Quantitative Impact of Integrals of Motion on the Eigenstate Thermalization Hypothesis

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Even though the eigenstate thermalization hypothesis (ETH) may be introduced as an extension of the random matrix theory, physical Hamiltonians and observables differ from random operators. One of the challenges is to embed local integrals of motion (LIOMs) within the ETH. Here we make steps towards a unified treatment of the ETH in integrable and nonintegrable models with translational invariance. Specifically, we focus on the impact of LIOMs on the fluctuations and structure of the diagonal matrix elements of local observables. We first show that nonvanishing fluctuations entail the presence of LIOMs. Then we introduce a generic protocol to construct observables, subtracted by their projections on LIOMs as well as products of LIOMs. The protocol systematically reduces fluctuations and/or the structure of the diagonal matrix elements. We verify our arguments by numerical results for integrable and nonintegrable models.

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Introduction.—Quantum simulators based on, e.g., quantum gases [1–5] provide an experimental platform to address a fundamental question of quantum mechanics, i.e., whether and how an initially nonequilibrium system reaches a thermal state [6–15]. The eigenstate thermalization hypothesis (ETH) is one of the main theoretical concepts that explains thermalization of local observables [16] in macroscopic quantum systems [10,17–21].

The ETH is most commonly expressed by the Srednicki ansatz [19] for the matrix elements of local observables $A_{nm} = \langle n|\hat{A}|m\rangle$ in the basis of eigenstates $\{|n\rangle\}$ of the Hamiltonian $\hat{H}$ (see, e.g., Refs. [10,11] for a review). The first part of the ansatz states that the diagonal matrix elements are smooth functions of energies, $A_{nn} \simeq A(E_n)$, and hence $A(E)$ coincides with the microcanonical average of $\hat{A}$ at energy E. The second part states that the off-diagonal matrix elements $A_{m\neq n}$ and the fluctuations of the diagonal elements $\delta A_{nn} = A_{nn} - A(E_n)$ (see Fig. 1 and Refs. [22–24]) decay exponentially with system size. Strong theoretical evidence supports validity of the ETH in various generic, nonintegrable models [20–48]. An important property of quantum systems that thermalize for any initial condition is that *all* eigenstates in the bulk of the spectrum satisfy the ETH [23,39,45].

Currently, a great scientific interest is devoted to gain deeper insight into nonergodic behavior of quantum many-body systems, along with the conditions for the ETH breakdown [11]. The most common evidence of the latter is slower-than-exponential decay of the fluctuations δA_{nn}. Indeed, this has been observed for a majority of eigenstates in clean integrable models [49–52] and strongly disordered finite systems [53–56], and for a vanishing fraction of eigenstates in some other classes of models [57–63]. This raises a question whether a modified version of the ETH, or a completely new theory, is necessary to describe properties of nonergodic quantum dynamics.

The goal of this Letter is to establish a unified framework of eigenstate thermalization that could be applied to both integrable and nonintegrable models with translational invariance. Previous works on integrable models in the context of the generalized ETH [50,51,64,65], the quench action approach [66–72], and the weak ETH [73–75], support the view that the ETH is violated due to the presence of local integrals of motion (LIOMs).

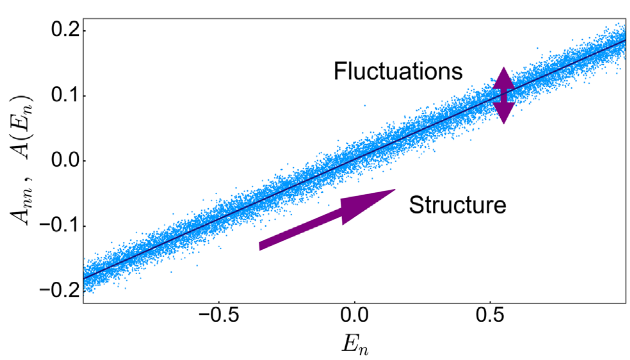

FIG. 1. Sketch of the diagonal matrix elements A_{nn} (dots) and the smooth function $A(E_n)$ (line). Here, $\hat{A}$ is the normalized kinetic energy of a nonintegrable model (15) at $L = 21$. Arrows denote two properties studied here: Structure, i.e., $A(E_n) \neq \text{const}$, and fluctuations of A_{nn} above $A(E_n)$.

Nevertheless, the quantitative role of integrals of motion in the ETH remains widely unexplored.

As a first result, we show that a specific form of fluctuations, δA_{nn}, which signals the violation of the ETH, also entails the presence of LIOMs. This motivates us to quantify the notion of similarity of a local observable to LIOMs via its projections on LIOMs. As a second result, we then outline a generic procedure, applicable to both integrable and nonintegrable models, in which observables are subtracted by their projections on LIOMs and products of LIOMs. This procedure reduces fluctuations of the diagonal matrix elements order by order, which we also verify numerically.

Preliminaries.—We study translationally invariant (TI) chains with L sites and discuss *traceless* TI observables, $\hat{A}=1/\sqrt{L}\sum_{j=1}^{L}\hat{a}_j$, where $\hat{a}_j$ is the density of $\hat{A}$. The choice of the prefactor $1/\sqrt{L}$ is uncommon but convenient since it yields the operators normalized, i.e., $0<\lim_{L\to\infty}\|\hat{A}\|<\infty$. The Hilbert-Schmidt norm $\|\hat{A}\|$ is defined as

$$\|\hat{A}\|^2 = \langle\hat{A}\,\hat{A}\rangle = \frac{1}{Z}\sum_n \langle n|\hat{A}^2|n\rangle = \frac{1}{Z}\sum_{n,m}|A_{nm}|^2, \quad (1)$$

where Z is the Hilbert space dimension. We introduce averaging over infinite time window, $\bar{\hat{A}}=\lim_{\tau\to\infty}\int_0^\tau dt\hat{A}(t)/\tau$, which projects out the off-diagonal matrix elements of the observable $\hat{A}$ such that $\bar{\hat{A}} = \sum_n A_{nn}|n\rangle\langle n|$. (For simplicity we assumed a nondegenerate energy spectrum; see the Supplemental Material [76] for a generalization to a degenerate spectrum.) Using the norm of the time-averaged operator we define the stiffness

$$\sigma_A^2 = \langle\bar{\hat{A}}\,\bar{\hat{A}}\rangle = \frac{1}{Z}\sum_n (A_{nn})^2. \quad (2)$$

Nonvanishing stiffness signals that the correlation function $\langle\hat{A}(t)\hat{A}\rangle$ remains nonzero for arbitrarily long t, hence it indicates absence of thermalization. Finally, we introduce fluctuations of diagonal matrix elements above their microcanonical average $A(E_n)$ [22,74],

$$\Sigma_A^2(\Delta) = \frac{1}{Z_\Delta}\sum_{E_n\in\Delta}[A_{nn} - A(E_n)]^2, \quad (3)$$

where Z_Δ is the number of eigenstates in the energy window Δ. If fluctuations are sampled over all eigenstates, then $Z_\Delta \to Z$ and $\Sigma_A^2(\Delta) \to \Sigma_A^2$.

An important feature beyond predictions of random matrix theory is that in general $A(E_n) \neq$ const. In other words, A_{nn} show some structure [10,15], e.g., the slope of $A(E_n)$ is nonzero as shown in Fig. 1. As a consequence, quantitative measures of fluctuations in finite systems, defined within energy windows that scale polynomially with system size, may be ambiguous (even though, e.g., subtracting a linear fit of A_{nn} in a sufficiently small energy window already reduces fluctuations [24]). The role of $A(E_n)$ is to remove the structure of the diagonal elements, if present. For observables with no structure, i.e., for $A(E_n) = 0$, the stiffness σ_A^2 becomes identical to Σ_A^2. Moreover, we note a property used throughout the Letter, namely, Σ_A^2 increases if the microcanonical average $A(E_n)$ is replaced by other smooth function $f(E_n)$,

$$\Sigma_A^2 \le \frac{1}{Z}\sum_n [A_{nn} - f(E_n)]^2. \quad (4)$$

The choice of observable normalization (1) and the use of stiffness (2) is physically justified by recalling the results for ballistic particle transport in one-dimensional integrable models. The ballistic transport shows up as nonvanishing charge stiffness $\sigma_I^2 > 0$ in the thermodynamic limit [77–94], defined for the observable $\hat{I} = 1/\sqrt{L}\sum_i \hat{j}_i$, where $\hat{j}_i$ is the charge current flowing between sites i and $i+1$. Since the microcanonical average $I(E_n)$ vanishes due to time-reversal symmetry, i.e., the observable $\hat{I}$ has no structure, the nonvanishing stiffness σ_I^2 also reflects nonvanishing fluctuations Σ_I^2 and the breakdown of the ETH [30].

We introduce a measure of violation of the ETH in integrable systems: fluctuations $\Sigma_A^2(\Delta)$ of a *normalized*, traceless TI observable $\hat{A}$ do not vanish,

$$\lim_{L\to\infty}\Sigma_A^2(\Delta) > 0, \quad (5)$$

for an arbitrary energy window Δ. In the Supplemental Material [76] we verify Eq. (5) for a structureless operator in an interacting integrable model.

Finally, we note that the majority of previous studies in integrable models focused on intensive observables $\hat{A}^{\rm int} = \hat{A}/\sqrt{L}$ [49–51,64,73,74]. Their fluctuations typically scale as $\Sigma_{A^{\rm int}}^2 \sim 1/L$. However, we argue that vanishing of $\Sigma_{A^{\rm int}}^2$ can be viewed as a consequence of the vanishing operator norm, $\|\hat{A}^{\rm int}\|^2 \sim 1/L$. In fact, if $A(E_n) = 0$, Eqs. (1) and (3) imply $\Sigma_{A^{\rm int}}^2 \le \|\hat{A}^{\rm int}\|^2$.

Violation of ETH entails existence of LIOMs.—Violation of the ETH, as defined in Eq. (5), implies [together with Eq. (4)] an inequality

$$0 < \frac{1}{Z}\sum_n [A_{nn} - f(E_n)]^2, \quad (6)$$

which in the thermodynamic limit holds for arbitrary smooth function of energy $f(E_n)$. We show in what follows that Eq. (6), i.e., violation of the ETH, entails the presence of LIOMs. We introduce a *projected* observable

$$\hat{A}_\perp = \hat{A} - p_A\hat{H}, \qquad p_A = \frac{\langle\hat{A}\,\hat{H}\rangle}{\langle\hat{H}\,\hat{H}\rangle}, \quad (7)$$

and argue that the time-averaged observable $\bar{\hat{A}}_\perp$ is a LIOM, orthogonal to the Hamiltonian $\hat{H}$.

We first recall that any time-averaged operator is conserved (but not necessarily local), and that time averaging is an orthogonal projection $\langle \bar{\hat{A}}\, \hat{B} \rangle = \langle \hat{A}\, \bar{\hat{B}} \rangle = \langle \bar{\hat{A}}\, \bar{\hat{B}} \rangle$, where $\langle \hat{A}\, \hat{B} \rangle$ is the Hilbert-Schmidt scalar product of operators $\hat{A}$ and $\hat{B}$, see Eq. (1). Then, orthogonality of $\bar{\hat{A}}_\perp$ to $\hat{H}$ follows from orthogonality of $\hat{A}_\perp$ to $\hat{H}$ in construction of Eq. (7), since $\langle \hat{H} \bar{\hat{A}}_\perp \rangle = \langle \bar{\hat{H}} \hat{A}_\perp \rangle = \langle \hat{H} \hat{A}_\perp \rangle = 0$.

The key step is to show locality of $\bar{\hat{A}}_\perp$. In general, testing locality of integrals of motion via analyzing their supports is a tough problem. Unexpectedly, it is easier to relax the constraint on strictly local LIOMs and test a broader concept of *quasilocality*. An additional support for such generalization is that quasilocal LIOMs play an important role in the integrable *XXZ* chains [95–101]. A conserved operator $\hat{Q}_\alpha$ is local or quasilocal when $\langle \hat{A} \hat{Q}_\alpha \rangle^2 / \langle \hat{Q}_\alpha \hat{Q}_\alpha \rangle > 0$ for some normalized TI and local observable $\hat{A}$ [101]. Using the identity $\langle \bar{\hat{A}}_\perp \bar{\hat{A}}_\perp \rangle = \langle \hat{A}_\perp \bar{\hat{A}}_\perp \rangle = \langle (\hat{A} - p_A \hat{H}) \bar{\hat{A}}_\perp \rangle = \langle \hat{A} \bar{\hat{A}}_\perp \rangle$ we get

$$\frac{\langle \hat{A} \bar{\hat{A}}_\perp \rangle^2}{\langle \bar{\hat{A}}_\perp \bar{\hat{A}}_\perp \rangle} = \langle \bar{\hat{A}}_\perp \bar{\hat{A}}_\perp \rangle = \sigma^2_{A_\perp}. \tag{8}$$

Inequality (6) implies that the stiffness of $\hat{A}_\perp$ is nonzero,

$$\sigma^2_{A_\perp} = \langle \bar{\hat{A}}_\perp \bar{\hat{A}}_\perp \rangle = \frac{1}{Z} \sum_n (A_{nn} - p_A E_n)^2 > 0, \tag{9}$$

hence the conserved operator $\bar{\hat{A}}_\perp$ is local or quasilocal.

Integrable systems.—The above analysis can straightforwardly be extended to a model containing an orthogonal set of LIOMs $\{\hat{Q}_\alpha\}$, $\langle \hat{Q}_\alpha \hat{Q}_\beta \rangle \propto \delta_{\alpha\beta}$. To this end we generalize the definition of projected observables, introduced in Eq. (7), to

$$\hat{A}_\perp = \hat{A} - \sum_\alpha p_{A\alpha} \hat{Q}_\alpha, \qquad p_{A\alpha} = \frac{\langle \hat{A} \hat{Q}_\alpha \rangle}{\langle \hat{Q}_\alpha \hat{Q}_\alpha \rangle}. \tag{10}$$

The observables $\hat{A}_\perp$ and $\bar{\hat{A}}_\perp$ are orthogonal to all LIOMs $\{\hat{Q}_\alpha\}$ since $\langle \bar{\hat{A}}_\perp \hat{Q}_\beta \rangle = \langle \hat{A}_\perp \hat{Q}_\beta \rangle = \langle \hat{A} \hat{Q}_\beta \rangle - p_{A\beta} \langle \hat{Q}_\beta \hat{Q}_\beta \rangle = 0$, and as a consequence, Eq. (8) is still valid. If the set of LIOMs is complete, then the norm of $\bar{\hat{A}}_\perp$ must vanish in the thermodynamic limit, and hence

$$\lim_{L\to\infty} \sigma^2_{A_\perp} = 0. \tag{11}$$

Otherwise, $\bar{\hat{A}}_\perp$ is local (or quasilocal) and it represents an additional LIOM which is missing in the set $\{\hat{Q}_\alpha\}$.

So far, the main message concerns the projected operators $\hat{A}_\perp$, which are still local operators but their stiffnesses vanish in the thermodynamic limit. They can be constructed for both integrable and nonintegrable cases which differ only by the number of LIOMs ($\hat{H}$ is typically the only LIOM in a nonintegrable case). We conjecture that the stiffnesses may be further reduced if operators are additionally subtracted by their projections on *products* of LIOMs. Below we provide analytical and numerical evidence for our conjecture.

Products of LIOMs in generic nonintegrable systems.—We first study a generic system where the only LIOM is the Hamiltonian $\hat{H}$, and hence the only products of LIOMs are the powers of $\hat{H}$. We start by finding the polynomial $f(E_n)$ that minimizes the right-hand side of Eq. (4). This is the best polynomial fit to the microcanonical average $A(E_n)$. We introduce a polynomial of degree k of the Hamiltonian

$$\hat{H}_{\perp k} = \hat{H}^k - \sum_{j=1}^{k-1} \frac{\langle \hat{H}^k \hat{H}_{\perp j} \rangle}{\langle \hat{H}_{\perp j} \hat{H}_{\perp j} \rangle} \hat{H}_{\perp j}, \tag{12}$$

where $\hat{H}_{\perp 1} \equiv \hat{H}$. Such polynomials are orthogonal by construction, $\langle \hat{H}_{\perp k} \hat{H}_{\perp l} \rangle \propto \delta_{k,l}$. Then, the central step is to construct projected observables $\hat{A}_{\perp k}$,

$$\hat{A}_{\perp k} = \hat{A} - \hat{f}_k(\hat{H}), \quad \hat{f}_k(\hat{H}) = \sum_{j=1}^{k} \frac{\langle \hat{A} \hat{H}_{\perp j} \rangle}{\langle \hat{H}_{\perp j} \hat{H}_{\perp j} \rangle} \hat{H}_{\perp j}, \tag{13}$$

which can be seen as a generalized form of Eq. (10), with $\hat{A}_{\perp 1} \equiv \hat{A}_\perp$. Using orthogonality of $\hat{H}_{\perp k}$ one easily finds an explicit form of the stiffness,

$$\sigma^2_{A_{\perp k}} = \sigma^2_A - \sum_{j=1}^{k} r_j, \qquad r_j = \frac{\langle \hat{A} \hat{H}_{\perp j} \rangle^2}{\langle \hat{H}_{\perp j} \hat{H}_{\perp j} \rangle}, \tag{14}$$

where the stiffness at $k = 1$ is $\sigma^2_{A_{\perp 1}} \equiv \sigma^2_{A_\perp}$. In the Supplemental Material [76] we show that $f_k(E_n) = \langle n | \hat{f}_k(\hat{H}) | n \rangle$ is indeed the best polynomial fit to $A(E_n)$ for a given degree k, hence $\lim_{k\to\infty} f_k(E_n) = A(E_n)$.

It follows from Eq. (14) that the stiffness σ^2_A is bounded from below by all r_k, i.e., by the projections of $\hat{A}$ on the kth power of the Hamiltonian. Using a Gaussian density of states one can show [76] that $r_k \sim \mathcal{O}(1/L^{k-1})$. Below, we demonstrate that the stiffness σ^2_A may be reduced order by order via subtracting these projections, i.e., via considering operators $\hat{A}_{\perp k}$ introduced in Eq. (13), for which the leading term of the stiffness $\sigma^2_{A_{\perp k}}$ is at most of the order $\mathcal{O}(1/L^k)$. The physical picture behind our construction is that the diagonal matrix elements of the projected observables $\hat{A}_{\perp k}$ become more structureless, i.e., they become closer to the ones typically used in the random matrix theory.

As an application, we study a nonintegrable periodic chain of interacting spinless fermions on L sites and with $N = L/3$ particles,

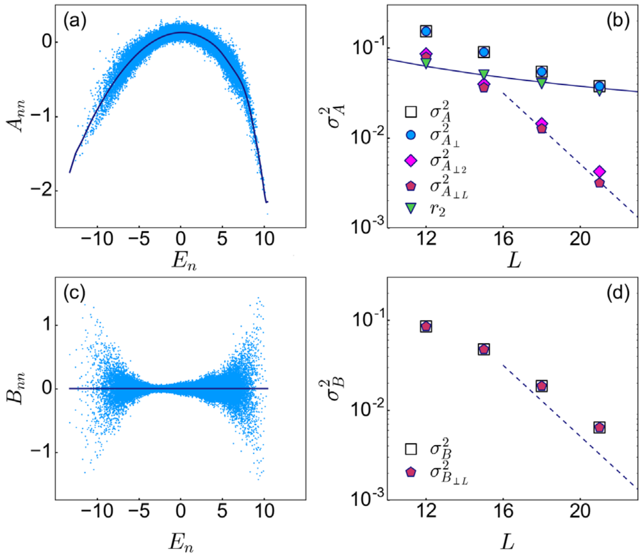

FIG. 2. Diagonal matrix elements of observables $\hat{A}$ and $\hat{B}$ (see text for definitions) for the generic Hamiltonian $\hat{H}$ (15). Symbols in (a) and (c) show A_{nn} and B_{nn}, respectively, versus E_n for $L = 21$ sites in all momentum sectors, while lines represent $f_{k=L}(E_n)$ (13). (b) and (d) Stiffnesses σ_A^2 and σ_B^2, respectively, and the corresponding stiffnesses $\sigma_{A_{\perp k}}^2$ and $\sigma_{B_{\perp k}}^2$, where $\sigma_{A_{\perp 1}}^2 \equiv \sigma_{A_\perp}^2$. In (b) we also plot the projection r_2 (14). Solid and dashed lines are guides to the eye, given by polynomial ($\sim 1/L$) and exponential ($\sim e^{-L/2.2}$) functions, respectively.

$$\hat{H} = -\sum_{j=1}^{L}(e^{i\phi}\hat{c}_{j+1}^\dagger \hat{c}_j + \text{H.c.}) + \sum_{j=1}^{L}(V\hat{\tilde{n}}_j\hat{\tilde{n}}_{j+1} + W\hat{\tilde{n}}_j\hat{\tilde{n}}_{j+2}). \tag{15}$$

Here, $\hat{n}_j = \hat{c}_j^\dagger \hat{c}_j$, $\hat{\tilde{n}}_j = \hat{n}_j - 1/3$, and we set $V = W = 1$. We remove degeneracies in all momentum sectors by introducing a flux $\phi = 2\pi/L$ and π/L for even and odd N, respectively.

We study two observables: the generalized hopping energy $\hat{A} = (1/\sqrt{L})\sum_j(\hat{\kappa}_j + \hat{\kappa}_j^\dagger)$, and the generalized current $\hat{B} = (1/\sqrt{L})\sum_j i(\hat{\kappa}_j - \hat{\kappa}_j^\dagger)$, with $\hat{\kappa}_j = e^{2i\phi}\hat{c}_{j+1}^\dagger(1 - 3\hat{n}_j)\hat{c}_{j-1}$. Figures 2(a) and 2(c) show their diagonal matrix elements A_{nn} and B_{nn}, respectively, and the corresponding polynomial fits $f_{k=L}(E_n)$ from Eq. (13).

The operator $\hat{A}$ has zero projection on $\hat{H}$, i.e., $r_1 = 0$ in Eq. (14), and hence $\sigma_A^2 = \sigma_{A_\perp}^2$. However, $\hat{A}$ has nonzero projection on $\hat{H}_{\perp 2}$ that includes the product $\hat{H}^2$. Figure 2(b) shows that this projection ($r_2 \propto 1/L$) approaches $\sigma_{A_\perp}^2$ for large L causing the power-law decay $\sigma_{A_\perp}^2 \propto 1/L$. After this projection is subtracted from $\hat{A}$, we observe a nearly exponential decay of $\sigma_{A_{\perp k}}^2$ with L for $k \geq 2$. In contrast to $\hat{A}$, the observable $\hat{B}$ has no projection on any power of the Hamiltonian, i.e., the diagonal matrix elements have no structure [see Fig. 2(c)], and σ_B^2 decays exponentially with L [see Fig. 2(d)].

Products of LIOMs in integrable systems.—We now turn our focus to integrable models, for which the set of products of LIOMs is much richer. The latter set should be built iteratively, in analogy to Eq. (12), to ensure that it contains mutually orthogonal operators. In the case of products of two LIOMs, denoted by $\hat{X}_{\gamma(\alpha,\beta)} = \hat{Q}_\alpha\hat{Q}_\beta - \langle\hat{Q}_\alpha\hat{Q}_\beta\rangle$, this is achieved by

$$\hat{X}_{\perp\gamma} = \hat{X}_\gamma - \sum_\alpha \frac{\langle\hat{X}_\gamma\hat{Q}_\alpha\rangle}{\langle\hat{Q}_\alpha\hat{Q}_\alpha\rangle}\hat{Q}_\alpha - \sum_{\gamma'=1}^{\gamma-1}\frac{\langle\hat{X}_{\perp\gamma}\hat{X}_{\perp\gamma'}\rangle}{\langle\hat{X}_{\perp\gamma'}\hat{X}_{\perp\gamma'}\rangle}\hat{X}_{\perp\gamma'}. \tag{16}$$

As an example, we study a chain of hard-core bosons (HCBs) with the Hamiltonian $\hat{H}_{\text{HCB}} = -\sum_j(\hat{b}_{j+1}^\dagger\hat{b}_j + \text{H.c.})$ using periodic boundaries and the onsite constraints $(\hat{b}_j^\dagger)^2 = (\hat{b}_j)^2 = 0$, where $\hat{b}_j^\dagger$ $(\hat{b}_j)$ creates (annihilates) a boson on site j. A complete set of LIOMs $\{\hat{Q}_\alpha\}$ is given by

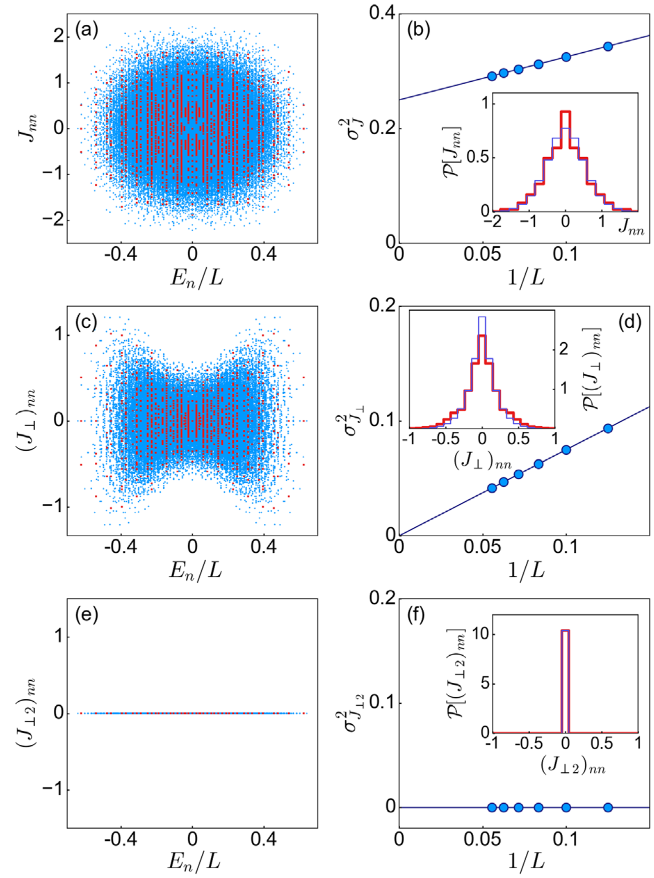

FIG. 3. Diagonal matrix elements of $\hat{J}$ and the projected observables $\hat{J}_\perp$ and $\hat{J}_{\perp 2}$, for the integrable HCBs Hamiltonian $\hat{H}_{\text{HCB}}$ that includes all particle sectors (see text for details). (a),(c), and (e) J_{nn}, $(J_\perp)_{nn}$ and $(J_{\perp 2})_{nn}$, respectively, for $L = 18$ (bright blue symbols) and $L = 12$ (dark red symbols). (b),(d), and (f) Stiffnesses σ_J^2, $\sigma_{J_\perp}^2 \equiv \sigma_{J_{\perp 1}}^2$ (for $k = 1$) and $\sigma_{J_{\perp 2}}^2$ (for $k = 2$), respectively. The insets show histograms of the corresponding distributions of the matrix elements. Lines in (b) and (d) are the functions $1/4 + (3/4)L^{-1}$ and $(3/4)L^{-1}$, respectively.

noninteracting spinless fermions onto which the HCBs are mapped (see the Supplemental Material [76] for details).

We construct a two-body structureless observable $\hat{J} = \sqrt{(2/L)} \sum_j (i\hat{b}^\dagger_{j+1}\hat{n}_j\hat{b}_{j-1} + \mathrm{H.c.})$, for which the microcanonical average vanishes, $J(E_n) = 0$. Figure 3(a) shows the diagonal matrix elements J_{nn} for two system sizes L, and Fig. 3(b) shows that the stiffness σ^2_J extrapolates to a nonzero value in the thermodynamic limit $L \to \infty$. These results signal violation of the ETH as stated in Eq. (5) and the existence of LIOMs.

We then construct a projected observable $\hat{J}_\perp$ according to Eq. (10) using a complete set of LIOMs. Figure 3(c) reveals that the diagonal matrix elements of $(J_\perp)_{nn}$ are reduced when compared to J_{nn} in Fig. 3(a). Moreover, Fig. 3(d) shows a vanishing stiffness $\sigma^2_{J_\perp} \propto 1/L$, in agreement with Eq. (11).

Finally, we construct the projected observable $\hat{J}_{\perp 2}$, which is a generalization of Eq. (13) for $k = 2$ to include all the possible products of two LIOMs $\hat{X}_{\perp\gamma}$ from Eq. (16). Remarkably, all the diagonal matrix elements of $(J_{\perp 2})_{nn}$ are exactly zero already in finite systems, as shown in Figs. 3(e) and 3(f). This reveals a special instance of the ETH, where the diagonal matrix elements form a well-defined function with zero fluctuations at any system size.

Conclusions.—The simplest classification of nonequilibrium dynamics contains two classes of quantum many-body systems: generic systems which thermalize after sufficiently long time, and integrable systems which do not thermalize. This difference is encoded in the diagonal matrix elements of observables. In this Letter, we introduced a quantitative framework to characterize these matrix elements. The central concept is the similarity of observables to LIOMs quantified via their projections on LIOMs and products of LIOMs. It can be applied to both integrable and generic systems, with the only difference consisting of the number and the structure of LIOMs. We thereby made steps towards a unified treatment of the ETH in integrable and generic quantum systems.

Previous works within the generalized ETH [50,51] studied the notion of similarity between *eigenstates* for a particular distribution of LIOMs after a quantum quench. Here, we established a framework that is quench independent and is based on the similarity of *observables* to LIOMs. Then, the infinite-time averages of observables are governed by the expectation values of LIOMs (and products of LIOMs) in the initial state and by the similarity of observables to LIOMs. So far our applications concerned translationally invariant systems with simple forms of LIOMs, and extensions to models with more involved structure of LIOMs, as well as to models without translational invariance, are desired for future work.

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