Standard Quantum Limit and Heisenberg Limit in Function Estimation

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Unlike well-established parameter estimation, function estimation faces conceptual and mathematical difficulties despite its enormous potential utility. We establish the fundamental error bounds on function estimation in quantum metrology for a spatially varying phase operator, where various degrees of smooth functions are considered. The error bounds are identified in the cases of the absence and the presence of interparticle entanglement, which correspond to the standard quantum limit and the Heisenberg limit, respectively. Notably, these error bounds can be reached by either position-localized states or wave-number-localized ones. In fact, we show that these error bounds are theoretically optimal for any type of probe states, indicating that quantum metrology on functions is also subject to the Nyquist-Shannon sampling theorem, even if classical detection is replaced by quantum measurement.

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Accurate estimation of signals with a limited amount of resources is a fundamental problem in physics. Quantum metrology has made a profound contribution to this problem by demonstrating a classically unattainable scaling of the estimation error [1–5]. This nonclassical accuracy, called the Heisenberg limit, can be achieved by various forms of quantum features including entanglement [2], quantum circuit [4], bosonic [6,7] and spin [8,9] squeezing, and quantum statistics [10]. Applications of the Heisenberg-limited measurement ranges from detection of fundamental signals such as atomic clocks [11,12] and gravitational waves [13,14] to experimental verification of nonclassical states such as scalable cat states [15] and polariton condensates [16,17].

The Heisenberg limit significantly surpasses the standard quantum limit (SQL) $\delta = O(N^{-1/2})$, which sets the accuracy bound arising from uncorrelated noises, where N is the size of the resource. While the Heisenberg limit $\delta = O(N^{-1})$ and the SQL $\delta = O(N^{-1/2})$ apply to both scalar estimation and vector estimation [18–23], it is generally considered that the continuous nature of the signal may alter the scaling law. Such a problem can be categorized as function estimation, which has attracted growing attention. For example, atomic clocks [24,25] and gravitational waves [14,26] involve time-varying signals, which offer richer information when treated as functions. More generally, exploration for new phenomena involves observing structures in a continuous space and/or time, which can be represented as functions. This indicates that functional structures play crucial roles in generic continuous systems including measurements on magnetometry [27,28], nanostructured materials [29–32], live cells [33,34], and event horizons [35,36]. Thus, it is not only a fundamental question but also relevant to a wide range of applications to ask how quantum metrology can contribute to the detection of functions with ultimate accuracy.

The quantum version of function estimation has been investigated in terms of the signal detection theory in Refs. [25,37–41]. In fact, weaker scaling laws are implied when the target parameter can change continuously in time, such as a Gaussian signal. The demonstration of such unconventional limits has recently come within experimental reach due to the realization of, e.g., high-N00N states [42] and optical phase tracking [43]. Although the detection theory is applicable to stochastic noises, it does not support the case where the relevant parameter is not inherently stochastic, which is often the case with quantum imaging and quantum signal processing [24,31,44,45].

In this Letter, we present a fundamental framework of quantum metrology on functions. Unlike parametric estimation, function estimation involves infinite degrees of freedom and inevitably requires further assumptions on the target function. Assuming only the smoothness of the function, we find the SQL of $O(N^{-q/(2q+1)})$ and the Heisenberg limit of $O(N^{-q/(q+1)})$, where q is a measure of the degree of smoothness of the function. Our framework allows analysis of estimation errors of data series under given smoothness, such as a bound on the amplitude of derivatives. This includes the previous results on Gaussian processes through computation of their smoothness [46], as demonstrated later. The data series requires neither to have a prior distribution nor even to be continuous, allowing, for example, a sample with a finite number of discontinuous points [31,32]. Moreover, we have found that the error bound can equally be saturated by states which are localized in position or wave number. This implies the

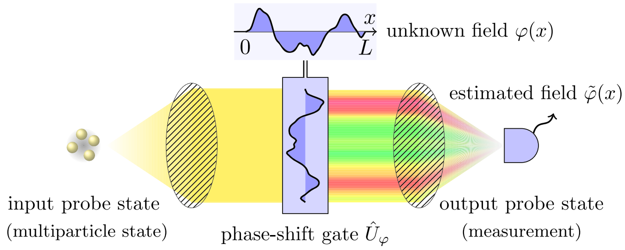

FIG. 1. Schematic illustration of quantum estimation of functions. First, some multiparticle state is prepared as an input probe state. It then passes through a phase-shift gate $\hat{U}_\varphi$ generated by a spatially varying field $\varphi(x)$ that we want to know. Finally, the output probe state is measured, whence the estimated field $\tilde{\varphi}(x)$ is computed.

equivalence between space discretization and momentum cutoff in quantum information processing, reminiscent of the Nyquist-Shannon sampling theorem in classical statistics.

Setup.—We consider the estimation of an unknown function $\varphi(x)$ defined over an interval $0 \le x \le L$ by using the position-dependent phase-shift gate $\hat{U}_\varphi$ shown in Fig. 1. For simplicity, we assume the periodic boundary condition.

To estimate φ, we prepare an N-particle state as a probe, which evolves according to the unitary operator $\hat{U}_\varphi$ and then is measured. These particles are distributed over the interval $[0, L]$ and have two internal states: one state $|+\rangle$ interacts with the phase-shift gate $\hat{U}_\varphi$ and the other state $|-\rangle$ does not.

We work in the first-quantization formalism and denote by $|x;\pm\rangle$ the position eigenstate at x with internal state $\pm$. Let the phase-shift gate act as $\hat{U}_\varphi|x;+\rangle = e^{i\varphi(x)}|x;+\rangle$ and $\hat{U}_\varphi|x;-\rangle = |x;-\rangle$. The unknown function φ can be estimated from measurement on this output. When the probe is composed of N separable particles, the error of the function estimation is bounded by the SQL. A probe with appropriately entangled N particles, on the other hand, leads to the Heisenberg limit.

We recall that the estimation error of a scalar parameter θ is computed as $\delta^2 = \mathbb{E}[|\tilde{\theta} - \theta|^2]$, where $\tilde{\theta}$ is the estimator depending on the stochastic nature of measurement outcomes. Similarly, we consider a stochastic estimator $\tilde{\varphi}$ for the function, and compute the mean-square periodic error (MSPE) [47] as

$$\delta^2 = \mathbb{E}\left[\int_0^L \frac{dx}{L}[\tilde{\varphi}(x) - \varphi(x)]_{2\pi}^2\right]. \tag{1}$$

In other words, the estimation error is averaged over x and the modulus is replaced by $[\tilde{\varphi}(x) - \varphi(x)]_{2\pi} \coloneqq \min_{n\in\mathbb{Z}}|\tilde{\varphi}(x) - \varphi(x) + 2\pi n|$, i.e., the minimal absolute value modulo 2π.

The main difficulty in function estimation lies in the fact that the problem involves infinite degrees of freedom. In particular, the lower bound on δ^2 cannot be established for an arbitrary function, since we cannot exclude any rapidly fluctuating functions from a finite number of measurements. Hence, we impose the following constraint on the target function φ:

$$\int_0^L \frac{dx}{L}|\varphi'(x)|^2 \le \frac{M^2}{L^2} \tag{2}$$

for some positive number $M > 0$. With this constraint, we can establish a suitable lower bound on sufficiently smooth and slowly varying functions φ.

The condition (2) can be applied only when the target function is differentiable. In our framework, we consider more general functions without differentiability: the Hölder continuity $|\varphi(x+\epsilon) - \varphi(x)| = O(\epsilon^q)$ for fixed $0 < q \le 1$ [48]. To be more precise, we impose a general constraint:

$$\sup_{0<\epsilon<a}\int_0^L \frac{dx}{L}\left|\frac{\varphi(x+\epsilon) - \varphi(x)}{\epsilon^q}\right|^2 \le \frac{M^2}{L^{2q}}, \tag{3}$$

where $a > 0$ is a constant that does not affect the estimation error in the limit of large N. The special case with $q = 1$ reduces to Eq. (2).

Estimation methods.—Given the target function φ under the constraint (3), there exist estimation methods that ensure a finite estimation error δ defined in Eq. (1). We here compare the two different methods. (i) In the position-state (PS) method, we estimate the individual phases $\varphi(x_j)$ at several positions x_j and then computationally reconstruct the entire function. (ii) In the wave-number-state (WS) method, we prepare a sufficiently large number of wave functions $\psi(x) \propto e^{i\varphi(x)}$ and estimate the function φ by reconstructing the quantum state $|\psi\rangle$ by the quantum tomography. We find that the numbers of particles N required for these two methods are the same up to a constant factor.

Position-state method.—The PS method can be used when the target function is relatively small, say, $|\varphi(x)| \le \pi/3$ for all x. In this case, we can circumvent the phase wrapping problem and employ a method analogous to the kernel density estimation [49].

In the first step, we sample n_1 positions $x_1, x_2, \ldots, x_{n_1}$ in the interval $0 \le x \le L$ with equal spacing. Then, the phase $\varphi_j = \varphi(x_j)$ at each position x_j is measured by using n_2 particles localized at x_j. The estimation error δ_{ind} of the individual phase φ_j is known [4,50] as it is the quantum metrology on a scalar parameter; the SQL $\delta_{\mathrm{ind}} = O(n_2^{-1/2})$ is established by the probe $(1/\sqrt{2})(|x_j;-\rangle + |x_j;+\rangle)^{\otimes n_2}$ and the Heisenberg limit $\delta_{\mathrm{ind}} = O(n_2^{-1})$ by $(1/\sqrt{2})(|x_j;-\rangle^{\otimes n_2} + |x_j;+\rangle^{\otimes n_2})$. Finally, the function estimator $\tilde{\varphi}(x)$ is computed from the individual estimators $\tilde{\varphi}_j$ by local linear smoothing [51]:

$$\tilde{\varphi}(x) = \sum_{j=1}^{n_1} \tilde{\varphi}_j f(x - x_j), \quad (4)$$

where f is a smoothing function. In the present case, we may just set $f(x) = 1$ for $x \leq (L/2n_1)$ and $f(x) = 0$ otherwise. This corresponds to the approximation by the value at the nearest site; i.e., we set $\tilde{\varphi}(x) = \tilde{\varphi}_j$, where x_j is the point nearest to x.

The estimation error can be decomposed into two parts: the statistical error δ_{stat} caused by the measurement and the deterministic error δ_{det} due to smoothing. The balance between these errors can be tuned by the width l of smoothing. The estimated value $\tilde{\varphi}_j(x)$ is of the same order as δ_{ind}, i.e., $\delta_{\text{stat}} = O(n_2^{-1/2})$ for the SQL and $\delta_{\text{stat}} = O(n_2^{-1})$ for the Heisenberg limit. On the other hand, the deterministic error δ_{det} is the variation of $\varphi(x)$ within the width $L/(2n_1)$, which turns out to be $\delta_{\text{det}} = O(n_1^{-q}M)$ by virtue of the constraint in Eq. (3).

For a given number of particles $N = n_1 n_2$, the optimal accuracy is determined by the trade-off between δ_{stat} and δ_{det}. As a consequence of Young's inequality, we obtain

$$\delta \geq O(n_1^{-q}M) + O(n_2^{-1/2}) \geq O((M^{1/q}N^{-1})^{q/(2q+1)}) \quad (5)$$

for the SQL and

$$\delta \geq O(n_1^{-q}M) + O(n_2^{-1}) \geq O((M^{1/q}N^{-1})^{q/(q+1)}) \quad (6)$$

for the Heisenberg limit. Therefore, the overall estimation error δ is significantly larger than the traditional quantum limit, which is an expected feature of the function estimation. We note that entanglement of particles in different positions is not necessary to achieve the Heisenberg limit; such intersite entanglement does not enhance the estimation of linear parameters, as suggested in studies of quantum network sensors [52,53].

Wave-number-state method.—In the WS method, we begin with the wave number eigenstate with zero eigenvalue: $\int_0^L (dx/\sqrt{2L})[|x;-\rangle^{\otimes n_p} + |x;+\rangle^{\otimes n_p}]$. We use the one-particle state ($n_p = 1$) for the SQL and a multipartite EPR state ($n_p > 1$) for the Heisenberg limit.

By the phase-shift gate $\hat{U}_\varphi$, one obtains the output probe state:

$$|S_\varphi\rangle = \int_0^L \frac{dx}{\sqrt{2L}} [|x;-\rangle^{\otimes n_p} + e^{in_p\varphi(x)}|x;+\rangle^{\otimes n_p}]. \quad (7)$$

The estimation is conducted by reconstructing $|S_\varphi\rangle$ as accurate as possible by measuring n_c copies of the probe state. For this purpose, we consider the projection P_K onto the subspace of wave numbers k such that $|k| \leq 2\pi K/L$. Since the postselected state $|S_\varphi^*\rangle \propto P_K|S_\varphi\rangle$ belongs to a $(2K+1)$-dimensional Hilbert space, it can be identified by the quantum tomography.

The error of the state reconstruction can be quantified by the infidelity $1 - |\langle S_\varphi|S_{\tilde{\varphi}}\rangle|$, where $|S_{\tilde{\varphi}}\rangle$ denotes the reconstructed state. In fact, we show in the Supplemental Material [54] that the MSPE has can be bounded by the expected infidelity as

$$\begin{aligned} \delta^2 &\leq \frac{\pi^2}{n_p^2}\mathbb{E}[1 - |\langle S_\varphi|S_{\tilde{\varphi}}\rangle|] \\ &\leq \frac{\pi^2}{n_p^2}(1 - |\langle S_\varphi|S_\varphi^*\rangle|^2) + \frac{\pi^2}{n_p^2}\mathbb{E}[1 - |\langle S_\varphi^*|S_{\tilde{\varphi}}\rangle|^2] \\ &= \delta_{\text{PS}}^2 + \delta_{\text{QT}}^2. \end{aligned} \quad (8)$$

Here, the error is divided into the postselection part δ_{PS}^2 and the quantum-tomography part δ_{QT}^2. The postselection error can be bounded by the constraint (3) as $\delta_{\text{PS}} \leq O(K^{-q}M)$ [54–56], while the results of the finite-dimensional tomography imply $n_p\delta_{\text{QT}} \leq O(K^{1/2}n_c^{-1/2})$ [57].

When $n_p = 1$ and $N = n_c$, we have the trade-off relation between δ_{PS} and δ_{QT} for the SQL:

$$\delta_{\text{PS}} = O(K^{-q}M), \qquad \delta_{\text{QT}} = O(K^{1/2}N^{-1/2}). \quad (9)$$

By setting $n_1 = K$ and $n_2 = N/K$, the errors δ_{PS} and δ_{QT} can be mapped to the errors δ_{det} and δ_{stat} in the PS method, respectively. Therefore, the SQL in the WS method reduces to that in Eq. (5) obtained by the PS method.

The error bound can be lowered for $n_p > 1$ and $N = n_p n_c$, while $K \leq O(n_c)$ must be maintained in order to robustly conduct the quantum tomography. Therefore, the optimal trade-off relation for the Heisenberg limit is

$$\delta_{\text{PS}} = O(n_c^{-q}M), \qquad \delta_{\text{QT}} = O(n_p^{-1}). \quad (10)$$

By setting $n_1 = n_c$ and $n_2 = n_p$, this trade-off relation corresponds exactly to the PS method, and we obtain the same Heisenberg limit as in (6). However, there is a caveat that in the output state (7) the phase ambiguity of $\varphi(x)$ modulo $2\pi/n_p$ must be removed. We show in the Supplemental Material [54] that this removal can be handled by analogy with the Kitaev method [50].

Optimality of the SQL.—We have preposed the SQL (5) and the Heisenberg limits (6) that can be achieved by both the PS and the WS methods. We show that these limits are in fact optimal; any theoretical method is subject to the same bounds on the estimation error.

We first derive the theoretical lower bound on the SQL. We consider the Fourier transform of the function $\varphi(x)$:

$$\varphi_k = \int_0^L \frac{dx}{L} e^{-2\pi ikx/L}\varphi(x). \quad (11)$$

On the wave number basis, the constraint (3) corresponds to the suppression of high-wave-number components:

$\varphi_k = o(k^{-q})$. In particular, the special constraint (2) is equivalent to $\sum_{k=1}^{\infty} k^2|\varphi_k|^2 \leq (M^2/8\pi^2)$, which can be seen from Perseval's equality. A generalization of this argument leads to a sufficient condition on the constraint (3):

$$\sum_{k=1}^{\infty} k^{2q}|\varphi_k|^2 \leq \frac{M^2}{2c_0^2}, \tag{12}$$

where $c_0 = 2\pi^q \max_{0\leq x\leq\pi}[x^{-q}\sin x]$ is a q-dependent constant.

To utilize the known results in the discrete parameter estimation [20,58], we focus on the functions with only some low-wave-number components. Using a K-dimensional vector $\mathbf{u} = (u_1, \ldots, u_K)$, we parametrize the function φ as

$$\varphi_{\mathbf{u}}(x) = \sum_{k=1}^{K} \sqrt{2}\, u_k \sin(2\pi i k x/L). \tag{13}$$

Such function $\varphi_{\mathbf{u}}$ meets the constraint if $\|\mathbf{u}\| \leq \rho$ is satisfied for $\rho = c_0^{-1} M K^{-q}$. Since $\rho = o(1)$ and $\sqrt{2}\sin(2\pi i k x/L)$ forms an orthonormal basis, the MSPE δ^2 of the function φ can be bounded by mean-square error $\delta_{\mathbf{u}}^2$ of the vector $\mathbf{u}$.

Hence, instead of the function estimation, we may consider the vector estimation in which case the error can be evaluated by the quantum Cramér-Rao bound (QCRB) [59,60]. For an unbiased estimation, the QCRB is given as

$$\delta_{\mathbf{u}} \geq \delta_{\mathrm{UUB}} \coloneqq K(\mathrm{tr}[J(\mathbf{u})])^{-1/2}, \tag{14}$$

where $[J(\mathbf{u})]$ is the Fisher information matrix defined for the output probe state $|\psi_{\mathbf{u}}\rangle$ as

$$[J(\mathbf{u})]_{jk} = 4\mathrm{Re}\left\langle \frac{\partial}{\partial u_j}\psi_{\mathbf{u}} \middle| [1 - |\psi_{\mathbf{u}}\rangle\langle\psi_{\mathbf{u}}|] \middle| \frac{\partial}{\partial u_k}\psi_{\mathbf{u}} \right\rangle. \tag{15}$$

Since the Fisher information is bounded from above by the SQL [61], i.e., $[J(\mathbf{u})]_{jj} \leq 8N$ for each $1 \leq j \leq K$, we obtain a uniform, unbiased bound $\delta_{\mathrm{UUB}} = (K/8N)^{1/2}$.

Although such a *uniform* bound is not applicable to the biased estimation, there exists the worst-case biased bound δ_{WBB} [54]. Within the region $\|\mathbf{u}\| \leq \rho$, one can find a vector $\mathbf{u}$ satisfying

$$\delta_{\mathbf{u}} \geq \delta_{\mathrm{WBB}}, \qquad \delta_{\mathrm{WBB}}^{-1} \coloneqq \delta_{\mathrm{UUB}}^{-1} + \rho^{-1}. \tag{16}$$

Since this biased version of the QCRB holds for any integer $K \geq 1$, we choose K that gives the maximal bound δ_{WBB}. This is satisfied when ρ and δ_{UUB} are comparable to each other—this case holds when we set $K = O((M^2N)^{1/(2q+1)})$. Hence the SQL is given as

$$\delta \geq c_1(M^{1/q}N^{-1})^{q/(2q+1)}, \tag{17}$$

where the constant factor c_1 is bounded from below by $\{[q]/[2(2q+1)^2]\}[c_0^{-1}q]^{2q/(2q+1)}$.

Optimality of the Heisenberg limit.—We consider the case in which entanglement between at most n_p (≥ 1) particles is allowed. It is known that the quantum information of a probe state is maximal when their wave function is completely symmetric [62]. With completely symmetric probe states, the problem becomes equivalent to the estimation of an effective phase $n_p\varphi(x)$ with $n_c = n_p^{-1}N$ separate particles; the probe states in Eq. (7) serve as an example for the WS method.

Since the function of interest φ is replaced by its effective one $n_p\varphi$, the MSPE δ^2 and the normalization constant M are replaced by $n_p^2\delta^2$ and n_pM, respectively. This argument leads to a generalized limit:

$$\begin{aligned} n_p\delta &\geq c_1[(n_pM)^{1/q}(n_p^{-1}N)^{-1}]^{q/(2q+1)} \\ &= c_1(Mn_p^{q+1}N^{-q})^{1/(2q+1)}. \end{aligned} \tag{18}$$

To restore the original function $\varphi(x)$ from the estimate of $n_p\varphi(x)$, we need to resolve the phase ambiguity by $2\pi/n_p$. For this purpose, the left-hand side of Eq. (18) should not exceed π, giving

$$n_p \leq [(\pi c_1^{-1})^{(2q+1)/2}M^{-1}N^q]^{1/(q+1)}. \tag{19}$$

With the maximal n_p substituted in Eq. (18), we obtain the Heisenberg limit

$$\delta \geq c_2(M^{1/q}N^{-1})^{q/(q+1)}, \tag{20}$$

where the constant c_2 is at least $(\pi^{-1}c_1)^{(2q+1)/(q+1)}$.

Extension to smoother functions.—The degree of smoothness can further be extended into $q > 1$, where the target function is known to be more than just differentiable. For an integer m and $0 < \sigma \leq 1$ satisfying $q = m + \sigma$, the constraint for smoother functions is given as

$$\sup_{0<\epsilon<a} \int_0^L \frac{dx}{L}\left|\frac{\varphi^{(m)}(x+\epsilon) - \varphi^{(m)}(x)}{\epsilon^{\sigma}}\right|^2 \leq \frac{M^2}{L^{2q}}. \tag{21}$$

Our results in the PS method and the optimality are also valid for $q > 1$, thus leaving the quantum limits (5) and (6) unchanged. On the other hand, the straightforward extension of the WS method into $q > 1$ does not work. Therefore, the asymptotic equivalence between the PS method and the WS method can be obtained only for $0 < q \leq 1$. See the Supplemental Material for more details [54].

Comparison with Gaussian signal estimation.—The error bounds we have obtained here are related to that of the Gaussian signal estimation [40,41,63], in which the time-dependent phase φ_t is subject to a Gaussian process with the power spectrum $I(\omega) \sim |\omega|^{-p}$. The estimation error of an instantaneous phase $\varphi_{t=0}$ is $O(\mathcal{N}^{-(p-1)/2p})$ for coherent states and $O(\mathcal{N}^{-(p-1)/(p+1)})$ for squeezed states, where $\mathcal{N}$ is the photon flux [41]. This can exactly be mapped into the SQL $O(N^{-q/(2q+1)})$ and the Heisenberg limit $O(N^{-q/(q+1)})$ in our study by setting $p = 2q + 1$. In fact, almost all sample functions of the Gaussian process φ_t satisfy Eq. (21) by taking a large time span [46,64]. Therefore, the estimation error of the Gaussian process is subject to the quantum metrology of function estimation. We suspect that this fact is related to the min-max theorem [65,66], which explains the consistency between the Bayesian and non-Bayesian estimation methods, though a clear connection is yet to be established.

Conclusion and outlook.—In this Letter, we have established the fundamental limits on function estimation subject to a bounded qth-order differentiability. The estimation error is bounded from below by $O(N^{-q/(2q+1)})$ in the standard quantum limit and $O(N^{-q/(q+1)})$ in the Heisenberg limit. These results reduce to the previous studies on the signal estimation in quantum optics [40,41] when the target function is an infinitely extended stochastic process. We have presented two theoretical methods of the functional quantum metrology, both of which saturate the fundamental limits for $0 < q \leq 1$.

This is a fundamental result for the efficient detection of functional structures—continuous signals and images, for example—which is a common target of estimation today. In fact, our results set fundamental theoretical bounds on various types of analysis relying on the function structure, such as model prediction or feature extraction [67,68]. On one hand, these bounds indicate the critical point where quantum methods outperform classical methods on functional data, with the scaling laws different from those obtained from parameter estimation. On the other hand, our result shows the optimal strategies for the quantum estimation of functions, such as an appropriate choice of temporal or spatial resolution and the size of entanglement. We note that choice of resolution is crucial in the real application [23,45,69,70], and what is more in the quantum case, we have seen that larger entanglement does not necessarily mean better accuracy.

The framework presented here enables further quantum information-theoretic analysis on functions, such as a quantum version of the Nyquist-Shannon sampling theorem which concerns the exact equivalence between the position and wave-number states in the signal detection, including the $O(1)$ prefactor that has remained undetermined.

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