Oriented Active Solids

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We present a complete analysis of the linearized dynamics of active *solids* with uniaxial orientational order, taking into account a hitherto overlooked consequence of rotation invariance. Our predictions include a purely active response of two-dimensional orientationally ordered solids to shear, the possibility of stable active solids with quasi-long-range order in two dimensions and long-range order in three dimensions, generic instability of the solid for one sign of active forcing, and the instability of the uniaxially ordered phase in momentum-conserved systems for large active forcing irrespective of its sign.

Active systems [1] are held away from thermal equilibrium [2,3] by a direct supply of free energy to individual constituent units. The isotropic supply of energy can couple nontrivially with the particle anisotropy leading to macroscopic phenomena such as coherent intracellular flows [4–6] and defect turbulence in motor-microtubule extracts [7]. Whereas theories of active oriented *fluids* are well established and understood [1,8–11], oriented active *solids* have received lesser theoretical attention. While multiple biological systems have been modeled as *uniaxial* active elastomers [12–16] and descriptions that couple phase-field crystals with the Toner-Tu equations for a polar flock have been constructed [17–20], this has so far been carried out without a theory of uniaxial active solids. In particular, all these studies ignore a symmetry-mandated coupling between orientation and strain, which in *passive* uniaxial elastomers would have led to the vanishing of the zero-frequency shear modulus [21–24] due to rotation invariance. This implies that an equilibrium uniaxial elastomer cannot exist in two dimensions and can only have quasi-long-range order (QLRO) in three dimensions. Can active orientationally ordered elastomers, unlike their passive counterparts, resist shear at zero frequency and thus acquire stability in two dimensions? To answer this question, and more, we present a complete hydrodynamic theory of an active solid with polar or nematic orientational order, in contact with a substrate or in a momentum-conserving permeating fluid medium. The former description has been employed to describe cell monolayers or tissues in an extracellular matrix [13,14,16] and an isotropic variant of the latter for the response of isotropic cross-linked active gels [25–28]. We start with a description in which the polar or apolar order parameter is coupled to an *isotropic* solid and, in the *ordered* phase, eliminate the nonhydrodynamic director modes to obtain the effective equations for polar or apolar active gels. Our treatment also applies to the systems considered in Ref. [29] in the limit of

infinite polymer relaxation time, the hydrodynamic limit of a Vicsek model with harmonic interparticle interactions [30], a continuum description of the slow variables of the motile solid phases in self-propelled Voronoi models [31], the hydrodynamics of traveling crystals [17–20], contractile polar filaments on substrates [32], and experiments and simulations of mixtures of passive beads and driven polar rods in the high-density limit [33].

Here are our main results. (i) Active polar or apolar elastomers, when dynamically stable, resist shear forces in all directions, leading to quasi-long-range and long-range translational order in two and three dimensions, respectively, in contrast to passive orientationally ordered elastomers which, lacking one shear modulus, can support only quasi-long-range order in three and short-range order in two dimensions. (ii) The dynamics of polar motile elastomers, at linear and nonlinear order, is qualitatively distinct from that of a solid driven in an externally imposed direction [34] and escapes the latter's transverse buckling instability. (iii) An active force proportional to $\propto \mathbf{Q} \cdot \nabla \cdot \mathbf{Q}$, where $\mathbf{Q}$ is the apolar order parameter, always destabilizes elastomers on a substrate when its magnitude is larger than the active force $\propto \nabla \cdot \mathbf{Q}$, irrespective of the sign, in contrast to its stabilizing role in incompressible active fluids on substrates [35]. (iv) For bulk momentum-conserving systems, ignoring inertia, extensile (contractile) stresses destabilize positively (negatively) uniaxial elastomers. (v) Uniaxial elastomeric gels are generically destabilized at high activities when a forcing rate given by the ratio of the coefficient of the active stress to the viscosity exceeds the passive orientational relaxation rate. (vi) As expected from symmetry considerations, director fluctuations in stable uniaxially ordered phases are finite and do not diverge in the limit of small wave vectors contrary to the expectation based on Refs. [14–16]. (vii) Like active smectics [36,37], stable uniaxial active elastomers have finite concentration fluctuations.

We start our demonstration of these results with polar permanently cross-linked elastomers on substrates. Our description involves the displacement field $\bar{\mathbf{u}}(\mathbf{x}, t)$ about a reference state with isotropic elasticity, the polar order parameter field $\mathbf{p}(\mathbf{x}, t)$, and the velocity field $\mathbf{v}(\mathbf{x}, t)$. We define a linearized strain field $\mathbf{W} = \mathbf{C} + \Phi\mathbf{I}$, $W_{ij} = \partial_j\bar{u}_i + \partial_i\bar{u}_j$ about the isotropic state, where $\mathbf{C}$ and Φ are the trace-removed and isotropic parts of $\mathbf{W}$, and $\mathbf{I}$ is the unit tensor. Since the elastomer is permanently cross-linked, there is no relative motion between mass and structure, and the density field ρ is slaved to structural dilations $\delta\rho/\rho_0 = -\nabla \cdot \bar{\mathbf{u}}$, implying that the dynamics of the density need not be explicitly considered, and the evolution equation for the displacement field is $\dot{\bar{\mathbf{u}}} = \mathbf{v}$.

We assume a purely relaxational dynamics for the polarization ignoring all advective and self-advective terms [38] $\dot{\mathbf{p}} = -\Gamma_p\delta F/\delta\mathbf{p}$, where

$$F = \int_\mathbf{x} \left[\frac{\lambda}{2}\left(\Phi - \frac{s}{\lambda}p^2\right)^2 + \mu\mathrm{Tr}\left(\mathbf{C} - \frac{t}{\mu}\mathbf{T}\right)^2 + f_p\right] \quad (1)$$

is the free-energy functional governing the dynamics in the absence of activity, with $T_{ij} = p_ip_j - (1/2)p^2\delta_{ij}$ and $f_p = (r/2)p^2 + (w/4)p^4 + (K/2)(\nabla\mathbf{p})^2$. The generic free-energy couplings s and t between the strain and the polarization lead to the development of a strain anisotropy either parallel ($t > 0$) or perpendicular to ($t < 0$) $\mathbf{T}$. The absence of these couplings, e.g., as in Refs. [13,17], would have implied an invariance of the uniaxial solid under *independent* rotations of the orientation and the elastic network, at least in the passive limit. Their inclusion reduces this symmetry to one under *joint* rotations of the two. The force balance equation $\Gamma\mathbf{v} = \mathbf{f}_p + \nabla \cdot \sigma^a - \delta F/\delta\bar{\mathbf{u}}$ for polar systems contains active propulsive forces $\mathbf{f}_p = (v\Gamma)\mathbf{p} + (v_1\Gamma)\mathbf{p} \cdot \mathbf{W}$, in addition to the divergence of the active stress which has the usual form $\sigma_{ij}^a = \zeta_1 T_{ij} + \zeta_2 p^2\delta_{ij}$ [1].

We assess the effects of the active motilities and stresses on a polarized phase with $\mathbf{p} = p_0\hat{x}$, where $p_0^2 = |r|/w$, $C_{ij}^0 = t/\mu T_{ij}^0$, and $\Phi^0 = (s/\lambda)p_0^2$, which would have been the equilibrium state in the absence of activity. Because of the active motility, this state has a steady velocity $\mathbf{v} = v_0\hat{x} = (vp_0 + v_1p_0W_{xx}^0)\hat{x}$. Expanding the fields in small fluctuations, $\mathbf{p} \approx (p_0 + \delta p)\hat{x} + p_0\theta\hat{y}$, $C_{ij} = C_{ij}^0 + \delta C_{ij}$, and $\Phi = \Phi^0 + \delta\Phi$ about the homogeneously oriented state, we find that *both* the magnitude $p_0\delta p = s\delta\Phi + 2t\delta C_{xx}/\bar{r}$, $\bar{r} = w + (2s^2/\lambda) + 2(t^2/\mu)$ and the angular fluctuations $\theta = [\mu/tp_0^2]\delta C_{xy}$ of the polarization are slaved to the elastic deformations. Because of the minimal couplings between $\mathbf{p}$ and $\mathbf{C}$, transverse fluctuations are not independently soft, despite the spontaneous breaking of rotation symmetry and are instead slaved to the strain fluctuations [54]. Ignoring the symmetry-mandated

free-energy coupling μ in Eq. (1) would have led to the incorrect conclusion that the relaxation rate of transverse fluctuations of $\mathbf{p}$ vanishes with the wave number. To analyze the displacement fluctuations about the orientationally ordered configuration, we transform to displacement $\mathbf{u}$ and strain η relative to the anisotropic state rather than $\bar{\mathbf{u}}$ and $\mathbf{C}$ and Φ, which were defined relative to an isotropic reference space. Upon integrating out the δp fluctuations, the free-energy (1) in terms of θ and $\mathbf{u}$ transforms to

$$F = \frac{1}{2}\int_\mathbf{x} B_1\eta_{xx}^2 + B_2\eta_{yy}^2 + B_3\eta_{xx}\eta_{yy} + B_4[\eta_{xy} - \beta(\theta - \Omega)]^2,$$

$$(2)$$

where $\eta_{ij} = (1/2)(\partial_iu_j + \partial_ju_i)$, the rotation angle $\Omega = (\partial_xu_y - \partial_yu_x)/2$. $\beta \in [-1, 1]$, whose sign is that of t, measures the degree of anisotropy of the solid. $\beta = 0$ is the isotropic case, and we refer to positive and negative anisotropy according to $\mathrm{sgn}(\beta)$. The standard analysis leading up to Eq. (2) as well as the expressions for B_i and β are presented in Ref. [38]. The angle θ appears with the rotation field of the solid [54] and the shear strain, and integrating it out leads to a vanishing of the shear modulus [21,22,55]. This is a consequence of rotation invariance and, in equilibrium, implies that a two-dimensional uniaxially ordered elastomer is not a true solid since it cannot resist shear.

In passive systems, $\dot{\mathbf{u}}$ would not contain terms of the form $\partial_y^2u_x\hat{x}$ and $\partial_x^2u_y\hat{y}$ upon integrating out the fast θ field. However, forces arising from the divergence of the *active* stress $\nabla \cdot \sigma^a \approx p_0^2\zeta_1(\partial_y\theta\hat{x} + \partial_x\theta\hat{y})$ yield terms of this form even when θ is eliminated in favor of $\Omega + \beta^{-1}\eta_{xy}$. The presence of these active forces implies that a shear experiment on an active solid should yield a nonzero value of the "shear modulus" (Fig. 1), which vanishes in the limit of 0 activity. Further, curiously, since the active stress depends on both Ω, which is *antisymmetric* in the indices x and y, and η_{xy} which is symmetric, the "modulus" measured by shearing along or transverse to the ordering direction yields different values. Turning to the propulsive forces and projecting these along and transverse to the polarization, we obtain

$$\hat{p} \cdot \mathbf{f}_p \overset{\mathrm{lin}}{=} \Gamma[v_0 + (v + v_1W_{xx}^0)\delta p + v_1p_0\delta W_{xx}]$$
$$= \Gamma v_0 + a_1\partial_xu_x + a_2\partial_yu_y, \quad (3)$$

where $\overset{\mathrm{lin}}{=}$ denotes equality to linear order in disturbances, $\hat{p}$ is the unit vector in the direction of polarization, a_1 and a_2 are obtained by replacing δp and δW_{xx} by their values in terms of the displacement fields [38], and $\hat{p}_\perp \cdot \mathbf{f}_p \overset{\mathrm{lin}}{=} \Gamma v_1p_0(\delta W_{xy} - 2\theta C_{xx}^0)$, where $\hat{p}_\perp$ is a unit vector perpendicular to $\hat{p}$. Since $C_{xx}^0 = (t/2\mu)p_0^2$ and
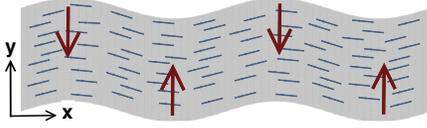
FIG. 1. An orientationally ordered solid with a sinusoidal distortion. The local direction of orientational ordering is denoted by blue bars. In passive solids, such distortions are "soft," with the leading order restoring force due to orientational elasticity $\sim \partial_x^4 u_y$ (i.e., no restoring force for a plane shear). Such fluctuations in passive systems destroy the orientationally ordered solid at arbitrarily low noise. Activity leads to an additional restoring force $\sim \partial_x^2 u_y$ whose direction when $b_4 > 0$ is shown with bold red arrows. This restoring force, which acts as an effective shear modulus (0 in passive systems), leads to the *existence* of active uniaxially ordered solid.

$\theta = (\mu/tp_0^2)\delta C_{xy} = (\mu/tp_0^2)\delta W_{xy}$, $\hat{p}_\perp \cdot \mathbf{f}_p = 0$. Thus, there is no propulsive force transverse to the polarization in an elastomer with spontaneous polar order. Physically, since the only anisotropy in the system is due to the broken rotation symmetry, the anisotropy of $\mathbf{W}$ aligns with $\mathbf{pp}$ on a fast timescale and at longer times, $\mathbf{W} \cdot \mathbf{p}$ is purely parallel to $\mathbf{p}$. Next, projecting the velocity along and transverse to the polarization, we obtain $\hat{p} \cdot \mathbf{v} \overset{\text{lin}}{=} \dot{u}_x$ and $\hat{p}_\perp \cdot \mathbf{v} \overset{\text{lin}}{=} \dot{u}_y - v_0\theta$, where the final term appears due to the mean motion of the solid along $\hat{x}$ [38,56,57]. Therefore, upon transforming to a frame moving with the mean velocity of the solid $v_0\hat{x}$,

$$\dot{u}_x = a_1\partial_x u_x + a_2\partial_y u_y + b_1\partial_x^2 u_x + b_2\partial_y^2 u_x + b_3\partial_x\partial_y u_y, \tag{4a}$$

$$\dot{u}_y = b_4\partial_x^2 u_y + b_5\partial_y^2 u_y + b_6\partial_x\partial_y u_x, \tag{4b}$$

where $b_2 = p_0^2\zeta_1(1-\beta)/2\Gamma\beta$, and $b_4 = p_0^2\zeta_1(1+\beta)/2\Gamma\beta$ and $b_3 - b_6$ are purely active (see Ref. [38] for expressions of b_i). The dynamical equations (4) yield the eigenfrequencies

$$\omega_\pm = \begin{cases} \left.\begin{aligned} &\left(a_1 + \frac{a_2 b_6}{|b_2 - b_5|}\right)q_x - ib_2 q_y^2 \\ &-\frac{a_2 b_6}{|b_2 - b_5|}q_x - ib_5 q_y^2 \end{aligned}\right\} q_x \ll q_y^2, \\[1em] \left.\begin{aligned} &a_1 q_x - ib_1 q_x^2 - i\left(b_2 + \frac{a_2}{a_1}b_6\right)q_y^2 \\ &-ib_4 q_x^2 - i\left(b_5 - \frac{a_2}{a_1}b_6\right)q_y^2 \end{aligned}\right\} q_x \gtrsim q_y^2. \end{cases} \tag{5}$$

This demonstrates that even in this overdamped system, there are propagating modes along the direction of motion with a dispersion $\omega \sim q$ which are damped at $\mathcal{O}(q^2)$. Since b_2 and b_4 are purely active, the polar solid is stable only when $\zeta_1\beta > 0$. Motility induces a further instability for a small enough value of b_2 if a_2 and a_1 have opposite signs.

However, in contrast to externally driven solids [34], where terms proportional to $\partial_y u_x$ and $\partial_x u_y$ appear in the $\dot{u}_y$ equation even in the comoving frame leading to eigenfrequencies that are either wavelike at small wave vectors or have an instability with a growth rate $\propto q$, the direction of motion of crawling solids is chosen spontaneously forbidding a transverse buckling instability [58]. This implies that active solids may be more stable than both their passive counterparts and externally driven ones, establishing our result (ii).

When $b_2 > 0$ and $b_4 > 0$, the static structure factor of both u_x and u_y fluctuations obtained by adding Gaussian white noise to Eq. (4) scales as $1/q^2$ in all directions, implying QLRO in two dimensions. In particular, $\langle u_x(q,t)u_x(-q,t)\rangle|_{q_x=0} \propto 1/(b_2 q_y^2)$ and $\langle u_y(q,t)u_y(-q,t)\rangle|_{q_y=0} \propto 1/(b_4 q_x^2)$, in contrast to passive uniaxial elastomers where these would be $\propto 1/q^4$, establishing our result (i) for polar systems. However, this conclusion of the linear theory may be invalidated by nine terms bilinear in $\nabla\mathbf{u}$ which are marginal in two dimensions. Though, unlike in externally driven solids [34], rotation invariance significantly constrains their coefficients, a treatment including these nonlinearities is beyond the scope of this paper. However, in a polar smectics with a displacement variable u, the two relevant nonlinearities $(\partial_x u)^2$ and $(\partial_y u)^2$ do not destroy the ordered phase [37] when they have opposite signs. Therefore, we believe that QLRO is possible even in polar solids in two dimensions for some range of parameter values.

We now turn to *apolar* elastomers on substrates. The ordering, in this case, is characterized by a rank-2 symmetric tensor $\mathbf{Q}(\mathbf{x}, t)$. We again assume a purely relaxational dynamics $\dot{\mathbf{Q}} = -\Gamma_Q \delta F_Q/\delta\mathbf{Q}$ for the apolar order parameter ignoring all couplings to velocity (which are irrelevant). Here, F_Q is of the form (1) with p^2 being replaced by $\text{Tr}[\mathbf{Q}^2]$ and $\mathbf{T}$ by $\mathbf{Q}$. Symmetry implies that apolar solids cannot self-propel, and the active forces are [1,35,61]

$$\mathbf{f}_a = \tilde{\zeta}_1 \nabla \cdot \mathbf{Q} + \tilde{\zeta}_2 \mathbf{Q} \cdot (\nabla \cdot \mathbf{Q}), \tag{6}$$

where the second is only allowed in systems not constrained by momentum conservation [35]. An analogous force with the same effect on stability is also allowed in polar systems but was not considered there for simplicity. Examining the effect of active forces on the state obtained in the absence of activity and repeating the analysis detailed for the polar case, we obtain the dynamical equations for $\dot{u}_x$ and $\dot{u}_y$, which have the same form as Eq. (4) with $a_1 = a_2 = 0$. Both b_2 and b_4 are purely active with $b_2 = S_0(\beta^{-1} - 1)(2\tilde{\zeta}_1 + \tilde{\zeta}_2 S_0)/4$ and $b_4 = S_0(\beta^{-1} + 1)(2\tilde{\zeta}_1 - \tilde{\zeta}_2 S_0)/4$, where S_0 denotes the degree of apolar ordering. This implies that a stable orientationally ordered solid is only realized when $\tilde{\zeta}_1 > |(\tilde{\zeta}_2 S_0/2)|$ for $\beta > 0$ and $\tilde{\zeta}_1 < -|(\tilde{\zeta}_2 S_0/2)|$ for $\beta < 0$. A negative $\tilde{\zeta}_1$,

which denotes extensile stresses in this convention, destabilizes a positively uniaxial solid with nematogens aligned along the deviatoric strain, while a positive $\tilde{\zeta}_1$ denoting contractile active stresses, destabilizes a solid in which the nematogens align perpendicular to the local deviatoric strain anisotropy. This should be compared to the situation in active smectics [36] and in cholesterics [62] or columnar systems. Interestingly, when $|\tilde{\zeta}_2| > 2\tilde{\zeta}_1/S_0$, the orientationally ordered state is destabilized; i.e., in contrast to incompressible active fluids on substrates, where, depending on the sign, it may play a *stabilizing* role, here $\tilde{\zeta}_2$ plays a *destabilizing* role irrespective of its sign, establishing our result (iii).

As in polar elastomers, $b_2 > 0$ and $b_4 > 0$ ensure that the variances of both u_x and u_y fluctuations scale as $1/q^2$ in all directions leading to QLRO in two dimensions establishing result (i) for the apolar case. Unlike the polar case, this conclusion cannot be invalidated by nonlinearities which, due to nematic symmetry, scale as $\sim \nabla(\nabla \mathbf{u})^2$ and are irrelevant in two-dimensional active apolar elastomers, in contrast to their passive counterparts, where $b_2 = b_4 = 0$ render them relevant in $d \leq 3$ [22,63].

Further, in three dimensions active polar and apolar elastomers have no relevant nonlinearity and have true long-range order unlike their passive counterparts. Interestingly [38], even active *biaxial* elastomers, whose passive analogs are softer than their uniaxial counterparts, have long-range order in three dimensions.

We now consider uniaxial elastomeric gels, i.e., (dilute) elastomers frictionally coupled to an incompressible permeating fluid, with conserved total momentum. The velocity of the elastomer is $\dot{\mathbf{u}} = \mathbf{v} + \mathbf{v}_r$, where $\mathbf{v}$ is the joint velocity of the network and the fluid, and $\mathbf{v}_r$ is the velocity of the network relative to their combined center of mass. The joint velocity $\mathbf{v}$ obeys the force balance equation $\eta \nabla^2 \mathbf{v} = \nabla \Pi - \tilde{\zeta}_1 \nabla \cdot \mathbf{Q} + \delta F/\delta \mathbf{u}$, where Π is the pressure that enforces incompressibility, η is the viscosity, and F is given by Eq. (2). Defining $v_x = \partial_y \psi$ and $v_y = -\partial_x \psi$, where ψ is a stream function, $u_t = (q_y u_x - q_x u_y)/q$ and $u_l = \mathbf{q} \cdot \mathbf{u}/q$, we obtain $\dot{u}_t = i|q|\psi + \mathcal{O}(q^2)$ and $\dot{u}_l \sim \mathcal{O}(q^2)$. The relaxation rate of u_t

fluctuations, which are the ones affected by the fluid, scale as q^0 to leading order in a wave vector. In passive elastomers, the relaxation rate of u_t to $\mathcal{O}(q^0)$ has to vanish for perturbations either purely along the ordering direction ($\mathbf{q} = q\hat{x}$) or purely transverse to it ($\mathbf{q} = q\hat{y}$). However, in active elastomers, the eigenfrequencies in these directions are nonzero

$$\omega(\mathbf{q} \to \mathbf{0}) = -i\frac{S_0 \tilde{\zeta}_1}{2\eta}(\beta^{-1} \pm 1), \tag{7}$$

where the $+$ $(-)$ sign is realized for wave vectors along $\hat{x}$ $(\hat{y})$ (see Ref. [38] for the expression for all wave vector directions). This implies that when $\beta\tilde{\zeta}_1 < 0$, the gel is unstable; i.e., extensile (contractile) stresses destabilize positively (negatively) uniaxial elastomers, establishing our result (iv). While the relaxation rate of u_t fluctuations is $\mathcal{O}(q^0)$, the static structure factor of a stable gel $\sim 1/q^2$ in all directions, as can be checked by adding momentum-conserving spatiotemporally white noise to the force balance equation.

We have assumed a fast relaxation of the angle field to its steady-state value determined by the displacement field. This may, however, be questionable in a gel—an orientationally ordered fluid is generically unstable with a wave-vector-independent growth rate [1,61]. When the growth rate of this generic instability is greater than the relaxation of the angle field to the value dictated by the local strain, the gel should be generically unstable *irrespective* of the sign of the active stress; i.e., even though a calculation purely in terms of displacement field predicts Eq. (7) that a contractile gel with $\tilde{\zeta}_1 > 0$ is stable for $\beta > 0$, it should be unstable beyond a critical active forcing when the coupled dynamics of angle $\dot{\theta} = (1 - \xi \cos 2\phi)q^2\psi/2 - \Gamma_\theta \delta F/\delta\theta$, with ξ being the flow-alignment parameter and Γ_θ the angular relaxation rate, and displacement fields are considered. For $\xi = 0$, an orientationally ordered contractile *fluid* is maximally unstable along q_y. For a contractile uniaxial gel with $\xi = 0$, the eigenfrequency of the coupled u_t and θ dynamics along $\hat{y}$ to leading order in wave numbers is

$$\omega_\pm = \frac{i}{8\eta}\left[2S_0\tilde{\zeta}_1 - B_4(1 + 4\beta^2\Gamma_\theta\eta) \pm \sqrt{\{2S_0\tilde{\zeta}_1 - B_4(1 + 4\beta^2\Gamma_\theta\eta)\}^2 - 32B_4(\beta - \beta^2)\Gamma_\theta S_0\tilde{\zeta}_1\eta}\right]. \tag{8}$$

This implies that when the active forcing $S_0\tilde{\zeta}_1/\eta$ exceeds the passive relaxation rate of angular fluctuations $B_4/2\eta + 2B_4\beta^2\Gamma_\theta$, the orientational order is destabilized even in a positively uniaxial contractile solid establishing our result (v).

Finally, since we have argued that both $\delta\rho$ and θ fluctuations scale as $\sim q u_q$ in all directions of the wave

vector space, their static structure factors $\langle|\theta_q|^2\rangle_{q\to 0} \sim q^2\langle|u_q|^2\rangle \to \text{const}$ and $\langle|\delta\rho_q|^2\rangle_{q\to 0} \sim q^2\langle|u_q|^2\rangle \to \text{const}$, implying nondiverging angular fluctuations at small wave vectors and normal (nongiant) number fluctuations, establishing our results (vi) and (vii).

We close with a brief discussion of the implications of our work for experiments and simulations. Reference [23]

showed that passive, periodically cross-linked actin gels display soft elasticity through detailed simulations. Our theory, which demonstrates that active orientationally ordered gels must have an active shear response, implies that the addition of myosin motors to this system must lead to either stiffening or supersoftening. Although disorder in passive uniaxial gels, biological or otherwise, rules out ideal soft elasticity, we predict a large change of the shear response of orientationally ordered gel with increasing activity. We also look forward to the construction of biomimetic gels, such as those composed of cross-linked microtubules and kinesin motors, where our predictions can be tested. Epithelial cellular layers [64] have been modeled as active elastic solids [14,16]. Our work demonstrates the approach required for modeling the motility of these systems, identifies an active shear response, and predicts that polarization fluctuations in these systems do not diverge at small wave vectors, which can be checked, for instance, in the assay of Ref. [16]. Our results are also applicable to *artificial* systems, in particular, solids composed of high densities of passive beads and low densities of uniaxial rods (high-density analogs of Ref. [33]) and, as discussed in Ref. [38], our predictions for the (in)stability of active uniaxial solids can be tested here. Further, we demonstrate in the Supplemental Material [38] how to modify current "traveling phase-field crystal" models [18–20] to obtain the correct motile solid limit. Such "traveling crystals" may be observed in either experiments or simulations of active Brownian particles with alignment interactions [65]. A further application of our theory, which predicts a singular change in the mechanical properties of liquid-crystalline elastomers with activity, is the construction of novel metamaterials. Such materials, constructed, for instance, by cross-linking [23] elongated photo- [66] or chemosensitive [67] colloids or actin or microtubule filaments, will have a fundamentally different strain response depending on the presence or absence of light, chemicals, or motors, respectively. These may be used to construct materials that flatten or crumple, or two-dimensional sheets that buckle out of plane in response to stimuli leading to stimuli-sensitive hinges or traps or materials that self-fold [38].

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