Finite Speed of Quantum Scrambling with Long Range Interactions

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In a locally interacting many-body system, two isolated qubits, separated by a large distance r, become correlated and entangled with each other at a time $t \geq r/v$. This finite speed v of quantum information scrambling limits quantum information processing, thermalization, and even equilibrium correlations. Yet most experimental systems contain long range power-law interactions—qubits separated by r have potential energy $V(r) \propto r^{-\alpha}$. Examples include the long range Coulomb interactions in plasma ($\alpha = 1$) and dipolar interactions between spins ($\alpha = 3$). In one spatial dimension, we prove that the speed of quantum scrambling remains finite for sufficiently large α. This result parametrically improves previous bounds, compares favorably with recent numerical simulations, and can be realized in quantum simulators with dipolar interactions. Our new mathematical methods lead to improved algorithms for classically simulating quantum systems, and improve bounds on environmental decoherence in experimental quantum information processors.

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Almost five decades ago, Lieb and Robinson proved that spatial locality implies the ballistic propagation of quantum information [1]. Intuitively defining a "scrambling time" $t_s(r)$ by the time at which an initially isolated qubit can significantly entangle with another a distance r away, locality implies that $t_s(r) \gtrsim r$. This result has deep implications in physics. Practical tasks such as information processing [2] are possible due to a lack of rapid decoherence with a noisy environment, thermalization occurs locally [3] and equilibrium correlation functions fall off sufficiently rapidly [4]. If quantum information can only propagate with a finite speed, a classical computer can efficiently approximate early time quantum dynamics [5]. Despite the exponentially large Hilbert space in many-body quantum systems, quantum information processors with short-range interactions cannot become entangled with an infinite environment arbitrarily quickly [6,7]. Lastly, emergent spacetime locality arising from microscopic quantum mechanics without manifest relativistic invariance may play a crucial role in understanding quantum gravity through the holographic correspondence [8].

However, the Lieb-Robinson theorem is not useful for a typical quantum information processor. A qubit in an experimental device is usually a spin or atomic degree of freedom, or Josephson junction. Such objects generically interact with long range interactions, and until now, whether locality of quantum scrambling necessarily persists in the presence of long range interactions has remained unclear. In 2005, Hastings and Koma used the canonical Lieb-Robinson theorem to prove that when $\alpha > d$, $t_s(r) \gtrsim \log r$ [4]; more recently, this bound has been improved for $\alpha > 2d$ to $t_s(r) \gtrsim r^{(\alpha-2d)/(\alpha-d)}$ [9–12]. If such bounds were tight, then insulating a quantum processor from its environment would be absolutely crucial. Yet numerical simulations cast into doubt the tightness of these formal bounds: two groups have recently shown that $t_s \gtrsim r$ in one-dimensional models with $\alpha \gtrsim 1.5$ [13] or even $\alpha > 1$ [14], depending on microscopic details.

In this Letter, we prove that $t_s(r) \gtrsim r$ whenever $\alpha > 3$, in all one-dimensional models with power law interactions. Our dramatic improvement over existing results is made possible by new mathematics [15]: identities for unitary time evolution expanded as a sum over flexibly chosen equivalence classes of sequences of couplings.

Our work has clear physical consequences. Scrambling in dipolar spin chains [16,17] is hardly faster than in a spin chain with nearest neighbor interactions; hence, it should be far more efficient to simulate numerically [5,10]. Nor does decoherence seriously limit the quantum information processing capabilities of a nuclear spin chain, no matter how large the environment. Quantum thermalization nearly proceeds as if interactions were local, as in typical theoretical models of scrambling [18,19].

Formal statement of theorem.—We now formally restate our theorem in a mathematically precise language. For simplicity, we will assume a one-dimensional chain of qubits (two-level systems); the generalization to all finite-dimensional quantum models in one dimension is contained in the Supplemental Material [20]. Thus, the Hilbert space is given by

$$\mathcal{H} = \bigotimes_{i\in\mathbb{Z}} \mathcal{H}_i = \bigotimes_{i\in\mathbb{Z}} \mathbb{C}^2. \tag{1}$$

Even though $\mathcal{H}$ is (uncountably) infinite dimensional, our bound on scrambling will reduce to a calculation on a finite segment of the chain.

The set of Hermitian operators on $\mathcal{H}$ forms a real vector space $\mathcal{B}$. Let the $U(2)$ generators $\{I, \sigma^x, \sigma^y, \sigma^z\}$ be our complete basis of Hermitian operators on $\mathcal{H}_i$; $\mathcal{B}$ is spanned by tensor products $\{I, \sigma_i, \sigma_i\sigma_j, \cdots\}$. Here and below, we can use a "bra-ket" notation with parentheses to emphasize that Hermitian operators on $\mathcal{H}$ are vectors in $\mathcal{B}$. We define $\|A\|$ as the maximal eigenvalue of A, the conventional operator norm [1].

We consider two-local Hamiltonians: i.e., those which may be expressed as a sum of terms that act on either a single site, or on two sites:

$$H = \sum_{i\in\mathbb{Z}} H_i + \sum_{i<j} H_{ij}. \tag{2}$$

We define the exponent α of long range interactions by demanding that

$$\|H_{ij}\| \le \frac{h}{|i-j|^\alpha}. \tag{3}$$

Let $i < j$ be integers. We define the scrambling time $t_s^\delta(r)$ to be the largest time such that

$$\sup_{A_{\le i}, B_{\ge j}} \frac{\|[A_{\le i}(t'), B_{\ge j}]\|}{\|A_{\le i}\|\|B_{\ge j}\|} < \delta, \quad \text{for } 0 < |t'| < t_s^\delta(|i-j|), \tag{4}$$

where $A_{\le i}$ denotes a bounded operator that acts trivially on any site $k > i$ and $B_{\ge j}$ acts trivially on any site $k < j$. While $A_{\le i}$ can act nontrivially on an infinite number of sites, we demand $B_{\ge j}$ acts nontrivially only on a finite number of sites [20]. Lastly, the operator $A_{\le i}(t') \coloneqq e^{iHt'}A_{\le i}e^{-iHt'}$ denotes a Heisenberg time-evolved operator. The definition of scrambling given in Eq. (4) bounds the growth in observable correlation functions, and the generation of entanglement between distant qubits [6,7]. We are now ready to state our main result:

Theorem 1.—For every $0 < \delta < 2$, there exists a constant $0 < K_\alpha < \infty$ for which

$$t_s^\delta(r) \ge K_\alpha \times \begin{cases} r^{\alpha-2} & 2 < \alpha < 3 \\ r(\log r)^{-2} & \alpha = 3 \\ r & \alpha > 3 \end{cases}. \tag{5}$$

Sketch of proof.—We now outline the proof of Theorem 1; details are found in Ref. [20]. For simplicity, we set $i = 1$ and $j = r$ in Eq. (4). We write A_1 and B_r below as shorthand for $A_{\le 1}$ and $B_{\ge r}$.

In the Heisenberg picture of quantum mechanics, operators evolve according to $\partial_t\mathcal{O} = i[H, \mathcal{O}]$. Just like the Schrödinger equation, this is *linear*: we write $\partial_t|\mathcal{O}) = \mathcal{L}|\mathcal{O})$ where $\mathcal{L}$, commutation with Hamiltonian H, generates time translations on the space of operators. The time evolved operator $|\mathcal{O}(t)) = e^{\mathcal{L}t}|\mathcal{O})$ is nothing more than a "rotated" operator of the same norm. We define the projection $\mathbb{P}_r$ onto the hyperplane Σ_r of $\mathcal{B}$ of all operators that act nontrivially on the support of B_r. This is a convenient object that bounds scrambling by the evolution of $|A_1)$ into Σ_r as a function of time:

$$\frac{\|[A_1(t), B_r]\|}{2\|A_1\|\|B_r\|} \le \frac{\|\mathbb{P}_r e^{\mathcal{L}t}|A_1)\|}{\|A_1\|}. \tag{6}$$

In the "canonical" form of the Lieb-Robinson theorem popularized by Hastings and Koma [4], one uses the triangle inequality $\partial_t\|[A_1(t), B_r]\| \le \|[A_1(t), [H, B_r]]\|$. Yet most of the terms on the right-hand side of this inequality sum do not contribute to $\|[A_1(t), B_r]\|$: they correspond to shifts in $A_1(t)$ that cannot grow $\|\mathbb{P}_r|A_1)\|$. We emphasize that this holds even though the operator norm $\|A_1(t)\|$ is not the "length" of the vector $|A_1(t))$.

Instead, we write

$$e^{\mathcal{L}t}|A_1) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{X_1,\ldots,X_n} \mathcal{L}_{X_n}\cdots\mathcal{L}_{X_1}|A_1), \tag{7}$$

where X_i corresponds to a term in the Hamiltonian, e.g., $\mathcal{L}_{\sigma_1^x\sigma_2^x} = i[\sigma_1^x\sigma_2^x, \cdot]$. $\mathbb{P}_r\mathcal{L}_{X_n}\cdots\mathcal{L}_{X_1}|A_1)$ is only nonzero if a subsequence of $\mathcal{L}$'s form a path from 1 to r. Our main technical development is expanding $\mathbb{P}_r e^{\mathcal{L}t}|A_1)$ in a controlled way: we classify all sequences with a path from 1 to r by a relatively small number of equivalence classes Γ. Generalizing the interacting picture, we obtain the following identity:

$$\mathbb{P}_r \sum_\Gamma \sigma(\Gamma) \int_0^t dt_\ell \int_0^{t_\ell} dt_{\ell-1} \cdots \int_0^{t_2} dt_1 e^{\mathcal{L}(t-t_\ell)}\mathcal{L}_\ell^\Gamma e^{\tilde{\mathcal{L}}_\ell^\Gamma(t_\ell - t_{\ell-1})} \cdots e^{\tilde{\mathcal{L}}_2^\Gamma(t_2-t_1)}\mathcal{L}_1^\Gamma e^{\tilde{\mathcal{L}}_1^\Gamma t_1}|A_1) = \mathbb{P}_r e^{\mathcal{L}t}|A_1). \tag{8}$$

Here Γ is a label for ℓ nontrivial sequential steps $\mathcal{L}_j^\Gamma$ and the time-ordered integral can intuitively be interpreted as the possible times $t > t_\ell > \cdots > t_1$ at which "critical" steps in the sequence of $\mathcal{L}_{X_i}$ occurred. In fact, the emergence of the integral over ℓ ordered times is analogous to the time ordered integrals which arise in time dependent perturbation theory. According to rules we will shortly state, $\sigma(\Gamma) = \pm 1$ is assigned to avoid double counting so that the terms match up across the equality. Applying the triangle inequality to (8), and noting $e^{\tilde{\mathcal{L}}_j^\Gamma t}$ is norm preserving, which resums superfluous terms in the series expansion (7):

$$\frac{\|\mathbb{P}_r e^{\mathcal{L}t}|A_1)\|}{2\|A_1\|} \le \sum_\Gamma \frac{t^\ell}{\ell!} \prod_{j=1}^{\ell} \|\mathcal{L}_j^\Gamma\|, \tag{9}$$

where

$$\|\mathcal{L}_j^\Gamma\| := \sup_{\mathcal{O}} \frac{\|\mathcal{L}_j^\Gamma \mathcal{O}\|}{\|\mathcal{O}\|}. \tag{10}$$

The physical content of Eq. (9) is interpreted as follows. If we can define an equivalence relation on sequences of couplings such that Eq. (8) holds, then only the nontrivial steps $\mathcal{L}_j^\Gamma$ need to be counted in the commutator bound (4). *Every* other term in the sequence $\tilde{\mathcal{L}}_i^\Gamma$ that shows up in the intermediate unitary evolution does not grow $\|\mathbb{P}_r e^{\mathcal{L}t}|A_1)\|$.

In general, the choice of Γ is quite flexible. For the Hamiltonian (2), our construction is depicted in Fig. 1. We start by regrouping all $\mathcal{L}_{mn} = i[H_{mn}, \cdot]$ by their scale $q \approx \lfloor \log_2 |m-n| \rfloor$ (if $1 \le m, n \le r$; the exact formula is in [20]). We write H as a sum of one-dimensional Hamiltonians, each consisting of terms of a given scale. At scale $q = 0$ ($q > 0$), these blocks form one-dimensional model of nearest (next-nearest) neighbor interactions between *blocks* of sites, see Fig. 1(a). Which couplings are grouped into which blocks is depicted in Fig. 1(b). At scale q, we denote the block (q, k) to be the k^{th} leftmost block in Fig. 1, starting with $k = 0$. We denote $\mathcal{L}_{(q,k)} = i[H_{(q,k)}, \cdot]$, where

$$H_{(q,k)} = \sum_{(i,j) \text{ in block } (q,k)} H_{ij}. \tag{11}$$

We now rewrite Eq. (7) as

$$\mathbb{P}_r e^{\mathcal{L}t}|A_1) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{(q_n,k_n),\ldots,(q_1,k_1)} \mathbb{P}_r \mathcal{L}_{(q_n,k_n)} \cdots \mathcal{L}_{(q_1,k_1)}|A_1). \tag{12}$$

The key observation is that any sequence above Eq. (12) that made it to r must traverse forward a distance $\gtrsim r/\log r$ on at least one of the $\lfloor \log_2 r \rfloor$ scales q (Fig. 2). For any sequence of $\mathcal{L}_{(q_i,k_i)}$'s in Eq. (12), we can read off the q-forward subsequence $\mathbb{P}_r \cdots \mathcal{L}_{(q,k_N)} \cdots \mathcal{L}_{(q,k_1)} \cdots |A_1)$, $k_N > \cdots > k_1$ by recursively finding the next (q, k_j) that exceeds the largest k so far. If $N > N_q \sim 2^{-q} r/\log_2 r$, then the sequence is "long." We organize equivalence classes Γ by the nonempty subset of the integers $\{0, 1, \ldots, \lfloor \log_2 r \rfloor\}$ that corresponds to the scales on which a long path from 1 to r exists.

For example, suppose a sequence in Eq. (12) at scales q_0 contains a long subsequence. Then it would be accounted by the Γ specified by the first N_{q_0} terms of the forward subsequence $0 \le k_1 < k_2 < \cdots < k_{N_{q_0}} < \cdots$. In Eq. (8), we take $\sigma(\Gamma) = 1$, $\mathcal{L}_m^\Gamma = \mathcal{L}_{(q,k_m)}$,

$$\mathcal{L}_{-q} = \mathcal{L} - \sum_k \mathcal{L}_{(q,k)} \tag{13}$$

and

$$\tilde{\mathcal{L}}_m^\Gamma = \mathcal{L}_{-q} + \sum_{k \le k_{m-1}} \mathcal{L}_{(q,k)}, \tag{14}$$

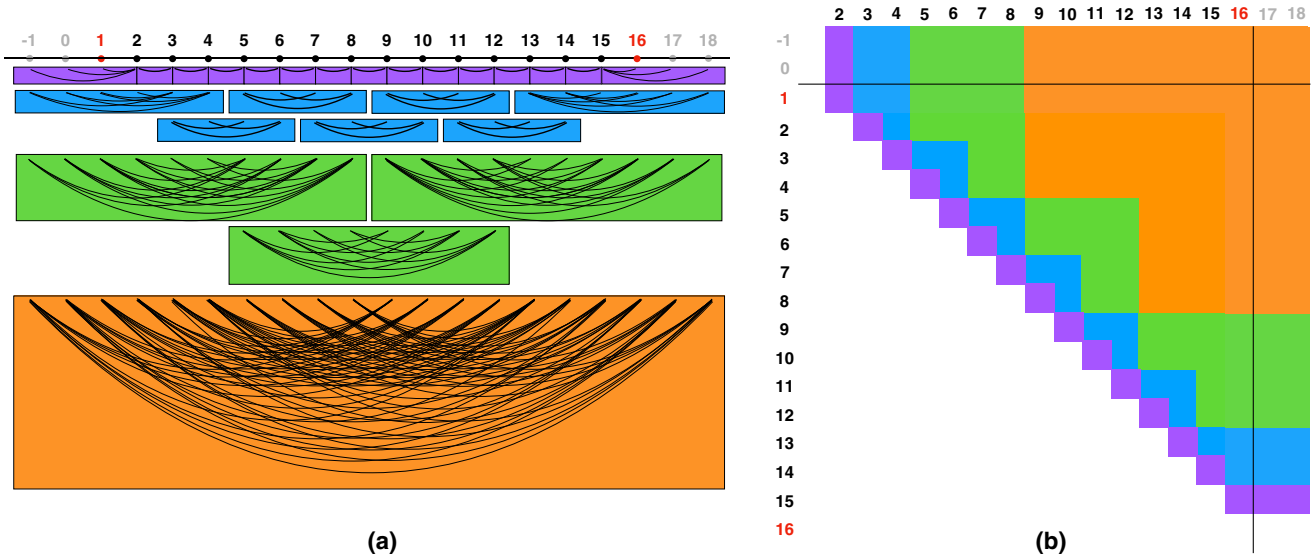

FIG. 1. Couplings $\mathcal{L}_{ij}$ $(i < j)$ can be broken up into the *scale* on which the coupling acts in a unique way. Intuitively, the scale q of a coupling is approximately $\lfloor \log_2(j-i) \rfloor$. Each scale with different values of q is denoted with a different color: from large (orange) to short (purple). In this example, we study $\|[A_1(t), B_{16}]\|$, and sites n obeying $n < 1$ or $n > 16$ are grouped in with these end sites when combining couplings. (a) Each scale or color would form a chain; (b) a more precise presentation. This L-shaped tiling ensures that the sum of each q block scales as $2^{-q(\alpha-2)}$ and can be extended to arbitrary large q.

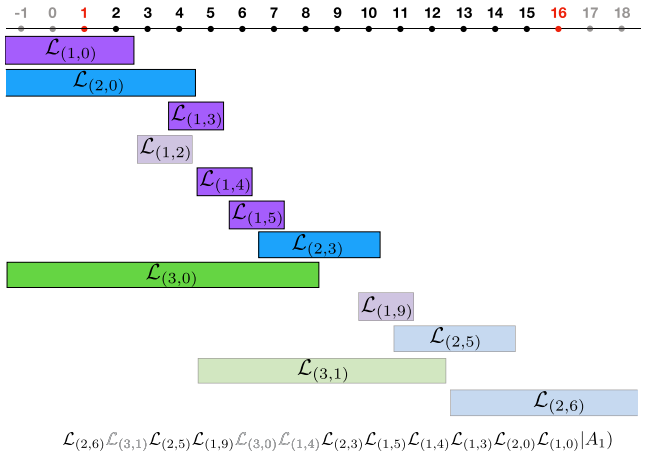

FIG. 2. Any sequence of $\mathcal{L}$ that grows $A_1(t)$ to the final site 16 must have a long sequence of couplings on at least one scale. For the particular sequence shown, there are three scales with sufficiently long sequences (no shorter than $16/\log 16 = 4$), and we bound the contribution of this sequence to $\|[A_1(t), B_{16}]\|$ by summing over the weight of all possible paths that contain the solid colored couplings (corresponding to $\mathcal{L}_j^\Gamma$) in a precise order. The lightly shaded couplings (corresponding to $\tilde{\mathcal{L}}_j^\Gamma$) do not contribute to Eq. (9).

where the "superfluous" terms $\tilde{\mathcal{L}}_m^\Gamma$ are exactly those terms that do not change the q-forward subsequence. To be precise, the terms allowed between $\mathcal{L}_m^\Gamma = \mathcal{L}_{(q,k_m)}$ and $\mathcal{L}_{m-1}^\Gamma = \mathcal{L}_{(q,k_{m-1})}$ are any $\mathcal{L}_{(q',k')}$ with $q \neq q'$, or any $\mathcal{L}_{(q,k'')}$ with $k'' \leq k_{m-1}$. The former are allowed because each scale q is treated separately; the latter because they are, by construction, not traversing forward.

In general, it may be the case that a single sequence in Eq. (12) contains multiple long paths on m distinct scales. The equivalence class Γ is then labeled by the m long sequences at m distinct scales. We generalize the construction of the previous paragraph, and set $\sigma(\Gamma) = (-1)^{1+m}$. The inclusion-exclusion principle then guarantees that Eq. (8) does not overcount the sequences which have multiple long subsequences, as in Fig. 2.

In fact, in order to prove Eq. (5), we have improved this argument in a few ways. (i) We tune N_q so that the contribution of all scales q to Eq. (9) is comparable. (ii) We demand that all long paths must increase the rightmost site on which the operator acts. We then evaluate Eq. (9), using that $\|\mathcal{L}_{(q,k)}\| \lesssim 2^{-q(\alpha-2)}$. Our results are summarized below.

When $\alpha > 3$, the dominant contribution to $\|\mathbb{P}e^{\mathcal{L}t}|A_1)\|$ comes from short length scales: a large fraction of the path from 1 to r often occurs in nearest neighbor hops. Scrambling proceeds as if interactions were nearest neighbor alone. The operator $|A_1(t))$ is largely supported on lattice sites $x < vt$, where v is a finite speed of quantum scrambling.

When $\alpha < 3$, the dominant contribution to $\|\mathbb{P}e^{\mathcal{L}t}|A_1)\|$ comes from few long hops across 1 to r. Counting the number of such long hops, we find $t_s(r) = O(r^{\alpha-2})$.

If $\alpha = 3$, we find that all scales are equally important, which leads to $t_s(r) = O(r/\log^2 r)$.

A final comment is that if the Hamiltonian is *frustrated*, we may replace $\alpha \to \alpha - 1$ in Theorem 1: namely, the linear light cone persists until $\alpha = 2$. A formal definition of frustration is that the maximal eigenvalue of $H_{(q,k)}$ is comparable to its magnitude in a randomly chosen state. This property is expected to hold for a self-averaging Hamiltonian, where each H_{ij} is multiplied by a zero-mean random variable. Frustration does not hold in a Hamiltonian where all 2-local terms H_{ij} in the Hamiltonian commute (e.g., $H_{ij} = \sigma_i^z\sigma_j^z/|i-j|^\alpha$).

Outlook.—We conclude the Letter with a discussion of the implications of our theorem. Recall that our new mathematical methods led to dramatic improvements over existing literature, where the previous optimal bound on scrambling in one-dimensional systems was $t_s(r) \gtrsim r^{(\alpha-2)/(\alpha-1)}$ for $\alpha > 2$ [10]. In fact, for any $\alpha > 3$, the speed of quantum scrambling is finite: entanglement [6,7] and quantum state transfer [2] proceed at a finite rate, and thermalization largely mimics that of a locally interacting system.

Our results for frustrated systems are very similar to the numerical simulations of Ref. [13], where it was argued that a finite speed of scrambling arises for $\alpha \gtrsim 1.5$ in a model with time-dependent random Hamiltonian. However, in another model with fixed Hamiltonian [14], it was found that $\alpha \gtrsim 1$ marked the onset of the finite scrambling speed. We conjecture that Eq. (5) holds with $\alpha \to \alpha - 1$ (hence, the light cone persists to $\alpha = 2$) for all models, including those which are not (by our definition) frustrated, whenever the operators in Eq. (4) act on a single site. It would be interesting if this can be proved rigorously.

The techniques developed in this Letter may generalize to other important problems in quantum information dynamics, including entanglement growth and quantum scrambling in finite temperature thermal ensembles, where the speed of quantum scrambling averaged over the thermal ensemble may depend on temperature [21]. We also hope to generalize our main theorem to any spatial dimension d. Lastly, we have also used similar techniques to constrain models of holographic quantum gravity [15]. Given the recent explosion of interest in realizing analog black holes in quantum simulators [22,23], our methods will constrain which experimental systems have the potential to achieve this ambitious goal.

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