Editors' Suggestion

Positivity of Amplitudes, Weak Gravity Conjecture, and Modified Gravity

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We derive new positivity bounds for scattering amplitudes in theories with a massless graviton in the spectrum in four spacetime dimensions, of relevance for the weak gravity conjecture and modified gravity theories. The bounds imply that extremal black holes are self-repulsive, $M/|Q| < 1$ in suitable units, and that they are unstable to decay to smaller extremal black holes, providing an S-matrix proof of the weak gravity conjecture. We also present other applications of our bounds to the effective field theory of weakly broken Galileons, axions, and $P(X)$ theories.

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Introduction.—The scattering matrix, or S matrix, is undoubtedly one of the most important observables in particle as well as gravitational physics, relating asymptotic states at past and future infinity. It has long been known that its general properties, unitarity, analyticity, and crossing symmetry, imply dispersion relations between forward elastic scattering amplitudes (i.e., at vanishing scattering angle) and total cross sections. Such relations in turn yield positivity bounds for amplitudes evaluated in the infrared (IR), those measured in experiments. Dispersion relations therefore provide nontrivial constraints on the operators' coefficients in the effective field theories (EFTs) that are used to calculate the amplitudes at low energy [1,2]. An EFT with operators entering the action with the "wrong" sign cannot arise as the low-energy limit of a consistent ultraviolet (UV) theory satisfying the S-matrix axioms, and thus it lives in the "swampland." The proof of the a theorem [3,4] is perhaps the prime example of an application of these positivity bounds.

In this Letter, new amplitudes' positivities are derived for theories with a massless graviton in the spectrum, despite the fact that the forward elastic 2-to-2 scattering is universally singular due to graviton exchange in the t channel. The new positivity bounds, and the way we circumvent the graviton forward singularity, are extremely important because they allow us to address the swampland program of quantum gravity and modified gravity theories, providing general and robust results.

As a notable application, we study the Einstein-Maxwell theory, the low-energy EFT of an Abelian $U(1)$ gauge theory coupled to gravity, and show that our positivity bounds imply certain inequalities among its leading higher-dimensional operators. Such operators are particularly relevant in the context of black hole physics—they affect the black hole's extremality condition, the minimal mass for which a charged black hole can exist. Our bounds imply that extremal black holes of mass M and $U(1)$ charge Q must satisfy $\sqrt{2}m_{\rm Pl}|Q|/M > 1$; thus they are self-repulsive and no longer kinematically forbidden from decaying into smaller extremal black holes. This result constitutes a proof of (the mild form of) the celebrated weak gravity conjecture (WGC) [5]: extremal black holes are themselves charged states in the theory for which gravity is the weakest force.

Another interesting application of our bounds is on weakly broken Galileons, hypothetical scalars whose special properties make them very interesting candidates for a modified theory of gravity. We show, however, that these states have in fact a tiny cutoff if they are to originate from a canonical microscopic S matrix, $\Lambda_{\rm UV} <$ a few$\times (H^3 m_{\rm Pl})^{1/4} \sim 1/(10^7\ {\rm km})$; i.e., their EFT breaks down at energies orders of magnitude smaller than the scale characterizing their interactions, the so-called strong coupling scale $\Lambda_3 = (H^2 m_{\rm Pl})^{1/3} \sim 1/(10^3\ {\rm km})$. Finally, we comment on how the EFTs of axions and $P(X)$ theories, of relevance in cosmology, are also subject to our constraints.

Regulating the forward limit.—The forward elastic amplitude of massless particles of polarizations z_i is dominated by the universal Coulomb singularity

$$\mathcal{M}^{z_1 z_2}(s, t \to 0) = -\frac{s^2}{m_{\rm Pl}^2 t} + O(s) \tag{1}$$

because of the equivalence principle or, equivalently, because of factorization of the amplitude at the pole into the soft emission of an on-shell massless graviton, which has universal strength given by the reduced Planck mass $m_{\rm Pl}$. Since the coefficient of s^2 in Eq. (1) would enter the dispersion relation in the forward limit for particles of any spin z_i [2], see Eq. (3), the naive application of the Cauchy integral theorem to $\mathcal{M}^{z_1 z_2}(s, t \to 0)$ yields $\infty = \infty$, which is consistent, but admittedly not very informative. However, this divergence is due to long-distance physics, i.e., vanishing exchanged momentum, corresponding to the graviton probing arbitrarily large macroscopic distances even for large center-of-mass energy squared s. In any EFT the presumption is that the IR physics is known; therefore, one should be able to track and resolve the source of the singularity. Indeed, we show below how to massage the dispersion relation into an effective, regulated expression $\infty - \infty = \text{finite} > 0$, returning something meaningful, free of ambiguity and in fact of a definite sign, which can be used for charting the swampland in gravitational theories.

The key observation is that the Coulomb singularity is due to the infinite flat-space volume: we regulate it by putting the system on a cylinder. We compactify one spatial direction on a circle of length L, while the other three spacetime dimensions remain flat and infinite. In this way we can still scatter 3D asymptotic states while at the same time getting rid of the Coulomb singularity. Indeed, there is no propagating massless graviton in $D = 3$, and hence no s^2/t term for any finite value of L [6]. The 4D graviton has not fully disappeared though—rather it has left three propagating avatars: $\hat{g}_{MN} \to \{\sigma, V_\mu, \text{KK modes}\}$, where σ is a massless dilaton, V_μ a massless (Abelian) graviphoton, and the Kaluza-Klein (KK) modes have masses $m_n^2 \sim n^2/L^2$. In the limit $L \to \infty$, which we take at the end after isolating the diverging terms, one recovers the 4D dynamics we are interested in.

In the following we will be interested in scattering nongravitational massless states, for example, the 3D photon A_μ and the 3D scalar Φ that live inside the 4D photon $\hat{A}_M$, in the 4D Einstein-Maxwell theory reduced to 3D. Since there is no 3D massless graviton and the states are either gapped, nonpropagating, or equivalent to simple scalars, we require a 3D Froissart-like bound

$$\lim_{s\to\infty} |\mathcal{M}^{z_1 z_2}(s, t = 0)/s^2| \to 0, \qquad z_i = \Phi, A, \tag{2}$$

where with a slight abuse of notation we are now using z_i to label the scattered 3D states. This is just the same assumption of polynomial boundedness that one accepts in 4D to derive dispersion relations and positivity bounds for the coefficient of, e.g., $(\partial\pi)^4$ or $(F_{\mu\nu}F^{\mu\nu})^2$, when 4D gravity is neglected or nondynamical [1,15].

Therefore, under exactly the usual assumptions that lead to the familiar positivity bounds for systems of spin-0 and spin-1 massless particles in 4D, and repeating similar steps to those outlined in, for example, Ref. [2], we obtain a (provisional) dispersion relation for our IR-regulated 4D gravitational theory

$$a^{z_1 z_2} = \frac{2}{\pi}\int_0^\infty \frac{ds}{s^3} \mathrm{Im}\mathcal{M}^{z_1 z_2}(s, t = 0) > 0, \tag{3}$$

where the low-energy scattering amplitude for the 3D states z_i is now regular in the forward elastic limit

$$\mathcal{M}^{z_1 z_2}(s, t \to 0) = a^{z_1 z_2} s^2 + \cdots. \tag{4}$$

The dimensional reduction to 3D has left a universal contribution from gravitational zero and KK modes. Each KK mode gives

$$a_{\rm KK}^{z_1 z_2} \propto \frac{1}{L^2 m_{\rm Pl}^4 m_{\rm KK}} \propto \frac{1}{L m_{\rm Pl}^4 |n|}, \tag{5}$$

where we used the fact that the nth KK-mode mass is $m_{\rm KK} \propto |n|\pi/L$. While each such contribution is subleading with respect to the terms that we want to bound in the following sections, their sum is actually logarithmically divergent. In addition, zero-mode loops generate $s^{3/2}$ terms in the amplitude, which dominate over the s^2 terms at low energy, seemingly swamping again the information about $a^{z_1 z_2}$. In fact, these problems can be easily solved because the right-hand side of the dispersion relation (3) reproduces the same growth, so these otherwise large terms cancel out between the two sides of Eq. (3). Indeed, since the integrand itself in Eq. (3) is positive by the optical theorem, schematically $\mathrm{Im}\mathcal{M}^{z_1 z_2}(s,t{=}0) = \sum_x |\mathcal{M}^{z_1 z_2 \to x}|^2 \times (\text{phase space})$, we can move to the left-hand side any contribution from intermediate states x in $|\mathcal{M}^{z_1 z_2 \to x}|^2$ and still get a positivity bound due to the remaining set of intermediate states. Specifically, we can move to the left-hand side the contributions from the intermediate IR states, such as the KK modes or anything that is calculable within the EFT (e.g., IR loops, that is, the light multiparticle intermediate states). The zero- and KK-mode contributions get subtracted and one is left to calculate just the contact terms suppressed by the cutoff $\Lambda_{\rm UV}$, that is, those that are generated by integrating out genuine UV states.

Just to illustrate this general point with a simple tree-level example, let us consider $\Phi\Phi \to \Phi\Phi$ scattering with the exchange of a scalar state S coupled to $(\partial\Phi)^2$,

$$\mathcal{M}_S^{\Phi\Phi}(s, t) = -\frac{2c}{m_{\rm Pl}^2 L}\left(\frac{s^2}{s - m_S^2 + i\epsilon} + \text{crossing}\right), \tag{6}$$

where c is a fixed $O(1)$ number. This contributes to $a^{z_1 z_2}$ in Eq. (4) by an amount $a_S^{\Phi\Phi} = 4c/(m_{\rm Pl}^2 L m_S^2)$. The imaginary part (associated with the production of S) is

$$\mathrm{Im}\mathcal{M}_S^{\Phi\Phi}(s,t=0)=\frac{2\pi c}{m_{\mathrm{Pl}}^2 L}m_S^4\delta(s-m_S^2)+\cdots,\quad (7)$$

precisely such that

$$a_S^{\Phi\Phi}-\frac{2}{\pi}\int_0^\infty\frac{ds}{s^3}\mathrm{Im}\mathcal{M}_S^{\Phi\Phi}(s,t=0)=0,\quad (8)$$

as expected on general grounds.

The KK-mode contributions to $a^{z_1z_2}$ in Eq. (5) actually arise at one loop, but the reasoning based on the optical theorem is completely general and works as in the previous example. This can be understood by discretizing the KK branch cut in a series of poles. Likewise for the contribution of the zero modes. The concrete details of how these contributions are subtracted are given in the Supplemental Material [7]. Here we note only that the KK modes, which grow the "extra" dimension as seen from a low-energy 3D observer, reproduce nicely the 4D universal gravitational contribution to the renormalization group running of $a^{z_1z_2}$. Since we can subtract it, which amounts to setting the renormalization scale at which $a^{z_1z_2}$ is evaluated at the cutoff where UV and IR amplitudes are matched, our final dispersion relation properly captures the UV physics that we are interested in.

All in all, our provisional dispersion relation (3) is rearranged into a much more informative expression

$$a^{z_1z_2}-a^{z_1z_2}_{\mathrm{KK,IR}}=\frac{2}{\pi}\int_0^\infty\frac{ds}{s^3}\mathrm{Im}\tilde{\mathcal{M}}^{z_1z_2}(s,t=0)>0,\quad (9)$$

where $\tilde{\mathcal{M}}$ is the amplitude with the aforementioned gravitational zero- and KK-mode loop contributions subtracted. The left-hand side is therefore obtained by taking into account only the s^2 contributions to the elastic z_1z_2 scattering due to the tree-level interactions with massless particles such as the graviphoton and the dilaton, as well as the UV generated contact terms. The two sides (factor L^{-1}) of the subtracted dispersion relation (9) are not only finite for $L\to\infty$ but also positive because of the optical theorem. We note that removing the IR modes from the positivity bound is always possible but is useful in practice only for UV completions that are not strongly coupled at Λ_{UV}, because it would become murky to assign what is IR (KK) and what is UV physics around the scale Λ_{UV}. The subtracted dispersion relation is instead sharp and useful for weakly coupled UV completions.

Einstein-Maxwell EFT.—Let us focus on the important example of the Einstein-Maxwell EFT, whose leading 4D operators are

$$S=\int d^4x\sqrt{|\hat{g}|}\left[\frac{m_{\mathrm{Pl}}^2}{2}\hat{R}-\frac{1}{4}\hat{F}^{MN}\hat{F}_{MN}+\frac{\alpha_1}{4m_{\mathrm{Pl}}^4}(\hat{F}^{MN}\hat{F}_{MN})^2+\frac{\alpha_2}{4m_{\mathrm{Pl}}^4}(\hat{\tilde{F}}^{MN}\hat{F}_{MN})^2+\frac{\alpha_3}{2m_{\mathrm{Pl}}^2}\hat{F}_{AB}\hat{F}_{CD}\hat{W}^{ABCD}\right],\quad (10)$$

where $\hat{W}^{ABCD}$ is the Weyl tensor and $\hat{\tilde{F}}_{MN}=\epsilon_{MNAB}\hat{F}^{AB}/2$. The dependence on the UV scale Λ_{UV} that generates the α_i is absorbed into their definitions. These are the most general (parity preserving) four-derivative operators, up to field redefinitions [17,18]. In order to regulate the 4D forward limit and apply the positivity bounds (9), we compactify the z direction as described in the previous section,

$$d\hat{s}_4^2[\hat{g}_{MN}]=e^{\sigma}ds_3^2[g_{\mu\nu}]+e^{-\sigma}(dz+V_\mu dx^\mu)^2,\quad (11)$$

$$\hat{A}_M dx^M=A_\mu dx^\mu+\Phi dz,\quad (12)$$

where all of the 3D fields are functions only of (t,x,y). Focusing on terms which contribute to the s^2 part of the amplitude for $\Phi\Phi\to\Phi\Phi$, $AA\to AA$, and $\Phi A\to\Phi A$ only, the terms in the action that we must retain are

$$S=L\int d^3x\sqrt{-g}\left\{\frac{m_{\mathrm{Pl}}^2}{2}\left(R-\frac{1}{2}(\partial\sigma)^2-\frac{1}{4}V^2\right)-\frac{1}{4}(1-\sigma)F^2-\frac{1}{2}(1+\sigma)(\partial\Phi)^2-\frac{1}{2}F_{\mu\nu}V^{\mu\nu}\Phi\right.$$
$$+\frac{\alpha_1}{4m_{\mathrm{Pl}}^4}(F^2+2(\partial\Phi)^2)^2+\frac{\alpha_2}{m_{\mathrm{Pl}}^4}(\epsilon^{\mu\nu\rho}F_{\mu\nu}\partial_\rho\Phi)^2+\frac{\alpha_3}{m_{\mathrm{Pl}}^4}\left[F_{\rho\mu}F^{\rho\nu}F^{\mu\sigma}F_{\nu\sigma}-\frac{1}{2}F^4-(\partial\Phi)^4+\frac{1}{2}F^2(\partial\Phi)^2\right]$$
$$\left.-\frac{\alpha_3}{m_{\mathrm{Pl}}^2}(F_{\rho\mu}F^{\rho}{}_{\nu}-\partial_\mu\Phi\partial_\nu\Phi)\nabla^\mu\nabla^\nu\sigma-\frac{\alpha_3}{m_{\mathrm{Pl}}^2}F_{\mu\nu}\partial_\rho\Phi(\nabla^\rho V^{\mu\nu}+g^{\mu\rho}\nabla_\alpha V^{\nu\alpha})\right\},\quad (13)$$

where $F^2=F_{\mu\nu}F^{\mu\nu}$, and the same holds for V. To arrive at this expression, we made a field redefinition $A_\mu\to A_\mu+\Phi V_\mu$ to make gauge invariance manifest. Since the $g_{\mu\nu}$ propagates no degrees of freedom in $D=3$, we integrated it out, which is effectively equivalent to plugging the lowest-order equations of motion $R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R=T^{\mu\nu}/(Lm_{\mathrm{Pl}}^2)$ into the interaction terms, generating new contact terms. Finally, we made a further field redefinition to remove interaction terms with $\Box\sigma$.

The associated subtracted forward elastic scattering amplitudes are

$$\tilde{\mathcal{M}}(\Phi\Phi \to \Phi\Phi)(s, t=0) = \frac{2s^2}{m_{\mathrm{Pl}}^4 L}(2\alpha_1 - \alpha_3) > 0, \quad (14)$$

$$\tilde{\mathcal{M}}(AA \to AA)(s, t=0) = \frac{2s^2}{m_{\mathrm{Pl}}^4 L}(2\alpha_1 + \alpha_3) > 0, \quad (15)$$

$$\tilde{\mathcal{M}}(\Phi A \to \Phi A)(s, t=0) = \frac{4s^2}{m_{\mathrm{Pl}}^4 L}\alpha_2 > 0. \quad (16)$$

Therefore, the associated positivity bounds read

$$2\alpha_1 - \alpha_3 > 0, \quad (17)$$

$$2\alpha_1 + \alpha_3 > 0, \quad (18)$$

$$\alpha_2 > 0, \quad (19)$$

or, equivalently, $\alpha_1 > |\alpha_3|/2$, $\alpha_2 > 0$. These new positivity bounds are one of the main results of this Letter.

Remarkably, these bounds are stronger, meaning more general, than just a pure 4D Euler-Heisenberg EFT without gravity [1,2,19], and they carry extra information about α_3, which enters the black hole extremality condition, as we discuss in the next section.

Moreover, our homogeneous bounds (17)–(19) are distinct from the order-of-magnitude causality bounds on $O(|\alpha_3|)$ [19,20], which are derived while assuming positivity of time delay and tree-level UV completion of the Einstein-Maxwell Lagrangian. See also Refs. [20,21] for a nice discussion of detectability of superluminal propagation within an EFT, and how the Euler-Heisenberg Lagrangian limit of real-world QED avoids superluminality [22].

It is interesting to compare the bounds (17)–(19) with the 4D calculation of the same processes retaining the Coulomb singularity in the $t \to 0$ limit,

$$\mathcal{M}_{4\mathrm{D}}^{\downarrow\downarrow} = -\frac{s^2}{m_{\mathrm{Pl}}^2 t} - \frac{s}{m_{\mathrm{Pl}}^2} + \frac{2s^2(2\alpha_1 - \alpha_3)}{m_{\mathrm{Pl}}^4}, \quad (20)$$

$$\mathcal{M}_{4\mathrm{D}}^{\uparrow\uparrow} = -\frac{s^2}{m_{\mathrm{Pl}}^2 t} - \frac{s}{m_{\mathrm{Pl}}^2} + \frac{2s^2(2\alpha_1 + \alpha_3)}{m_{\mathrm{Pl}}^4}, \quad (21)$$

$$\mathcal{M}_{4\mathrm{D}}^{\uparrow\downarrow} = -\frac{s^2}{m_{\mathrm{Pl}}^2 t} - \frac{s}{m_{\mathrm{Pl}}^2} + \frac{4s^2\alpha_2}{m_{\mathrm{Pl}}^4}, \quad (22)$$

where the up and down arrows represent the two choices of real linear polarizations (corresponding to crossing symmetric amplitudes [2,23]). The lesson is that our 4D-regulated calculation, which works with 3D Lorentz invariance of the cylinder, teaches us which finite parts we are allowed to retain for the positivity bounds: throw away the s^2/t singularity, the finite $O(s)$ term, but retain precisely the $O(s^2)$ term.

This immediately prompts us to expect a continuous set of positivity bounds associated with arbitrary linear polarizations $|c_{1,2}\rangle = (c_{\theta_{1,2}}|\uparrow_{1,2}\rangle + s_{\theta_{1,2}}|\downarrow_{1,2}\rangle)$, namely,

$$\alpha_3(c_{2\theta_1} + c_{2\theta_2}) + 4\alpha_1 c^2_{\theta_1+\theta_2} + 4\alpha_2 s^2_{\theta_1+\theta_2} > 0, \quad (23)$$

where $c_\theta = \cos\theta$ and $s_\theta = \sin\theta$.

WGC and extremal black holes.—The leading higher-dimensional corrections α_i in the 4D Einstein-Maxwell EFT (10) modify the black hole extremality condition to [24]

$$\left(\frac{\sqrt{2}|Q|}{M/m_{\mathrm{Pl}}}\right)_{\mathrm{extr}} = 1 + \frac{4}{5}\frac{(4\pi)^2 m_{\mathrm{Pl}}^2}{M^2}(2\alpha_1 - \alpha_3) > 1, \quad (24)$$

where M is the black hole mass and Q its charge (including the gauge coupling), and we work around $M \simeq Q m_{\mathrm{Pl}}\sqrt{2}$. Remarkably, on the right-hand side of this expression, one finds the same combination, $2\alpha_1 - \alpha_3$, bounded to be positive by Eq. (17). Therefore, positivity bounds imply a greater charge-to-mass ratio for extremal black holes than in pure general relativity coupled minimally to an Abelian $U(1)$ gauge theory. The lighter the extremal black hole, the larger the charge-to-mass ratio. Extremal black holes within the validity of the 4D EFT, i.e., whose Schwarzschild radius $r_s = M/4\pi m_{\mathrm{Pl}}^2$ is larger than $1/\Lambda_{\mathrm{UV}}$, are therefore self-repulsive.

The positivity bound (17) implies the mild form of the WGC [5], which posits that a consistent theory of quantum gravity must contain massive charged states in the spectrum with $|q| > m/(\sqrt{2}m_{\mathrm{Pl}})$: the extremal black holes of Eq. (24) are such states. As a result, the paradox of stable extremal black holes has evaporated since extremal black holes are no longer kinematically forbidden to decay. Indeed, an extremal black hole of mass M and charge Q cannot decay into states that all have larger mass-to-charge ratio since the spectrum of masses and charges (m_i, q_i) is constrained by $M > \sum_i m_i$ and $Q = \sum_i q_i$, whereas $\sum_i m_i = \sum_i |q_i| m_i/|q_i| > M$, which would be a contradiction. This argument is evaded precisely by decay products that contain one smaller extremal black hole, which has smaller mass-to-charge ratio (24) because of the positivity bound (17).

Since the same combination of EFT coefficients, $2\alpha_1 - \alpha_3$, enters the Wald entropy shift [18,19], our positivity bound (17) implies a larger black hole entropy as well. (Positivity bounds on even higher-derivative terms [23] imply a positive shift of the Kerr black hole entropy [25].)

Bounds on Galileons, axions, and $P(X)$.—Our IR-regulated positivity bounds can now be used to constrain other interesting theories that have a massless graviton in the spectrum, e.g., those that are formulated in the context of modified gravity such as the weakly broken Galileons [26,27]

$$\mathcal{L} = -\frac{1}{2}(\partial\pi)^2 - \frac{1}{2\Lambda_3^3}(\partial\pi)^2\Box\pi + \frac{1}{4\Lambda_2^4}(\partial\pi)^4 + \cdots, \quad (25)$$

where one can imagine the natural situation where $\Lambda_2 \gg \Lambda_3$ since $(\partial\pi)^4$ weakly breaks the Galilean symmetry, whereas $(\partial\pi)^2\Box\pi$ is an invariant. It was shown indeed that the hierarchy

$$\Lambda_2^4 \simeq H^2 m_{\rm Pl}^2, \qquad \Lambda_3^3 \simeq H^2 m_{\rm Pl}, \quad (26)$$

where H is the Hubble constant, is stable under the loop corrections due to gravity that break the Galileon symmetry [27].

However, as estimated in Refs. [2,28] and calculated in detail in Ref. [29], scales Λ_2 and Λ_3 cannot be arbitrarily separated while keeping the cutoff $\Lambda_{\rm UV}$ fixed in a theory without gravity, parametrically $1/\Lambda_2^4 > \Lambda_{\rm UV}^8/\Lambda_3^{12}$. We can now see that a similar bound survives even when the graviton is dynamical if the UV completion is assumed to be weakly coupled (at least up to Λ_2). Following the arguments of the previous sections, we can subtract the gravitational KK modes after the 3D compactification, and we then extract the following bound,

$$\frac{1}{\Lambda_2^4 L} > \frac{2}{\pi}\int^{\Lambda_{\rm UV}^2}\frac{ds}{s^3}\,{\rm Im}\tilde{\mathcal{M}}^{\pi\pi}(s) > \frac{c}{16\pi^2}\frac{\Lambda_{\rm UV}^8}{L\Lambda_3^{12}}, \quad (27)$$

where we used the positivity of the integrand in Eq. (9) and retained from the optical theorem only the inelastic cross section into pairs of Galileon KK modes π_k, $\sum_{k,m_k<\Lambda_{\rm UV}}\sigma(\pi\pi\to\pi_k\pi_k)$. The constant $c = O(10^{-4})$ is an inessential numerical factor resulting from integrating over the phase space and then along the branch cut. The bound (27) nicely reproduces the scaling from the calculation without gravity [29]. As a consequence, hierarchy (26) between Λ_2 and Λ_3, which is stable because of symmetry, in fact requires an extremely small cutoff,

$$\Lambda_{\rm UV} < (H^3 m_{\rm Pl})^{1/4}\left(\frac{16\pi^2}{c}\right)^{1/8} \sim \frac{1}{10^7\ {\rm km}}, \quad (28)$$

in order to be consistent with the beyond positivity bound (27) that applies in a gravity theory.

We can also consider

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{a}{4f^4}(\partial\phi)^4 + \cdots, \quad (29)$$

describing axions, or $P(X)$ theories with $X = (\partial\phi)^2$ and $f \sim \Lambda_2$ in Eq. (26). Our positivity bounds imply that $a > 0$, even in the presence of gravity—such a constraint was not possible before because of the inherent long-distance sensitivity, $t \sim H^2$, of their forward scattering amplitudes.

Conclusions.—In this Letter we derived new amplitudes' positivities in quantum gravity in four dimensions. We showed how to regulate and subtract the gravitational Coulomb singularity in the forward elastic limit by putting the theory on a cylinder, using its 3D scattering states and then restoring 4D spacetime. This method allowed us to extract positivity bounds on the s^2 coefficient of the EFT amplitudes removing the t-channel graviton singularity in a controlled way. Remarkably, the resulting positivity bounds are generically different than those obtained in flat space without gravity. This is due to the contribution to the amplitudes from the dilaton and the graviphoton (on top of the contact terms from the nondynamical metric), which remain dynamical even on the cylinder and leave their finite gravitational footprint in the 4D limit.

As an important application, we studied the Einstein-Maxwell EFT and showed that the positivity bounds imply that extremal black holes have a charge-to-mass ratio larger than 1, which is approached from above as the mass is increased. This provides an S-matrix proof of the mild form of the WGC since it implies that extremal black holes are self-repulsive, $|Q| > M$ in suitable units. The amplitudes' positivity implies as well that the Wald entropy shift due to the leading higher-dimensional operators is always positive.

In the context of the swampland program, these are perhaps somewhat negative results since they lower the expectations that the WGC is useful for charting the landscape of consistent theories of quantum gravity. *Any* weakly coupled UV completion with a canonical S matrix gives rise to unstable extremal black holes. Of course, it may be that string theory is the only such UV completion.

We considered other important applications in cosmology within the context of modified gravity. In particular, we found that perturbative UV completions of Galileons can be consistent with the (beyond) positivity bounds in a theory with a massless graviton only if the cutoff of the theory is at least as small as a few $\times\,(H^3 m_{\rm Pl})^{1/4}$.

This work opens the door to a future understanding of other quantum gravity conjectures in terms of amplitudes' positivity.

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[7] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.123.251103 for a detailed discussion of the contributions to the dispersion relation from the zero and KK modes (see also Ref. [14]), and of some of the features of 3D scattering amplitudes.

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