Localization, Topology, and Quantized Transport in Disordered Floquet Systems

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We investigate the effects of disorder on a periodically driven one-dimensional model displaying quantized topological transport. We show that, while instantaneous eigenstates are necessarily Anderson localized, the periodic driving plays a fundamental role in delocalizing Floquet states over the whole system, henceforth allowing for a steady-state nearly quantized current. Remarkably, this is linked to a localization-delocalization transition in the Floquet states at strong disorder, which occurs for periodic driving corresponding to a nontrivial loop in the parameter space. As a consequence, the Floquet spectrum becomes continuous in the delocalized phase, in contrast with a pure-point instantaneous spectrum.

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Introduction.—Thouless pumping [1,2] provides one of the simplest manifestations of topology in quantum systems, and has attracted a lot of recent interest, both theoretically [3–11] and experimentally [12–17]. Since the seminal works by Thouless and Niu [1,2], it has been argued that the quantization of the pumped charge is robust against weak disorder, but a clear characterization of the localization properties of the relevant states, and the breakdown of quantized transport at strong disorder, is still missing.

Thouless pumping is also the first example of a topological phase emerging in a periodically driven system with no static analogue. Such phases have been the subject of many recent proposals [4,6,8,18–22]. In this respect, understanding the role of disorder has a twofold purpose: on one hand, it is important to understand the robustness to disorder of the topology of driven systems [17,23] *per se*; on the other hand, localization properties in the topological phase are relevant for the possibility of stabilizing topological pumping in *interacting* systems [24,25] by means of many-body localization [26,27].

Restricting ourselves to the noninteracting case, a puzzling aspect regards the nature of the Floquet states. While quantized transport over a single period of the driving is expected at a small disorder [2], its robustness over many driving cycles is not trivial, since it would imply the existence of *extended* Floquet states. But in the adiabatic limit, where charge is strictly quantized, Floquet states for a generic driving should coincide with the Hamiltonian eigenstates, which are Anderson localized in 1D. So, how can Thouless pumping in Anderson insulators be stable in the long-time limit? Previous studies of periodically driven 1D Anderson insulators in the low-frequency regime [28,29] have found a generic increase of the localization length of Floquet states compared to the static case, without any evidence of truly extended states nor a clear link between the localization properties and topology.

In this Letter, we address these questions by inquiring the effects of disorder on Thouless pumping from the point of view of Floquet theory. We focus on the finite-size scaling of the localization length of Floquet states, the long-time dynamics and the winding of Floquet quasienergies, and show that Thouless pumping is associated to extended Floquet states. Remarkably, as disorder increases these states undergo a true delocalization-localization transition at a critical disorder strength W_c, which reflects itself in the breakdown of quantized transport. Crucially, topology plays a fundamental role in the existence of such extended states and on the character of the phase transition, as we prove by explicit comparison with the case of a trivial adiabatic driving protocol.

Model.—We consider a disordered version of the driven Rice-Mele model [30]. For a system of spinless fermions on a chain of $L = 2N$ sites, with $\hat{c}_j^\dagger$ creating a fermion on the jth site, the Hamiltonian reads

$$\hat{H}(t) = -\sum_{j=1}^{N} [J_1(t)\hat{c}^\dagger_{2j-1}\hat{c}_{2j} + J_2(t)\hat{c}^\dagger_{2j}\hat{c}_{2j+1} + \text{H.c.}] - \sum_{l=1}^{L} [(-1)^l\Delta(t) + W\zeta_l]\hat{c}^\dagger{}_l\hat{c}_l. \quad (1)$$

Here $J_{1(2)}(t)$ and $\Delta(t)$ describe hopping amplitudes and on site energies for the clean model, while $W\zeta_l$ describes the on site disorder of strength W, with $\zeta_l \in [-\frac{1}{2}, \frac{1}{2}]$ uniformly distributed random numbers. We assume periodic boundary conditions. In absence of disorder, $W = 0$, and for generic $J_{1(2)}$ and Δ, the instantaneous spectrum is split in two bands, separated by a gap. Thus, at half filling, the charge pumped in one period is equal in the adiabatic limit to the Chern number of the occupied band [1]. This integer is different from 0 when the driving is topologically nontrivial, e.g., when the path in the space $(J_1 - J_2, \Delta)$ encloses the gapless point (0,0).

To characterize the topological phase we compute the average number of particles pumped over an infinite number of periods [7,31]

$$\bar{Q} = \lim_{M\to\infty} \frac{1}{M} \int_0^{M\tau} dt \langle \Psi(t)|\hat{\mathrm{J}}(t)|\Psi(t)\rangle, \qquad (2)$$

from the quantum-average of the current density operator $\hat{J}(t)$ [1,32]. Here, $\tau = 2\pi/\omega$ is the driving period, and the system is initially prepared in the N-particle ground state $|\Psi_0\rangle$ of $\hat{H}(t = 0)$.

Since the Hamiltonian is time periodic, we can exploit the Floquet representation [35] of the evolution operator $\hat{U}(t, 0) = \sum_\nu e^{-i\mathcal{E}_\nu t/\hbar} |\Phi_\nu(t)\rangle\langle\Phi_\nu(0)|$, where $|\Phi_\nu(t)\rangle = |\Phi_\nu(t + \tau)\rangle$ are N-particle Floquet modes and $\mathcal{E}_\nu$ are the many-body quasienergies. $\bar{Q}$ can be computed directly in the Floquet diagonal ensemble [7,36]:

$$\bar{Q} = Q_d = \sum_\nu \mathcal{N}_\nu \int_0^\tau dt \langle \Phi_\nu(t)|\hat{\mathrm{J}}(t)|\Phi_\nu(t)\rangle, \qquad (3)$$

where $\mathcal{N}_\nu = |\langle\Psi_0|\Phi_\nu(0)\rangle|^2$ is the occupation number of the νth Floquet state. For noninteracting fermions, it suffices to know the single-particle (SP) Floquet states $|\phi_\alpha(t)\rangle$ and their occupation number n_α to explicitly calculate the diagonal pumped charge [7,31].

Results.—Figure 1 shows the disorder average $[Q_d]_{\mathrm{av}}$, computed with Eq. (3), as a function of the disorder strength W, in a topological driving cycle with $J_{1(2)}(t) = J_0 \pm \delta_0 \cos(\omega t)$, and $\Delta(t) = \Delta_0 \sin(\omega t)$. Observe that topological pumping persists for sufficiently small $W \lesssim 3J_0$. The regime of large $W \gtrsim 8J_0$ is also rather clear: $Q_d = 0$. The intermediate region $W/J_0 \approx 4$ shows large sample-to-sample fluctuations: the inset shows that the drop of $[Q_d]_{\mathrm{av}}$ starts when disorder closes the instantaneous gap $\Lambda_N \equiv \min_{t\in[0,\tau]}[E_{N+1}(t) - E_N(t)]$, where $E_N(t)$ is the N-particle ground state energy at time t.

To understand why quantized transport survives in spite of the instantaneous eigenstates being Anderson localized [37], we now show that, crucially, a significant fraction of the SP Floquet states remain *delocalized* even for very low frequency, due to a driving-induced mixing of localized states [28,29]. We analyzed localization-delocalization of states through the real-space inverse participation ratio

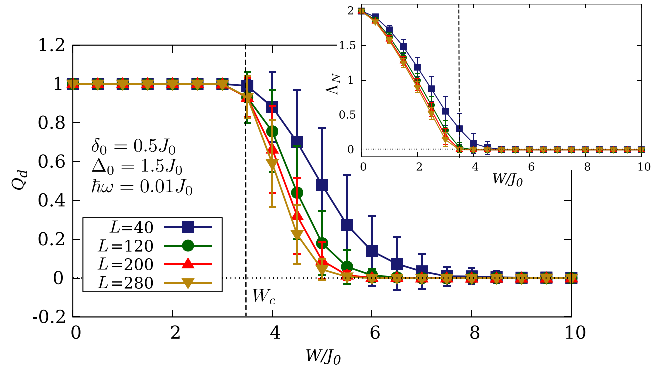

FIG. 1. Disorder average of the diagonal pumped charge plotted against disorder strength. The transition between the quantized charge regime and the trivial one $Q_d = 0$ is linked to the closing of the minimum energy gap due to the disorder, shown in the inset, as highlighted by the vertical dashed line.

(IPR) [38] of the single-particle Floquet modes $|\phi_\alpha(0)\rangle$, $\mathrm{IPR}_\alpha = \sum_l |\langle l|\phi_\alpha(0)\rangle|^4$, with $|l\rangle = \hat{c}_l^\dagger|0\rangle$ being a particle localized at site l. For a finite system $\mathrm{IPR}_\alpha \in [L^{-1}, 1]$, where $\mathrm{IPR}_\alpha \sim L^{-1}$ signals a completely delocalized (plane-wave-like) state, while $\mathrm{IPR}_\alpha = 1$ corresponds to a perfect localization on a single site. Figure 2(a) shows the distribution of IPRs of Floquet states for three values of the disorder strength W. Notice the presence of a very sharp peak in the IPR distribution which we find to scale as $\mathrm{IPR}_\alpha \sim L^{-1}$ for $W/J_0 = 2$ and 4, suggesting that the *mode* of the IPR distribution corresponds to extended states [39]. We find, however, that very similar distributions (not shown) would emerge—for the same disorder strength—when the driving protocol is topologically *trivial*.

To better analyze the size dependence of the IPR peak, and its correlation with the topology of the driving, we estimate a characteristic localization length for a chain of size L from the inverse of the peak's position in the IPR distribution (inverse of the mode), $\bar{\xi}_L(W) = \{\mathrm{argmax}[P(\mathrm{IPR}_\alpha)]\}^{-1}$. Figures 2(b) and 2(c) show the size-scaling of $\bar{\xi}_L(W)$ for a trivial and topological driving, respectively. When the driving is trivial, our data suggest that $\bar{\xi}_L(W)$ scales as $W^{-\beta}$ with $\beta \simeq 2.5$ for a large W, see Fig. 2(b), while it saturates to the system size at $\sim L$ when W is small. Hence we can extract a crossover disorder strength $W^* \sim L^{-1/\beta}$ separating these two regimes, vanishing in the thermodynamic limit: here truly extended Floquet states appear only at zero disorder. By rescaling the data, we see a very good collapse of $\bar{\xi}_L(W)/L$ vs $L^{1/\beta}W$ [Fig. 2(b), inset]. On the other hand, when the driving is topological, the same phenomenology holds with a *finite critical disorder strength* W_c [Fig. 2(c)]: For $W > W_c \simeq 3.5J_0$, we observe that $\bar{\xi}_L(W) \sim (W - W_c)^{-\beta}$, with $\beta \simeq 2$, while again the localization length saturates to L when $W < W_c$, thus indicating the presence of an actual localization-delocalization phase transition. The critical exponent is in good agreement with bosonization

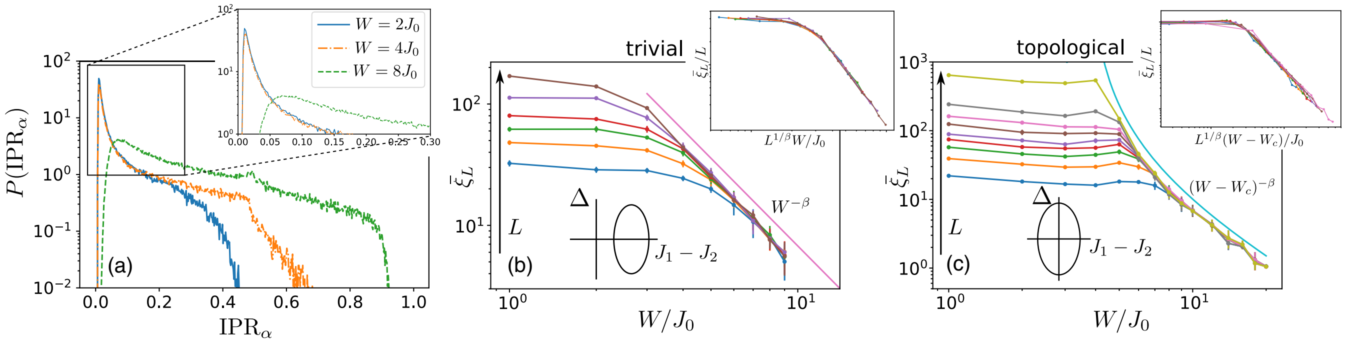

FIG. 2. (a) Disorder averaged IPR distribution of SP Floquet states for several values of W and $L = 280$. The inset shows the sharp peaks almost superimposed at small IPR for $W = 2J_0$ and $W = 4J_0$. (b) Characteristic localization length $\bar{\xi}_L(W)$ as a function of disorder, for a trivial driving cycle $[J_1 = 4J_0 + \delta_0 \sin(\omega t)]$. The inset shows the collapse obtained by $\bar{\xi}_L \to \bar{\xi}_L/L$ and $W \to WL^{1/\beta}$, with $\beta \simeq 2.5$. (c) Characteristic localization length when the system exhibits topological transport $[J_1 = J_0 + \delta_0 \sin(\omega t)]$. The inset shows the collapse $\bar{\xi}_L \to \bar{\xi}_L/L$ and $W \to (W - W_c)L^{1/\beta}$, with $W_c \simeq 3.5J_0$ and $\beta \simeq 2$. The parameters used in the simulation are $\delta_0 = 0.5J_0$, $\Delta_0 = 1.5J_0$, $\hbar\omega = 0.01J_0$.

calculations [40], while the value of W_c extracted by our scaling analysis is compatible with the breaking of quantization in Fig. 1.

To understand the mechanism behind the delocalization-localization transition, we study the relation between the time-averaged energy of the SP Floquet states $\langle E \rangle_\alpha = \frac{1}{\tau}\int_0^\tau dt \langle \phi_\alpha(t)|\hat{H}(t)|\phi_\alpha(t)\rangle$ and the corresponding IPR_α (Fig. 3). For weak disorder, extended states carrying charge in the positive (negative) direction lays in the middle of the lower (higher) band, while localized ones stay closer to the edges.

Floquet states from different bands are separated in energy by a gap—closely related to the instantaneous energy gap—and by a mobility edge. The presence of localized states at the band edges suggests that the driving-induced mixing occurs mainly at the band center, as long as $W \lesssim W_c$ (see left inset of Fig. 3, with $W = 3.5J_0$). This implies an additional robustness against nonadiabatic effects—indeed, we observe that our results do not depend on the precise value of ω [32], even when the gap is almost completely closed. When $W \gg W_c$ the two bands merge into a single one where extended states transporting opposite charges hybridize into localized states (see right inset with $W = 8J_0$), and the current stops flowing.

This phenomenology is similar to what happens in integer quantum Hall effect (IQHE) in 2D systems, where there *must be* spectral regions of extended states [41–43], in order to have a nonzero quantized transverse conductivity. Also the exponent $\beta \simeq 2$, found for $W > W_c$, is in good agreement with a similar scaling analysis performed on the density of extended states in IQHE in a disordered sample [44]. Even though the parallelism between the physics of clean 1D topological charge pumping and 2D integer quantum Hall effect is well established [2,45,46], at the best of our knowledge this is the first time where a localization-delocalization transition in a one-dimensional driven Anderson insulator is associated to topological properties, as it happens in IQHE [42,43].

Winding of quasienergies.—In a clean system, quantized pumping corresponds to a nontrivial winding of the quasienergy of the occupied Floquet bands in k space [4,7,31,47,48]. When translational invariance is broken, a common procedure is to introduce a phase twist $\theta \in [0, 2\pi)$ between site 1 and site L and then take the average of Q_d over θ [2,4]. This operation is justified because when the state projector is exponentially localized, the dependence of observables on the twisted boundary decays exponentially with L [49]. Equation (3) can then be equivalently written as [32]

$$Q_d = \int_0^{2\pi} \frac{d\theta}{2\pi} Q_d(\theta) = \frac{\tau}{\hbar} \sum_\nu \int_0^{2\pi} \frac{d\theta}{2\pi} \mathcal{N}_\nu(\theta) \partial_\theta \mathcal{E}_\nu(\theta). \quad (4)$$

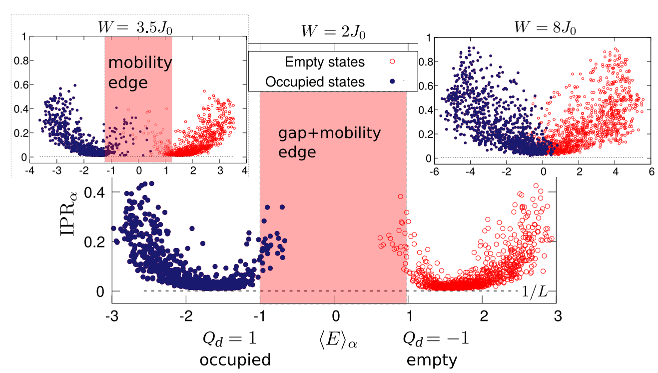

FIG. 3. IPR_α vs the time-averaged energy of the corresponding Floquet state $\langle E \rangle_\alpha$. The dashed lines indicates the value $1/L$ associated to extended states; the data refer to several realizations of a chain with 200 sites and disorder strength $W = 2J_0$. In the insets the same data are shown for a larger disorder. $Q_d = \pm 1$ is the charge transported when a single band is completely filled.

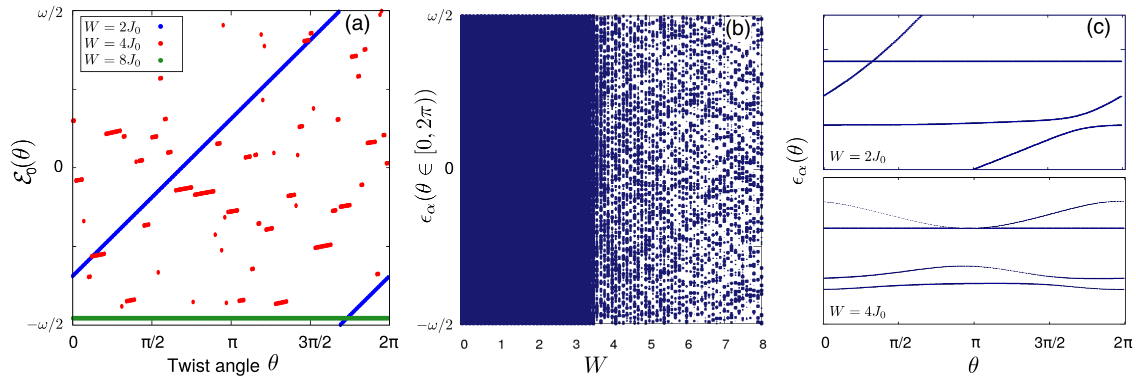

FIG. 4. (a) Quasienergy $\mathcal{E}_0(\theta)$ of the many-body Floquet state with lowest energy. The winding is well defined only for $W = 2J_0$ and $W = 8J_0$, when $|\Psi_0\rangle$ has a nonvanishing projection on a single MB Floquet state. (b) SP quasienergy spectrum for all possible angles θ as a function of the disorder. (c) typical behaviors of SP $\epsilon_\alpha(\theta)$. When $W < W_c$ most of the states are extended and sensitive to the boundary condition (upper panel), while for $W > W_c$ many quasienergies do not contribute to the winding, being periodic in θ (lower panel). All data correspond to a single disordered chain of $L = 80$ sites. The size of the dots is proportional to $n_\alpha(\theta)$.

Here $\mathcal{E}_\nu = \epsilon_{\alpha_1} + \cdots + \epsilon_{\alpha_N}$ is the N particle quasienergy associated to the Floquet state $|\Phi_\nu\rangle$ given by a Slater determinant of the SP states $|\phi_{\alpha_1}\rangle, \ldots, |\phi_{\alpha_N}\rangle$; $\mathcal{N}_\nu = |\langle\Psi_0|\Phi_\nu\rangle|^2$ is the occupation number. In this context, the winding number is the number of times that $\mathcal{E}_\nu(\theta)$ wraps around the first Floquet-Brillouin zone as θ goes from 0 to 2π. Besides nonadiabatic corrections that depend on the initial state $|\Psi_0\rangle$ [7,50], Q_d is quantized when a single many-body Floquet state is occupied, e.g., $\mathcal{N}_\nu \simeq \delta_{0,\nu}$ independently of θ, and that state has a nontrivial winding number. Henceforth we focus on the Floquet state with the lowest initial energy $|\Phi_0(\theta)\rangle$, computed as the Slater determinant of the N SP Floquet states with highest projection on the ground state in which the state is initially prepared.

We report in Fig. 4(a) $\mathcal{E}_0(\theta)$ in the first Floquet-Brillouin zone for three different disorder strengths $W/J_0 = 2, 4$, and 8, which are, respectively, below, close to the transition value, and above it. In Fig. 4(b) the SP quasienergy spectrum is plotted with respect to W as θ spans the interval $\in [0, 2\pi)$, while Fig. 4(c) shows some details of $\epsilon_\alpha(\theta)$ that help to understand the localization transition. A localized state is characterized by a quasienergy ϵ_α *periodic* in θ, while extended ones with positive winding satisfy the relation $\epsilon_\alpha(2\pi) = \epsilon_{\alpha+1}(0)$. Hence we distinguish between three situations. (i) $W < W_c$: $|\Phi_0\rangle$ coincides essentially with $|\Psi_0\rangle$ ($\mathcal{N}_\nu \simeq \delta_{0,\nu}$), because adiabaticity is preserved at a many-body level, and has winding number equal to 1, the blue line in Fig. 4(a). The SP quasienergy spectrum is continuous in θ and there are no gaps in the Floquet-Brillouine zone, Fig. 4(b). Most of the SP states feel the twist at the boundary, obey $\epsilon_\alpha(2\pi) = \epsilon_{\alpha+1}(0)$ and contribute to the winding of $\mathcal{E}(\theta)$ [upper panel of Fig. 4(c)]. (ii) $W \gtrsim W_c$: the initial ground state $|\Psi_0\rangle$ has relevant projections over many MB Floquet states and the SP occupation numbers n_α depend nontrivially on θ. Hence the set $\alpha_1, \ldots, \alpha_N$ of SP Floquet states in the Slater determinant $|\Phi_0\rangle$, changes with θ, thus making $\mathcal{E}_0(\theta)$ discontinuous [red dots in Fig. 4(a)]. Gaps start to appear in the SP quasienergy spectrum [Fig. 4(b)], and the occupation number itself depends nontrivially on θ. The SP Floquet states with opposite transported charge start to be mixed in pairs of localized states, with quasienergies periodic in θ [lower panel of Fig. 4(c)]. (iii) $W > W_c$: both SP Floquet states and Hamiltonian eigenstates are strongly localized and there is no current. Again $\mathcal{N}_\nu = \delta_{0,\nu}$, but the winding number is trivial [green line in Fig. 4(a)] because SP quasienergy spectrum has only a pure-point contribution and localization makes the system insensitive to the boundary twist.

Conclusions.—We analyzed in detail the steady-state current flowing in a one dimensional Floquet-Anderson insulator: the topological periodic driving mixes localized Hamiltonian eigenstates to give extended Floquet modes. The dynamics is adiabatic only at a many-body level, but not at the SP one, where driving-induced mixing of localized states occurs even at very low frequencies. Delocalization makes quantized pumping robust, until extended Floquet states with opposite winding coalesce for large disorder. Even though the physics of quantum pumping in clean systems is the same of 2D IQHE, this analogy is not trivial in the presence of disorder, since the 1D periodically driven chain would be mapped in an *extremely anisotropic* disordered 2D model.

A subtle point emerges in the adiabatic limit $\omega \to 0$. In a truly adiabatic evolution the SP, Floquet states would coincide with the Hamiltonian eigenstates, thus being localized, at least when the disorder-induced SP level crossings are actually *avoided crossings* [51]. This happens generally when one takes the adiabatic limit in a finite system, where the energy levels are protected by finite gaps, although exponentially small in L. Quantized pumping still works because disorder induces resonances in the spectrum, allowing for large distance tunneling [53,54].

If the thermodynamic limit $L \to \infty$ is taken first, the spectrum becomes *dense* and the driving mixes SP levels at an arbitrarily small frequency, leading again to a quantized pumping. These arguments suggest that quantized pumping is obtained regardless of which of the two limits ($\omega \to 0$ or $L \to \infty$) is taken first, although the physical mechanisms are different.

Finally, we point out that the interplay between disorder, topology (and possibly interaction) in Floquet systems can be investigated in cold atom experiments, where a disordered or quasiperiodic potential can be easily engineered [55].

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