnessed. The stellar robustness inherits the property of invariance under Gaussian operations of the stellar rank, because the trace distance is invariant under unitary operations. It is related to the fidelity by the following result:

Lemma 2.—Let $|\psi\rangle \in \mathcal{H}_\infty$, then

$$\sup_{r^\star(\rho)<r^\star(\psi)} F(\rho, \psi) = 1 - [R^\star(\psi)]^2, \tag{9}$$

where F is the fidelity.

We give a proof in the Supplemental Material [47]. Certifying that a (mixed) state ρ has a fidelity greater than $1 - [R^\star(\psi)]^2$ with a given target pure state $|\psi\rangle$ thus ensures that the state ρ has stellar rank equal or greater that $r^\star(\psi)$.

We characterize hereafter the topology of the stellar hierarchy, with respect to the trace norm. Formally, this topology is summarized by the following result for states of finite stellar rank:

Theorem 5.—For all $N \in \mathbb{N}$,

$$\overline{R_N} = \bigcup_{0\le K\le N} R_K, \tag{10}$$

where $\overline{X}$ denotes the closure of X for the trace norm in the set of normalized states $\mathcal{H}_\infty$.

The proof of this result, given in the Supplemental Material [47], is quite technical, and obtained by showing double inclusion, by considering converging sequences of states and studying their limit.

This result implies that the set on the right-hand side, containing the states of stellar rank smaller than N, is a closed set in $\mathcal{H}_\infty$ for the trace norm. In particular, since all ranks of the stellar hierarchy are disjoint, for any state of finite rank N, there is no sequence of states of strictly lower rank converging to it, and this holds for all N. Each state of a given finite stellar rank is thus isolated from the lower stellar ranks, i.e., there is a ball around it in trace norm which only contains states of equal or higher stellar rank. On the other hand, with the other inclusion, no state of a given finite stellar rank is isolated from any higher stellar rank, i.e., one can always find a sequence of states of any higher rank converging to this state in trace norm. Hence, Theorem 5 implies that for all states $|\psi\rangle \in \bigcup_{N\in\mathbb{N}} R_N$, we have $R^\star(\psi) > 0$, i.e., states of finite stellar rank are robust. We show in the Supplemental Material [47] that the robustness of a single photon-added squeezed state is given by $[1 - 3\sqrt{3}/(4e)]^{1/2} \approx 0.72$ as an example, and we reduce computing the robustness of any finite stellar rank state to a generic optimization problem.

For states of infinite stellar rank, we have the following result:

Lemma 3.—The set of states of finite stellar rank is dense for the trace norm in the set of normalized single-mode pure states:

$$\overline{\bigcup_{N\in\mathbb{N}} R_N} = \mathcal{H}_\infty, \tag{11}$$

where $\overline{X}$ denotes the closure of X for the trace norm in the set of normalized states $\mathcal{H}_\infty$.

This result is easily proven by considering the sequence of normalized truncated states for any given state in $\mathcal{H}_\infty$. We refer to the Supplemental Material [47] for details.

In particular, this means that states of infinite stellar rank are not isolated from lower stellar ranks, unlike states of finite stellar rank. Lemma 3 thus implies that for all states $|\psi\rangle \in R_\infty$ of infinite stellar rank, $R^\star(\psi) = 0$, i.e., states of infinite stellar rank are not robust.

Preparing a state $|\psi\rangle$ with precision better than $R^\star(\psi)$ ensures that the obtained state has rank greater or equal to $r^\star(\psi)$. For example, engineering a state that has a trace distance less than $[1 - (3\sqrt{3}/4e)]^{1/2} \approx 0.72$ with any single photon-added squeezed state implies that this state has a stellar greater or equal to 1. When considering imperfect single-mode non-Gaussian state engineering, one may thus restrict to states of finite stellar rank, which are obtained uniquely by a finite number of single-photon additions to a Gaussian state, by Theorem 2. In particular, cat states, being states of infinite stellar rank, can be approximated to arbitrary precision by finite rank states

[58]. Alternatively, one may also describe such states using Theorem 4 as finite superposition of displaced squeezed number states. Engineering of such states has recently been considered in Ref. [59], by photon detection of Gaussian states.

Smoothed non-Gaussianity of formation.—The topology of the stellar hierarchy obtained previously motivates the following definition:

Definition 6.—Let ρ be a normalized single-mode state, and let $\epsilon > 0$. The ϵ-smoothed non-Gaussianity of formation $\mathcal{NGF}_\epsilon(\rho)$ is defined as the minimal stellar rank of the states σ that are ϵ-close to ρ in trace distance. Formally,

$$\mathcal{NGF}_\epsilon(\rho) = \inf_\sigma \{r^\star(\sigma), \text{s.t.} D_1(\rho, \sigma) \leq \epsilon\}, \qquad (12)$$

where D_1 denotes the trace distance.

The infimum is also a minimum, since the set considered only contains integer values and is lower bounded by zero. That minimum is not necessarily attained for the energy cut-off state (consider, e.g., a Gaussian state). The smoothed non-Gaussianity of formation is a smoothed version of the stellar rank. By Theorem 2, it quantifies the minimal number of single-photon additions that need to be applied to a Gaussian state in order to obtain a state ϵ-close to a target state. As mentioned in the introduction, universal qubit quantum computing with continuous variables may be achieved using specific non-Gaussian resource states together with Gaussian operations and measurements [13]. On the other hand, the trace distance between two states provides a meaningful measure in the context of quantum computing, because a small trace distance ensures that any computation done with the states will yield similar results, with high probability [60]. In that context, the smoothed non-Gaussianity of formation provides an operational cost measure for non-Gaussian resource states, which is invariant under Gaussian operations.

Summary and discussion.—Based on the stellar representation of single-mode continuous variable quantum states, we have defined the stellar rank as the number of zeros of the stellar function, or, equivalently, of the Husimi Q function. Using the analytical properties of the stellar function, we have shown that this rank is invariant under Gaussian operations, and induces a hierarchy over the space of normalized single-mode states. We have characterized the states of finite stellar rank as the states obtained by successive single-photon additions to a Gaussian state, or, equivalently, as finite superpositions of (equally) displaced and squeezed states. Additionally, we have given the stellar rank an operational meaning, as the minimal non-Gaussian cost for engineering a state, in terms of single-photon additions. We have derived the equivalence classes for Gaussian convertibility using the notion of core states, and we have studied in detail the robustness of the ranks of the stellar hierarchy. Finally, we have introduced the smoothed non-Gaussianity of formation as a robust alternative to the stellar rank, in the context of approximate state engineering, and quantum computing with continuous variables.

The robustness of the genuine n-photon non-Gaussian hierarchy has been investigated numerically in Ref. [37]. We demonstrated analytically this robustness in this Letter, and provided an explicit method for computing the stellar robustness of any finite stellar rank state. This allows for the computation of the threshold required for successfully certifying nonzero stellar ranks. We expect that the robustness decreases with the rank. Thanks to the robustness of the stellar hierarchy, we have shown that the stellar rank can be experimentally witnessed by direct fidelity estimation with non-Gaussian target pure states. While the target state is pure, we emphasize that this certification method does not require the tested state itself to be pure. Deriving other simple experimentally observable conditions would also be interesting, for example, based on sampling from the Husimi Q function with heterodyne detection [61], given its relation with the stellar function. Another interesting perspective is to extend the stellar formalism to the case of multimode states. However, it is likely to be a challenging problem, as the stellar function for multimode states is a multivariate analytical function, which prevents the use of the factorization theorem, crucial in the derivation of our results.

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*ulysse.chabaud@gmail.com

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