Goychuk Replies: The author of the preceding Comment [1] attempts to prove that the non-Markovian Fokker-Planck equation or NMFPE [Eqs. (3), (4)] in our Letter [2] does not correspond to the generalized Langevin equation or GLE therein with $m = 0$. With this goal in mind, he points out that the same master equation also corresponds to a Markovian process upon a transformation of the time variable. This observation is, however, not correct because such a master equation as NMFPE does not define *uniquely* the stochastic process [3], apart from its single-time probability density $P(x, t)$. Moreover, this NMFPE is exact [3–5] for the inertialess GLE dynamics in a parabolic well. Indeed, in this case, $x(t)$ is a non-Markovian Gaussian process, which is fully characterized by its first two moments—what allows finding the exact NMFPE [2–6]. Moreover, Gaussian approximation for the escape problem is well justified for $L \gg l_T$ in Ref. [2], and the analytical solution of the NMFPE yields escape kinetics much slower than the relaxation process, which is consistent with the main assumptions of the rate theory [7].

Next, the author of Ref. [1] numerically confirms in Fig. 1 an important analytical result on the fractional Fokker-Planck equation (FFPE) escape also obtained in Ref. [2]. Furthermore, he uses our approach [5,8] to numerically integrate the GLE and presents a result which disagrees with the NMFPE result in Ref. [2]. His result is, however, not new. A methodological issue with using the NMFPE approach to describe escape kinetics was clarified at length in our subsequent work [5,8,9]; see, e.g., in review Ref. [5] on pp. 221–229 and Ref. [8], pp. 7 and 8. Indeed, the application of the NMFPE to the escape problem is based on the assumption that all the relaxation modes of the environment, which lead to the memory friction in the GLE, are *fast* on the time scale of escape (or barrier passage). The Markovian embedding approach to GLE dynamics of Refs. [5,8], which superseded some of the results in Ref. [2], makes this exceptionally clear. The corresponding numerics revealed many fast escape events occurring on the background of slow, quasifrozen relaxation modes. This fact is the reason why the NMFPE approach fails in such a situation [5,7–9]. Indeed, the GLE escape kinetics is often better described by a stretched exponential dependence [5,8,9], rather than a power law. Hence, the fact that GLE escape can occur much faster than the NMFPE-based theory [2] predicts is firmly established in the prior literature. Nevertheless, the NMFPE can concurrently describe relaxation kinetics in a bistable potential remarkably well [9]. Indeed, electron transfer in the adiabatic limit of a non-Markovian generalization of Zusman theory in Ref. [9] corresponds to fractional kinetics in a double parabolic well potential with a cusp. The analytical results stemming from a NMFPE-based approach agree well with the GLE based numerics for the relaxation kinetics in Fig. 1 therein. However, the escape kinetics in Figs. 2(f) and 2(e) of Ref. [9] reveals a failure of the NMFPE approach. This disagreement of the *ensemble-based* NMFPE approach with the *trajectory-based* GLE results provides also a manifestation of nonergodic features [9].

Moreover, this very subtle issue with use of the NMFPE was not originated in Ref. [2]. The akin idea to apply an exact NMFPE to a passage through a parabolic barrier can be traced back to Ref. [10] and was used in a number of subsequent papers. Its success was demonstrated, e.g., by reproducing the non-Markovian rate also obtained by other methods in the limit of *high* potential barriers [7].

To conclude, for $P(x, t)$, the NMFPE in Ref. [2] exactly corresponds to non-Markovian GLE dynamics in a parabolic potential well. Its application to the escape problem is, however, subjected to certain failures, indicated above and clarified in Refs. [5,7–9]. Nevertheless, it nicely describes, e.g., a bistable relaxation dynamics [9]. Hence, the NMFPE of Refs. [3,4,6] used in Ref. [2] was and remains one of the gems of the theory of non-Markovian stochastic processes and its applications. However, it must be used with care.

Funding of this research by the Deutsche Forschungsgemeinschaft (German Research Foundation), Grant No. GO 2052/3-1 is gratefully acknowledged.

Igor Goychuk
Institute of Physics and Astronomy
University of Potsdam
Karl-Liebknecht-Straße 24/25
14476 Potsdam-Golm
Germany

Received 29 October 2019; published 6 December 2019
DOI: 10.1103/PhysRevLett.123.238902

[1] R. K. Singh, preceding Comment, Phys. Rev. Lett. **123**, 238901 (2019).
[2] I. Goychuk and P. Hänggi, Phys. Rev. Lett. **99**, 200601 (2007).
[3] P. Hänggi and H. Thomas, Z. Phys. B **26**, 85 (1977); P. Hänggi, H. Thomas, H. Grabert, and P. Talkner, J. Stat. Phys. **18**, 155 (1978); P. Hänggi, Z. Phys. B **31**, 407 (1978).
[4] J. T. Hynes, J. Phys. Chem. **90**, 3701 (1986).
[5] I. Goychuk, Adv. Chem. Phys. **150**, 187 (2012).
[6] S. A. Adelman, J. Chem. Phys. **64**, 124 (1976).
[7] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).
[8] I. Goychuk, Phys. Rev. E **80**, 046125 (2009).
[9] I. Goychuk, Phys. Rev. E **99**, 052136 (2019).
[10] P. Hänggi and F. Mojtabai, Phys. Rev. A **26**, 1168 (1982).

