Convergence of Nonperturbative Approximations to the Renormalization Group

Ivan Balog,[1] Hugues Chaté,[2,3,4] Bertrand Delamotte,[4] Maroje Marohnić,[5] and Nicolás Wschebor[6]
[1]*Institute of Physics, Bijenička cesta 46, HR-10001 Zagreb, Croatia*
[2]*Service de Physique de l'Etat Condensé, CEA, CNRS Université Paris-Saclay, CEA-Saclay, 91191 Gif-sur-Yvette, France*
[3]*Computational Science Research Center, Beijing 100094, China*
[4]*Sorbonne Université, CNRS, Laboratoire de Physique Théorique de la Matière Condensée, LPTMC, F-75005 Paris, France*
[5]*Visage Technologies AB, Diskettgatan 11A SE-583 35 Linköping, Sweden*
[6]*Instituto de Física, Facultad de Ingeniería, Universidad de la República, 11000 Montevideo, Uruguay*

(Received 4 July 2019; published 13 December 2019)

We provide analytical arguments showing that the "nonperturbative" approximation scheme to Wilson's renormalization group known as the derivative expansion has a finite radius of convergence. We also provide guidelines for choosing the regulator function at the heart of the procedure and propose empirical rules for selecting an optimal one, without prior knowledge of the problem at stake. Using the Ising model in three dimensions as a testing ground and the derivative expansion at order six, we find fast convergence of critical exponents to their exact values, irrespective of the well-behaved regulator used, in full agreement with our general arguments. We hope these findings will put an end to disputes regarding this type of nonperturbative methods.

DOI: 10.1103/PhysRevLett.123.240604

Wilson's renormalization group (RG) is an extraordinary means of understanding quantum and statistical field theories. Its perturbative implementation [1–3], in particular under the form of the ϵ expansion, has been a very efficient toolbox. Real-space renormalization [4] and the $1/N$ expansion [5] have been important conceptually, but remained unable to yield accurate results in most physically relevant cases. However, 25 years ago, an alternative formulation of Wilson's RG allowing for new "nonperturbative" approximation schemes [6]—which are anyway needed to solve the exact RG equation—has led to remarkable results on problems that are very difficult or fully out of reach of the perturbative approach [7–11].

This nonperturbative approach to the RG (NPRG hereafter) is versatile, allowing us to treat equilibrium and nonequilibrium problems, disordered systems, with access to both universal and nonuniversal quantities. To list a few successes just within statistical physics, let us mention the random field Ising model (spontaneous supersymmetry breaking and the associated breaking of dimensional reduction in a nontrivial dimension) [12,13], the Kardar-Parisi-Zhang equation in dimensions larger than one (identification of the strong coupling fixed point) [14–16], the glassy phase of crystalline membranes [17], the phase diagram of reaction-diffusion systems [18,19], etc.

Most of these results were obtained using an NPRG approximation scheme known as the derivative expansion (DE). In a nutshell, the underlying ideas are as follows: The exact NPRG equation governs the evolution of an effective action Γ_k (in the field theory language the generating functional of one-particle irreducible correlation functions) with the RG momentum scale k. In the NPRG approach, a regulator function $R_k(q^2)$ ensures that the large wave number modes (with $q^2 > k^2$) are progressively integrated over while the others are frozen. When $k = 0$, all statistical fluctuations have been integrated and $\Gamma_{k=0} = \Gamma$, the Gibbs free energy of the model. The DE consists in approximating the functional $\Gamma_k[\phi]$, where ϕ represents all the fields of the problem, by its Taylor expansion in gradients of ϕ truncated at a finite order.

In spite of its undeniable successes, the DE—and the NPRG in general—has often been criticized. Two main points are usually raised, the (apparent) lack of a small parameter controlling its convergence and the arbitrariness induced by the choice of the regulator function R_k. Indeed, within any approximation scheme, the end results do carry a residual influence of R_k. This has been often invoked against the NPRG approach, even though the dependence on R_k is similar to the renormalization scheme dependence in perturbation theory [20].

In this Letter, we aim to put an end to this controversy. We use the Ising model as a testing ground both because its relative simplicity allows us to study the sixth order of the DE and because its critical exponents are accurately known [21–23]. We provide numerical evidence and analytical arguments showing that the DE not only converges, but does so rapidly. Contrary to usual perturbative approaches (for a previous perturbative study of the convergence of the DE, see [24]), we find that the DE has (i) a finite radius of convergence and (ii) a fast convergence, even at low orders, when the anomalous dimension is small. We also discuss the respective quality of regulators $R_k(q^2)$ and propose empirical rules for selecting optimal ones, without prior knowledge of the problem at stake. We argue that our conclusions are most likely generic.

We start with a brief review of the NPRG, specialized here to the ϕ^4 model for convenience [25]. A one-parameter family of models indexed by a scale k is defined such that only the short wavelength fluctuations, with wave numbers $q = |\mathbf{q}| > k$, are summed over in the partition function $\mathcal{Z}_k$. The decoupling of the slow modes, $\varphi(|\mathbf{q}| < k)$, in $\mathcal{Z}_k$ is performed by adding to the original Hamiltonian H a quadratic (masslike) term which is nonvanishing only for these modes:

$$\mathcal{Z}_k[J] = \int D\varphi \exp\left(-H[\varphi] - \Delta H_k[\varphi] + \int_x J\varphi\right), \quad (1)$$

where $\Delta H_k[\phi] = 1/2 \int_\mathbf{q} R_k(q^2)\varphi(\mathbf{q})\varphi(-\mathbf{q})$. The form of the regulator function $R_k(q^2)$ is discussed in detail below [see Eqs. (7a) and (7c) for examples used here]. The k-dependent Gibbs free energy $\Gamma_k[\phi]$ [with $\phi(x) = \langle\varphi(x)\rangle$] is defined as the (slightly modified) Legendre transform of $\log \mathcal{Z}_k[J]$:

$$\Gamma_k[\phi] + \log \mathcal{Z}_k[J] = \int_x J\phi - \frac{1}{2}\int_\mathbf{q} R_k(q^2)\phi(\mathbf{q})\phi(-\mathbf{q}). \quad (2)$$

The exact RG flow equation of Γ_k reads [7–9]

$$\partial_t\Gamma_k = \frac{1}{2}\int_\mathbf{q} \partial_t R_k(q^2)(\Gamma_k^{(2)} + R_k)^{-1}[\mathbf{q}, -\mathbf{q}; \phi], \quad (3)$$

where $t = \log(k/\Lambda)$ and $\Gamma_k^{(2)}[\mathbf{q}, -\mathbf{q}; \phi]$ is the Fourier transform of the second functional derivative of $\Gamma_k[\phi]$. The DE consists in solving Eq. (3) in a restricted functional space where $\Gamma_k[\phi]$ involves a limited number of gradients of ϕ multiplied by ordinary functions of ϕ. Zeroth order is the commonly used local potential approximation (LPA): only the momentum dependences present in H are kept in the correlation functions. For the φ^4 model, $\Gamma_k[\phi]$ is then approximated by $\int_x [U_k(\phi) + \frac{1}{2}(\nabla\phi)^2]$: only a running potential term is retained. At order $s = 6$, the ansatz for Γ_k involves 13 functions (see [26]):

$$\Gamma_k[\phi] = \int d^d x\left[U_k(\phi) + \frac{1}{2}Z_k(\phi)(\partial_\mu\phi)^2 + \frac{1}{2}W_k^a(\phi)(\partial_\mu\partial_\nu\phi)^2 + \cdots + \frac{1}{96}X_k^h(\phi)[(\partial_\mu\phi)^2]^3\right]. \quad (4)$$

The flow of all functions is obtained by inserting the ansatz (4) in Eq. (3) and expanding and truncating the right-hand side on the same functional subspace. In practice, this is implemented in Fourier space. For instance, we obtain from Eq. (4) that $Z_k(\phi) = \partial_{p^2}\Gamma_k^{(2)}(p;\phi)|_{\mathbf{p}=0}$ with ϕ a constant field. Thus, the flow of $Z_k(\phi)$ is given by the p^2 term of the flow of $\Gamma_k^{(2)}(p;\phi)$.

At criticality—the regime of interest here—the RG flow reaches a fixed point. In practice, the fixed point is reachable when using dimensionless and renormalized functions denoted below by lowercase letters. We proceed as usual [10] by rescaling fields and coordinates. Here $\tilde{x} = kx$, $\tilde{\phi}(\tilde{x}) = \sqrt{Z_k^0}k^{(2-d)/2}\phi(x)$. Functions are then rescaled according to their canonical dimension and renormalized by $(Z_k^0)^{n/2}$ where n is the number of fields they multiply in the ansatz (4). This leads to $Z_k(\phi) = Z_k^0 z_k(\tilde{\phi})$. The absolute normalization of both Z_k^0 and $z_k(\tilde{\phi})$ is defined only once their value is fixed at a given point. We use the (re)normalization condition: $z_k(\tilde{\phi}_0) = 1$ at a fixed value $\tilde{\phi}_0$. The running anomalous dimension is then defined by $\eta_k = -\partial_t \log Z_k^0$. It becomes the anomalous dimension η at the fixed point [10].

Let us now give analytical arguments in favor of the fast convergence of the DE. We continue using the φ^4 theory here, but our results are more general. The key remark is that the momentum expansion applied away from criticality, either in the symmetric or broken phase, is known to be convergent with a finite radius of convergence. For instance, calling m the mass, that is, the inverse correlation length, the c_n in

$$\frac{\Gamma^{(2)}(p,m)}{\Gamma^{(2)}(0,m)} = \frac{\Gamma_{k=0}^{(2)}(p,m)}{\Gamma_{k=0}^{(2)}(0,m)} = 1 + \frac{p^2}{m^2} + \sum_{n=2}^{\infty} c_n\left(\frac{p^2}{m^2}\right)^n \quad (5)$$

are universal close to criticality and behave at large n as $c_{n+1}/c_n \sim -1/9$ and $-1/4$ in the symmetric and broken phases respectively (see, e.g., [3]). These behaviors follow from the fact that the singularity nearest to the origin in the complex p^2 plane is $9m^2$ ($4m^2$) because the Minkowskian version of the theory has a three-particle (two-particle) cut in the symmetric (broken) phase respectively [27]. Any regulator acts as a (momentum dependent) mass term. Thus, the critical theory regularized by $R_k(q^2)$ should be similar to the noncritical (massive) theory and should therefore also have a convergent expansion in p^2/k^2—which is nothing but the DE—with a finite radius of convergence that we call $\mathcal{R}$ typically between 4 and 9 as we show below.

At criticality and for $s = 6$, the analog of Eq. (5) is

$$\frac{\Gamma_k^{(2)}(p,\phi) + R_k(0)}{\Gamma_k^{(2)}(0,\phi) + R_k(0)} = 1 + \frac{Z_k p^2 + W_k^a p^4 + X_k^a p^6}{U_k'' + R_k(0)} \underset{[k\to 0]}{\to} 1 + \frac{p^2}{m_{\rm eff}^2} + \frac{w_a^* v^{*\prime\prime}}{z^{*2}}\frac{p^4}{m_{\rm eff}^4} + \frac{x_a^* v^{*\prime\prime 2}}{z^{*3}}\frac{p^6}{m_{\rm eff}^6}, \quad (6)$$

where the field dependence on the right-hand side has been omitted, primes denote derivation with respect to ϕ or $\tilde{\phi}$, u^*, z^*, w_a^*, x_a^* stand for the dimensionless functions of $\tilde{\phi}$ at the fixed point, and $m_{\text{eff}}^2 = k^2 v^{*\prime\prime}/z^*$ with $v^{*\prime\prime} = u^{*\prime\prime} + R_k(0)/Z_k^0 k^2$.

If the two expansions (5) and (6) are indeed similar then m_{eff}^2 must be the mass generated by the regulator and the coefficients of p^4/m_{eff}^4 and p^6/m_{eff}^6 must be analogous to c_2 and c_3 in Eq. (5). As for m_{eff}, if it is indeed generated by the regulator, it must be of order $R_k(q^2 = 0) \simeq \alpha k^2$ [see discussion below and Eq. (7)].

It is known that the c_n in Eq. (5) form an alternating series and that they are very small [3]: In the symmetric phase, $c_2 = -4 \times 10^{-4}$ and $c_3 = 0.9 \times 10^{-5}$ and in the broken phase, $c_2 \simeq -10^{-2}$ and $c_3 \simeq 4 \times 10^{-3}$. Together with the fact that the series in Eq. (5) has a finite radius of convergence, this suggests that it not only converges but that it does so rapidly.

Let us now discuss the role and shape of the regulator function $R_k(q^2)$. The DE—like any approximation scheme—introduces an influence of the choice of R_k on the end results [28]. There exist some constraints and *a priori* guidelines to choose R_k so that its influence stays minimal. First, R_k must freeze the small momentum modes $\varphi(|\mathbf{q}| < k)$ in $\mathcal{Z}_k[J]$ [Eq. (1)], so that they decouple from the long-distance physics. It must also leave unchanged the large momentum modes $\varphi(|\mathbf{q}| > k)$. Second, because the DE is a Taylor expansion of the $\Gamma_k^{(n)}(\{\mathbf{p}_i\})$ in powers of $\mathbf{p}_i \cdot \mathbf{p}_j/k^2$ (around 0), it is valid provided $\mathbf{p}_i \cdot \mathbf{p}_j/k^2 < \mathcal{R}$. This implies that whenever the $\Gamma_k^{(n)}$'s are replaced in a flow equation by their DE, the momentum region beyond $\mathcal{R}$ must be efficiently cut off. This is the role of the $\partial_t R_k(q^2)$ term in Eq. (3). It suppresses this kinematic sector in the integral over the internal momentum $\mathbf{q}$ if $R_k(q^2)$ almost vanishes for $|\mathbf{q}| \gtrsim k$. On the other hand, modes $\varphi(|\mathbf{q}| < k)$ are almost frozen if $R_k(q^2)$ is of order k^2 for $|\mathbf{q}| < k$. These two characteristic features give the overall shape of $R_k(q^2)$. Note also that if a nonanalytic regulator is chosen, one must make sure that the nonanalyticities thus introduced in the complex plane of q^2 are further than $\mathcal{R}$ from the origin. Finally, at order s of the DE the flow equations of the functions involve $\partial_t R_k(q^2)$ and $\partial_{q^2}^n R_k(q^2)$ from order $n = 1$ to $s/2$. Since the DE is performed around $q = 0$, it is important that these derivatives decrease monotonically: if not, a "bump" at a finite value $q^2 = q_0^2 > 0$ would magnify a region around q_0 which is less accurately described by the DE [30].

Taking into account all the prerequisites above, we have considered either regulators that are C^∞ in the complex plane of q^2, decay rapidly but do not vanish for $q > k$, or functions that vanish identically for $q > k$, are not C^∞ but are sufficiently differentiable for regularizing the DE at the order s studied, and have their derivatives as small as possible for $q \simeq k$. Specifically, we used

$$W_k(q^2) = \alpha Z_k^0 k^2 y/[\exp(y) - 1], \tag{7a}$$

$$\Theta_k^n(q^2) = \alpha Z_k^0 k^2 (1-y)^n \theta(1-y) \quad n \in \mathbb{N}, \tag{7b}$$

$$E_k(q^2) = \alpha Z_k^0 k^2 \exp(-y), \tag{7c}$$

where $y = q^2/k^2$. We show in [26] that $\Theta_k^\infty(q^2)$ is equivalent to $E_k(q^2)$.

We now present our results [31]. We focus here on the three-dimensional (3D) case, for which near-exact results are provided by conformal bootstrap [21–23], but we have obtained similar results in two dimensions [29]. For each regulator function R_k, we have calculated the critical exponents ν (associated with the divergence of the correlation length) and η [40], as well as the different ratios appearing in Eq. (6), at orders $s = 0$ (LPA), 2, 4, and 6. (Numerical details can be found in [26]) These quantities depend on the parameters of R_k, that is, for the regulators (7), on α that we typically vary in the range [0.1, 10].

Each regulator function we studied yielded very similar results. We first discuss those obtained with (7c). In Fig. 1, we show the curves $\nu(\alpha)$ and $\eta(\alpha)$ for orders $s = 2$ to 6. At each order, exponent values exhibit a maximum or a minimum at some value α_{opt} as α varies. Following a "principle of minimal sensitivity" [32,41], we select the values $\nu(\alpha_{\text{opt}})$ and $\eta(\alpha_{\text{opt}})$ taken at the extrema as our best estimates. Note that this is the situation closest to the exact theory, for which there is no dependence on the regulator. At a given order s, $\alpha_{\text{opt}}^{(\nu)}$ and $\alpha_{\text{opt}}^{(\eta)}$ are close but different, and their difference decreases fast with increasing s; see Fig. 1.

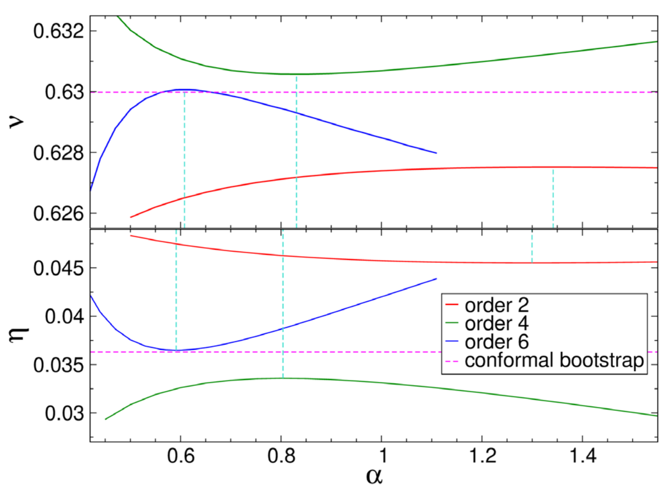

FIG. 1. Exponent values $\nu(\alpha)$ and $\eta(\alpha)$ at different orders of the DE for regulator (7c). Vertical lines indicate α_{opt}. LPA ($s = 0$) results do not appear within the narrow ranges of values chosen here (see Table I).

TABLE I. Three-dimensional Ising critical exponents obtained with regulator (7c) at orders $s = 0$ (LPA) to 6. Absolute distances between these values and the near-exact conformal bootstrap [23] ones are given by $|\delta\nu|$ and $|\delta\eta|$. Monte Carlo [42], high-temperature expansion [43], and six-loop perturbative RG values [2] are also given for comparison.

DE	ν	$\vert\delta\nu\vert$	η	$\vert\delta\eta\vert$
$s = 0$	0.651 03	0.021 06	0	0.036 30
$s = 2$	0.627 52	0.002 45	0.045 51	0.009 21
$s = 4$	0.630 57	0.000 60	0.033 57	0.002 73
$s = 6$	0.630 07	0.000 10	0.036 48	0.000 18
Conformal bootstrap	0.629 971(4)		0.036 297 8(20)	
Six loop	0.6304(13)		0.0335(25)	
High T	0.630 12(16)		0.036 39(15)	
MC	0.630 02(10)		0.036 27(10)	

Important remarks are in order. For each exponent, increasing s: (i) extrema alternate between being given by a maximum and a minimum; (ii) the local curvature at α_{opt} increases; (iii) strikingly, the exponent values at α_{opt} essentially alternate around and converge very fast to values very close to the conformal bootstrap "exact" ones. (At order 6, the optimal value of ν "crosses" the exact value, but these two numbers coincide up to three or four significant digits; see Fig. 1 and Table I.) The increase of curvature at α_{opt} and the accompanying faster variations of exponent values with α as s is increased imply that it is crucial to work with the optimal values given by the extrema, that is $\nu(\alpha_{\mathrm{opt}}^{(\nu)})$ and $\eta(\alpha_{\mathrm{opt}}^{(\eta)})$. This fast, alternating convergence is due to the alternating nature of the series of coefficients c_n. The speed of convergence is also in agreement with our considerations above about the radius of convergence $\mathcal{R}$ of the DE at criticality: the amplitude of the oscillations of the optimal values considered as functions of s decreases typically by a factor between 4 and 10 at each order s (Table I).

As mentioned above, all regulators we studied yield very similar results. For each exponent, the dispersion of values (over all regulators studied) typically also decreases by a factor 4 to 8 when going from one order s to the next, something we interpret as another manifestation of the radius of convergence of the DE; see Table II. We also noticed that regulators not satisfying our prerequisites very well typically yield "worse" results, somewhat away from those given by the set of good regulators (7) [29]. Our extensive exploration of regulators, including some multi-parameter ones not described here, thus leads us to conjecture the existence, for a given exponent and a given order of the DE, of an optimum-optimorum value, a "ceiling"—or a "floor", depending on the exponent and the order considered—that cannot be passed by any regulator (taken at its optimal parameter value α_{opt}). In particular, at LPA level, we did not find any regulator able to yield a ν value below the one given by the Wilson-Polchinski approach $\nu_{\mathrm{WP}} = 0.6496$ [33]. We recall that this value is the one given by regulator Θ_k^1, which thus appears, under our conjecture, as *the* optimal regulator at LPA level [34,44].

The above conjecture, if adopted, allows us to order regulators by increasing quality. Pending a proof, or, better, a constructive method to determine optimal regulators, we propose to use, at each order s, the range of exponent values over a family of "reasonable" regulators to define typical values (given by the middle of this range) and error bars (given by the half-range, which may appears as a conservative estimate). The resulting numbers are in Table II. To estimate asymptotic ($s \to \infty$) values for a given problem treated by the DE, we propose to extrapolate results obtained at low orders taking into account the facts uncovered above: The exponents, considered as functions of s, should have a monotonic as well as an oscillating contribution. For instance $\nu(s) = \nu_\infty + a_\nu \beta^{-s/2} + b_\nu (-1)^{s/2} \beta^{-s/2}$ where, typically, $4 \le \beta \le 9$ (given by the radius of convergence) and a and b are unknown coefficients. By considering all the regulators we have studied as well as all values of β between 4 and 9 we obtain a dispersion of asymptotic estimates, whose mean and maximal extent give us the following final numbers and associated error bars (Table II, see also [26]):

TABLE II. Exponent values given by the middle of the range of values observed over the family of regulators (7a)–(7c). Error bars are simply given by the half range. Extrapolation to asymptotic ($s \to \infty$) values are obtained by fitting the finite-s ones (see text).

Derivative expansion	ν	η
$s = 0$ (LPA)	0.651(1)	0
$s = 2$	0.6278(3)	0.0449 (6)
$s = 4$	0.630 39(18)	0.0343(7)
$s = 6$	0.630 12(5)	0.0361 (3)
$s \to \infty$	0.6300(2)	0.0358(6)
Conformal bootstrap	0.629 971(4)	0.036 297 8(20)

$\nu = 0.6300(2)$ and $\eta = 0.0358(6)$. Remarkably, these are in excellent agreement with conformal bootstrap values $\nu = 0.629971(4)$ and $\eta = 0.0362978(20)$, and better than perturbative six-loop ones.

We now come back to the momentum expansion of $\Gamma_k^{(2)}(p,\phi) + R_k(0)$ in the light of our results. We emphasized above that if $m_{\rm eff}$ in Eq. (6) is the mass generated by the regulator then it must be of order $R_k(q^2=0) \simeq \alpha k^2$, which implies that we should have $u^{*\prime\prime}/z \propto \alpha$. Remarkably, this relation is satisfied to a high accuracy for all regulators we have studied when using our optimal values $\alpha_{\rm opt}$; see [26]. We have also checked for all regulators and all $\tilde\phi$ that $w_a^* v^{*\prime\prime}/z^{*2} < 0$ and $x_a^* v^{*\prime\prime 2}/z^{*3} > 0$ in agreement with the signs of c_2 and c_3. The ratio $r = x_a^* u^{*\prime\prime}/(w_a^* z^*)$, which plays a role analogous to c_3/c_2, varies between $-\frac{1}{20}$ for $\tilde\phi$ around $\tilde\phi_{\rm min}$, the minimum of the potential, and $-\frac{1}{4}$ at large $\tilde\phi$ and is largely regulator independent (see [26]). These values typically correspond to what is found in the symmetric and broken phases, respectively, which is expected for a regularized theory at criticality.

We now go a step further and explain the behavior of the coefficients of p^4 and p^6 in Eq. (6). At criticality, when $p \gg k$, $\Gamma_k^{(2)}(p,0) \simeq \Gamma_{k=0}^{(2)}(p,0) \propto p^{2-\eta}$. On the other hand, when $p \ll k$, $\Gamma_k^{(2)}(p)$ is given by Eq. (6) at $\phi = 0$. Matching these two expressions for $p \sim k$, we find a simple analytic representation of the form $\Gamma_k^{(2)}(p) \simeq A p^2 (p^2 + bk^2)^{-\eta/2} + m_k^2$ where A and b are constants and $m_{k=0} = 0$. Expanded in powers of p^2/k^2, this expression yields an alternating series with a negative coefficient starting from p^4 and a positive one for p^6 as in Eq. (5). Moreover, all coefficients of the series from p^4 are proportional to η, which makes all of them naturally small, again as in Eq. (5). We therefore conclude that the DE is a convergent expansion with (i) a finite radius of convergence typically between 4 and 9 and, (ii) a rapid convergence because all the coefficients of the $(p^2/k^2)^n$ terms with $n \geq 2$ are proportional to η, which is small for the 3D Ising model [45].

In summary, we have shown that the derivative expansion often used in NPRG studies has a finite radius of convergence and we provided guidelines for choosing the regulator function at the heart of the procedure. Using the Ising model in three dimensions as a testing ground, we find fast convergence of critical exponents to their exact values, irrespective of the well-behaved regulator used, in full agreement with our general arguments. Our findings naturally extend to many other models—those having a unitary Minkowskian extension—and to other NPRG approximations such as the Blaizot-Mendéz-Wschebor scheme [46–48]. This establishes that the NPRG approach is not only versatile, being able to deal with any equilibrium or nonequilibrium model, but also quantitative, providing accurate results even at low orders.

This work was supported by Grant No. 412FQ293 of the CSIC (UdelaR) Commission and Programa de Desarrollo de las Ciencias Básicas (PEDECIBA), Uruguay and ECOS Sud U17E01. I. B. acknowledges the support of the Croatian Science Foundation Project IP-2016-6-7258 and the QuantiXLie Centre of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund—the Competitiveness and Cohesion Operational Program (Grant No. KK.01.1.1.01.0004). I. B. also thanks the LPTMC for its hospitality and the CNRS for funding. B. D. thanks L. Canet, N. Dupuis, C. Fleming, J. Pawlowski and M. Tissier for discussions and remarks about an early version of the manuscript.

[1] K. G. Wilson and J. B. Kogut, Phys. Rep. **12**, 75 (1974).
[2] R. Guida and J. Zinn-Justin, J. Phys. A **31**, 8103 (1998).
[3] A. Pelissetto and E. Vicari, Phys. Rep. **368**, 549 (2002).
[4] T. Niemeijer and J. M. J. Van Leeuwen, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic Press, New York, 1976), Vol. 6, pp. 425–505.
[5] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 3rd ed., International Series of Monographs on Physics (Clarendon Press, Oxford, 1996).
[6] By “nonperturbative” we mean an approximation not based on the smallness of a coupling constant.
[7] C. Wetterich, Phys. Lett. B **301**, 90 (1993).
[8] U. Ellwanger, Z. Phys. C **58**, 619 (1993).
[9] T. R. Morris, Int. J. Mod. Phys. A **09**, 2411 (1994).
[10] J. Berges, N. Tetradis, and C. Wetterich, Phys. Rep. **363**, 223 (2002).
[11] B. Delamotte, Lect. Notes Phys. **852**, 49 (2012).
[12] G. Tarjus and M. Tissier, Phys. Rev. Lett. **93**, 267008 (2004).
[13] M. Tissier and G. Tarjus, Phys. Rev. Lett. **107**, 041601 (2011).
[14] L. Canet, H. Chaté, B. Delamotte, and N. Wschebor, Phys. Rev. Lett. **104**, 150601 (2010).
[15] L. Canet, H. Chaté, B. Delamotte, and N. Wschebor, Phys. Rev. E **84**, 061128 (2011).
[16] L. Canet, H. Chaté, B. Delamotte, and N. Wschebor, Phys. Rev. E **86**, 019904(E) (2012).
[17] O. Coquand, K. Essafi, J. P. Kownacki, and D. Mouhanna, Phys. Rev. E **97**, 030102(R) (2018).
[18] L. Canet, H. Chaté, and B. Delamotte, Phys. Rev. Lett. **92**, 255703 (2004).
[19] L. Canet, B. Delamotte, O. Deloubrière, and N. Wschebor, Phys. Rev. Lett. **92**, 195703 (2004).
[20] A set of prescriptions, either implicit or explicit, is necessary to parametrize perturbation theory in terms of physical quantities. In principle, results should be independent of this renormalization scheme, but truncating the perturbative series induces a dependence.
[21] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Phys. Rev. D **86**, 025022 (2012).

[22] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, J. Stat. Phys. **157**, 869 (2014).
[23] F. Kos, D. Poland, and D. Simmons-Duffin, J. High Energy Phys. 11 (2014) 109.
[24] T. R. Morris and J. F. Tighe, J. High Energy Phys. 08 (1999) 007.
[25] Having the same universal properties, we use along the Letter interchangeably Ising and ϕ^4 models.
[26] See Supplemental Material Sec. C at http://link.aps.org/supplemental/10.1103/PhysRevLett.123.240604, for the proof.
[27] The situation is slightly more complicated in the broken phase, see [3], but this has no qualitative impact on our arguments.
[28] The choice of the normalization point $\tilde{\phi}_0$ also intervenes, but in a way that is exactly the same as that of the overall prefactor α in front of R_k [29].
[29] I. Balog, B. Delamotte, H. Chaté, and Marohnić (to be published).
[30] We have numerically checked this by comparing two very similar regulators except that one has a bump in $\partial_t R_k(q^2)$ and its derivatives with respect to q^2 around $q^2 = k^2$ [29].
[31] Previous works at LPA and order 2 [10,11,32–38] and order 4 [39] exist, but often considered slightly different flow equations and sometimes resorted to truncated Taylor expansions of the fields; see [26]. A detailed discussion will be given in [29].
[32] L. Canet, B. Delamotte, D. Mouhanna, and J. Vidal, Phys. Rev. D **67**, 065004 (2003).
[33] A. Hasenfratz and P. Hasenfratz, Nucl. Phys. **B270**, 687 (1986).
[34] D. F. Litim, Nucl. Phys. **B631**, 128 (2002).
[35] F. J. Wegner and A. Houghton, Phys. Rev. A **8**, 401 (1973).
[36] N. Tetradis and C. Wetterich, Nucl. Phys. **B422**, 541 (1994).
[37] T. R. Morris, Phys. Lett. B **329**, 241 (1994).
[38] S. Seide and C. Wetterich, Nucl. Phys. **B562**, 524 (1999).
[39] L. Canet, B. Delamotte, D. Mouhanna, and J. Vidal, Phys. Rev. B **68**, 064421 (2003).
[40] Determinating the critical exponent ω at order $s = 6$ is ongoing work [29]. At order $s = 4$ we obtain $\omega = 0.83(1)$.
[41] P. M. Stevenson, Phys. Rev. D **23**, 2916 (1981).
[42] M. Hasenbusch, Phys. Rev. B **82**, 174433 (2010).
[43] M. Campostrini, A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. E **65**, 066127 (2002).
[44] J. Pawlowski, M. Scherer, R. Schmidt, and S. Wetzel, Ann. Phys. (Amsterdam) **384**, 165 (2017).
[45] In $2D$, η is about 7 times larger, which in turn explains why the DE does not converge as fast then [29].
[46] J. P. Blaizot, R. M. Galain, and N. Wschebor, Phys. Lett. B **632**, 571 (2006).
[47] F. Benitez, J. P. Blaizot, H. Chaté, B. Delamotte, R. Mendez-Galain, and N. Wschebor, Phys. Rev. E **80**, 030103(R) (2009).
[48] F. Benitez, J. P. Blaizot, H. Chaté, B. Delamotte, R. Mendez-Galain, and N. Wschebor, Phys. Rev. E **85**, 026707 (2012).