Bound State Soliton Gas Dynamics Underlying the Spontaneous Modulational Instability

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We investigate the fundamental phenomenon of the spontaneous, noise-induced modulational instability (MI) of a plane wave. The statistical properties of the noise-induced MI, observed previously in numerical simulations and in experiments, have not been explained theoretically. In this Letter, using the inverse scattering transform (IST) formalism, we propose a theoretical model of the asymptotic stage of the noise-induced MI based on N-soliton solutions of the focusing one-dimensional nonlinear Schrödinger equation. Specifically, we use ensembles of N-soliton bound states having a special semiclassical distribution of the IST eigenvalues, together with random phases for norming constants. To verify our model, we employ a recently developed numerical approach to construct an ensemble of N-soliton solutions with a large number of solitons, $N \sim 100$. Our investigation reveals a remarkable agreement between spectral (Fourier) and statistical properties of the long-term evolution of the MI and those of the constructed multisoliton, random-phase bound states. Our results can be generalized to a broad class of strongly nonlinear integrable turbulence problems.

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Integrable partial differential equations such as sine-Gordon, Korteweg–de Vries (KDV) and the one-dimensional nonlinear Schrödinger equation (1D-NLSE), are considered as universal models of nonlinear physics [1]. They describe, at leading order, various nonlinear systems and can be integrated using the inverse scattering transform (IST), often seen as a nonlinear analogue of the Fourier transform [2–6]. The traditional IST theory deals with decaying fields [6,7], but constant nonzero boundary conditions at infinity can also be considered [8,9]. Periodic and quasiperiodic fields can finally be examined in the framework of the finite gap theory [10–12]. Deterministic initial conditions have been widely studied within these three different boundary conditions.

In contrast, the propagation of random waves in integrable systems remains an open theoretical problem of significant applied interest due to the complexity of many real world nonlinear phenomena modeled by integrable equations. For this reason, integrable turbulence (IT) has been introduced as a "new chapter of turbulence theory" by V. Zakharov [13] and is now an active theoretical and experimental field of research [14–23].

In particular, the development of the noise-induced modulational instability (MI), also known as the Benjamin-Feir instability [24] and arising in the focusing regime of the 1D-NLSE, represents a prominent example of the IT phenomena [15]. This instability can be observed in many physical systems, such as deep water waves [12], Bose-Einstein condensates [25], or nonlinear optical waves [26]. In the traditional formulation, the development of MI is seen as the amplification of an initially small sinusoidal perturbation of a plane wave—the condensate [11,27]. In this case, the nonlinear stage of MI is described by homoclinic solutions of the 1D-NLSE—the Akhmediev breathers [28–31].

When the small initial perturbation of the condensate is a random process, the numerical simulations of the focusing 1D-NLSE show that the long-time evolution is characterized by a stationary single-point statistics, which is Gaussian despite the presence of highly nonlinear breatherlike structures [15,17,32]. It has also been shown, recently, that this long-time (stationary) statistics is typified by a quasi-periodic structure of the spatial autocorrelation function $g^{(2)}$ of the wave field intensity [33]. While all these remarkable features of IT have been recently demonstrated experimentally in optical fiber [33], they are still not understood theoretically.

IT can be approached from a completely different perspective which is close to classical statistical mechanics. In 1971, V. Zakharov introduced the concept of soliton gas (SG) as an infinite collection of interacting KDV solitons that are randomly distributed in space [34]. SGs have been studied theoretically in different contexts [35–38] and generated very recently in shallow water experiments [39]. Originally introduced for the case of small density [34], the notion of SG has been extended to gases of finite density for both KDV and the focusing 1D-NLSE [36]. Importantly, the macroscopic properties of dense SGs are determined by pairwise collisions of solitons accompanied by phase shifts that are accumulated at long time leading to significant corrections to the average soliton velocities. The key role in the SG theory is played by the spectral (IST) distribution function which has the meaning of the density of states $f(\lambda, x, t)$, so that $f(\lambda_0, x_0, t)d\lambda dx$ is the number of solitons with the spectral parameter λ in the interval $[\lambda_0, \lambda_0 + d\lambda]$ found in the space interval $[x_0, x_0 + dx]$. For a spatially nonuniform SG, the evolution of f is described by a kinetic equation [34,36].

Given the above two approaches, one can naturally pose a question about a possibility of describing IT phenomena, for instance, the development of MI, by considering the SG dynamics for some special spectral (IST) distribution. In fact, the idea to explain the nonlinear stage of MI using soliton interactions was put forward as early as 1972 by V. Zakharov and A. Shabat [2]. However, rather paradoxically, up to now, this possible link between soliton interactions and the MI of the plane wave has not been explored.

In this Letter, we provide a bridge between the two fundamental phenomena of nonlinear physics by showing that SG dynamics explains the salient features of the nonlinear stage of the noise-induced MI. More precisely, we demonstrate a remarkable agreement between the spectral (Fourier) and statistical properties of an unstable plane wave in the long-time evolution and those of a SG representing a random infinite-soliton bound state. The two key ingredients in our analysis are (i) a special choice of the spectral (IST) distribution in the SG and (ii) random phases of the so-called norming constants.

Without loss of generality, we consider the focusing 1D-NLSE in the standard dimensionless form

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + |\psi|^2\psi = 0, \tag{1}$$

and the plane wave solution—the condensate—of unit amplitude $\psi_c(t, x) = \exp it$. Here, ψ represents the complex field, and x and t are space and time. The classical formulation of the noise-induced MI problem is to consider the initial condition composed of the condensate with some additional noise

$$\psi(t = 0, x) = 1 + \eta(x), \tag{2}$$

where η is a small noise, $\langle|\eta|^2\rangle \ll 1$, with zero average, $\langle\eta\rangle = 0$. The condensate is unstable with respect to long-wave perturbations with the growth rate $\Gamma(k) = k\sqrt{1 - k^2/4}$. The spectral width of the noise is assumed to be much larger than the larger unstable wave number $k = 2$. Previously, this problem was widely investigated both numerically, by using periodic boundary conditions in a box of large size [15,33,40,41], and experimentally [33,42].

Nonlinear wave fields ruled by Eq. (1) can be characterized by the so-called scattering data (the IST spectrum). In the general case, the IST spectrum of spatially localized wave fields $\psi(x)$—with zero boundary conditions—consists of the continuous and discrete components. A special class of solutions, the N-soliton solutions (N-SS), exhibits only a discrete spectrum consisting of N complex-valued eigenvalues λ_n, $n = 1, \ldots, N$, and complex coefficients C_n, called norming constants, defined for each λ_n. The key result of the IST theory is that, while the wave field ψ exhibits a complex dynamics, the IST spectrum evolves trivially in time [7]

$$\forall\, n\colon \lambda_n = \text{const}, \qquad C_n(t) = C_n(0)e^{-2i\lambda_n^2 t}. \tag{3}$$

The asymptotic evolution of the N-SS for $t \to \infty$ generally leads to a superposition of N moving solitons (one-soliton solutions) separated in space. Each soliton corresponds to a point λ_n of the discrete spectrum so that $\mathrm{Im}\lambda_n$ is proportional to the soliton's amplitude and $\mathrm{Re}\lambda_n$ to its velocity. The value C_n determines the soliton phase and position.

There is a special class of N-SSs with zero velocities, $\mathrm{Re}\lambda_n = 0$, the so-called bound states. In the following, we consider an ensemble of N-SSs with the eigenvalues located on the imaginary axis and random phases in Eq. (3), i.e., $C_n(0) = |C_n(0)|e^{i\theta_n}$, where θ_n are uniformly distributed in $[0, 2\pi)$. For $N \gg 1$, we assume the following (Weyl's) normalized distribution of the IST eigenvalues $\lambda_n = i\beta_n$:

$$\varphi(\beta) = \beta/\left(\sqrt{1 - \beta^2}\right). \tag{4}$$

We shall call the limit at $N \to \infty$ of the described random N-SS ensemble a bound state SG.

The assumption of random norming constant phases in the long-term evolution of a stochastic field is natural because the phase rotations $-2i\lambda_n^2 t$ for large t introduce an effective randomization. Random phases of norming constants are proposed here to describe IT in the framework of IST. Note that, similarly, in wave turbulence theory, the so-called "random phase approximation" corresponds to random phases of the Fourier components of weakly dispersive waves [43,44].

Our motivation behind the eigenvalues distribution (4) is the Bohr-Sommerfeld quantization rule for the discrete spectrum of the semiclassical Zakharov-Shabat scattering problem with the potential $\psi(x)$ in the form of a real-valued rectangular box of unit amplitude and width $L_0 \gg 1$, see, e.g., [2,7,45],

$$\lambda_n = i\beta_n = i\sqrt{1 - \left(\frac{\pi(n-\frac{1}{2})}{L_0}\right)^2}, \quad n = 1, 2, \ldots, N, \qquad (5)$$

where $N = \mathrm{int}[L_0/\pi]$. Importantly, for a "semiclassical" box, the contribution of the "nonsoliton" part of the field, i.e., of the continuous IST spectrum, decays exponentially with L_0 and, so, can be neglected [7]. The continuous limit of Eq. (5) with $N \to \infty$, $\beta_n \to \beta$ yields the Weyl's distribution (4) for $\varphi(\beta) = (1/N)dn/d\beta$. In this case, the density of states is simply $f(\beta) = (1/L_0)dn/d\beta = (1/\pi)\varphi(\beta)$. In fact, this distribution has been recently shown to follow from a certain criticality condition for the bound state SG [46].

The derivation of the general N-SS of the 1D-NLSE is a classical result of the IST theory [2]. However, the numerical computation of N-SS with large $N \sim 100$ has been realized for the first time only in 2018 [47]. Here, we use the same approach based on the specific implementation of the so-called dressing method [48] combined with 100-digits arithmetics (see [47] and the Supplemental Material [49]).

In the following, we shall compare dynamical and statistical properties of the nonlinear stage of MI with those of a specific bound state SG. The SG is built as a random ensemble of 10^3 realizations of N-SS with $N = 128$, the spectral distribution (4) and soliton phase parameters θ_n uniformly distributed in the interval $[0, 2\pi)$. In the results reported in the main Letter, the discretization of the Weyl distribution (4) is taken at the Bohr-Sommerfeld quantization points (5). Similar results are obtained when soliton eigenvalues are randomly distributed using the probability function (4) (see Supplemental Material [49]).

The density of the gas, i.e., the number of solitons per unit length, plays a crucial role in the dynamics. Our numerical investigations have shown that the higher the density is, the better is the agreement between SG and the stationary state of MI. In the dressing method, the density is empirically controlled by "space position parameters" x_{0n} (see [47]) and reaches its maximum when all $x_{0n} = 0$. Note that, when all $x_{0n} = 0$, all $|C_n| = 1$, and the N-SS is symmetric; in order to avoid this artificial symmetry, we use a random uniform distribution of space position parameters x_{0n} in a narrow interval $[-2, 2]$, so that the soliton density remains practically unchanged.

Finally, it is important to note that the norming constants used in the dressing methods somewhat differ from those appearing in the standard IST formulation (see, e.g., [50]). However, the uniform random phase distribution employed in our dressing construction of N-SS also translates to uniform distribution for the traditional IST phases.

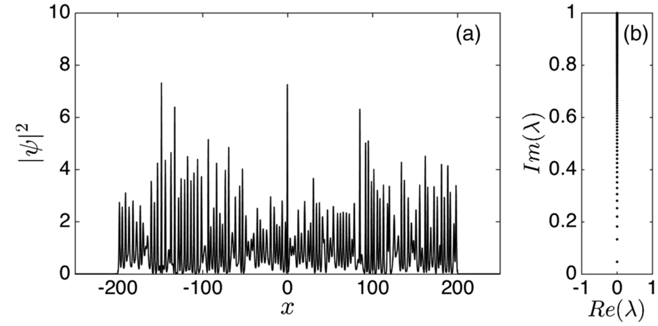

FIG. 1. Example of one realization of 128-SS with random norming constant phases. (a) Intensity profile $|\psi(x, t=0)|^2$. (b) Soliton eigenvalues computed from Eq. (5).

An example of the bound state N-SS with $N = 128$ constructed according to the above described rules is shown in Fig. 1. First, we qualitatively compare the temporal evolutions of this bound state N-SS and of an unstable plane wave (Fig. 2). The evolution of the two states is simulated with the pseudospectral Runge-Kutta fourth-order method as described in [15] in a periodic box of size $L \simeq 570$; for the MI, the noise amplitude is $\sqrt{\langle|\eta|^2\rangle} = 10^{-5}$ (see Supplemental Material [49]). Remarkably, the features characterizing the spatiotemporal dynamics of the N-SS and of the noise-induced MI, which is shown in Fig. 2, are qualitatively very similar; in particular, one can recognize the emergence of the well-known structures resembling the Akhmediev breathers.

Now, we quantitatively compare the statistical properties of SG and of the asymptotic stage of MI. In particular, the long-term evolution of the noise-induced MI can be characterized by stationary values of the potential H_{nl} and kinetic H_l energy and, also, by stationary shapes of the (Fourier) spectrum, the probability density function (PDF) of the intensity $I = |\psi|^2$, and the autocorrelation function of the intensity $g^{(2)}$ (see [15,51]). H_{nl} and H_l are the two parts

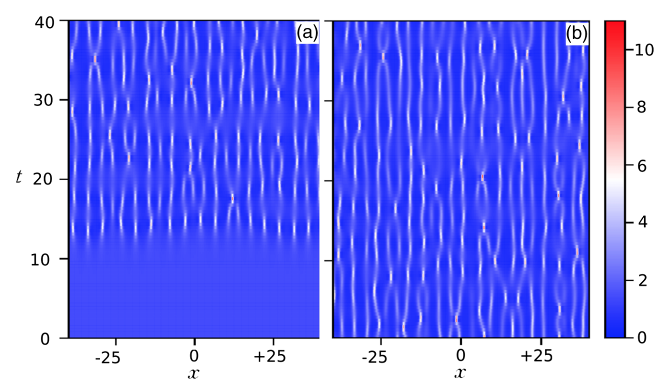

FIG. 2. Numerical simulations of 1D-NLSE: Space-time diagrams of $|\psi(x,t)|^2$. (a) Noise-induced modulational instability of a plane wave. (b) Dynamics of the random phase bound N-SS [the initial condition $|\psi(x,0)|^2$ is shown in the Fig. 1(a)].

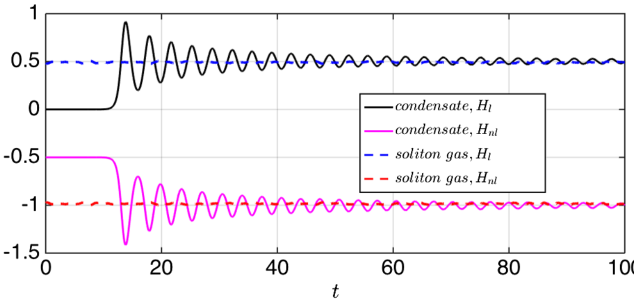

FIG. 3. Time evolution of the ensemble-averaged kinetic $\langle H_l(t)\rangle$ and potential $\langle H_{nl}(t)\rangle$ energies for the noise-induced MI (black and magenta solid lines) and random phase SG (blue and red dashed lines).

of the total energy (Hamiltonian) E, which is one of the infinite constants of motion of the 1D-NLSE [7]

$$E = H_l + H_{nl}, \qquad H_l = \frac{1}{2}\frac{1}{L}\int_{-L/2}^{L/2} |\psi_x|^2 dx,$$
$$H_{nl} = -\frac{1}{2}\frac{1}{L}\int_{-L/2}^{L/2} |\psi|^4 dx. \tag{6}$$

In the case of the MI, the kinetic and potential energies reach, after some oscillatory transient, the stationary values of $\langle H_l\rangle = 0.5$ and $\langle H_{nl}\rangle = -1$, see [15] and Fig. 3. Meanwhile, the considered bound state SG is characterized by the same stationary values of $\langle H_l\rangle$ and $\langle H_{nl}\rangle$ (dashed lines in Fig. 3) from the start of the evolution. Note that $\langle \ldots \rangle$ means an ensemble-averaging over 1000 random phase realizations here.

In the following, for practical reasons, we perform ensemble averaging together with temporal averaging, both for the noise-induced MI and for the N-SS. In the case of the condensate, the temporal averaging is performed when the system is sufficiently close to the asymptotic stationary state by its statistical properties ($t \in [160, 200]$). In the case of the N-SS, time averaging is started from the very beginning of the system evolution. It is extremely important to note that the results obtained solely using the ensemble averaging over random phase realizations are the same as those obtained using both ensemble and time averaging (see Supplemental Material [49]).

As demonstrated in Fig. 4(a), the wave-action spectrum,

$$S_k \propto \langle |\psi_k|^2 \rangle, \qquad \psi_k = \frac{1}{L}\int_{-L/2}^{L/2} \psi e^{-ikx} dx, \tag{7}$$

of the asymptotic state of the MI and of the considered SG coincide with excellent accuracy. Note that, for the SG case, S_k is renormalized proportionally to the spatial extension of the wave field (see Supplemental Material [49]).

Moreover, both the SG and the noise-induced MI exhibit nearly identical PDF $\mathcal{P}(I)$ of the field intensity $I = |\psi|^2$ [Fig. 4(b)]. Thus, the PDF of the N-SS quantitatively reproduces the exponential distribution discovered

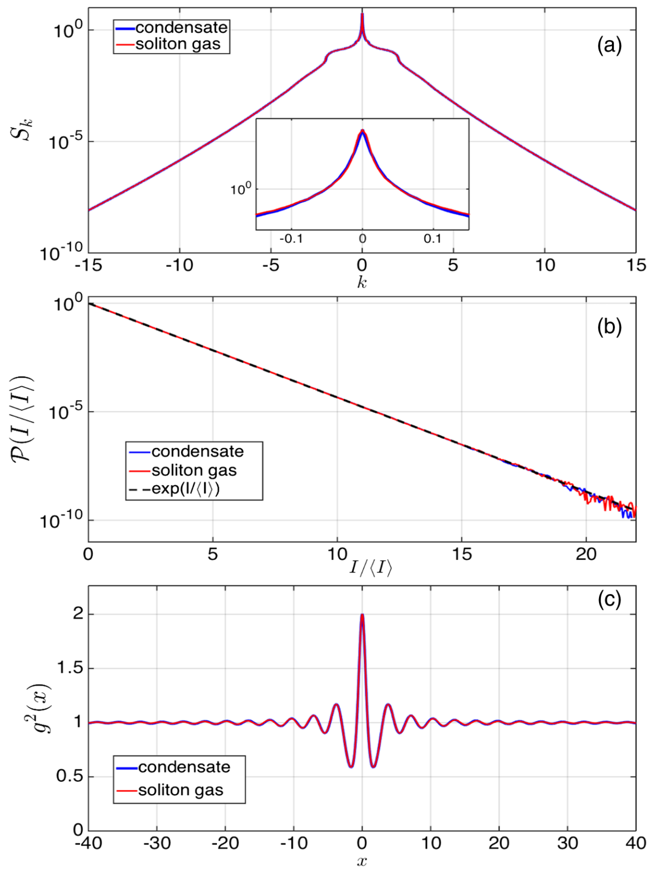

FIG. 4. Comparison of ensemble averaged statistical characteristics of the asymptotic state of the MI development and of random phase 128-SSs. (a) Wave action spectrum S_k. (b) The PDF $\mathcal{P}(I)$. (c) Autocorrelation function of intensity (second order degree of coherence) $g^{(2)}(x)$.

earlier as the asymptotic statistics of the unstable condensate [15,33].

As has been shown very recently in [33], the long-term evolution of the MI is typified by an oscillatory structure of autocorrelation of the intensity $g^{(2)}(x)$

$$g^{(2)}(x) = \frac{\langle I(y,t) I(y-x,t)\rangle}{\langle I(y,t)\rangle^2}, \tag{8}$$

which represents the second-order degree of coherence. The SG accurately reproduces this remarkable oscillatory shape [see Figs. 4(c)].

Note that H_l, H_{nl}, the PDF, and the $g^{(2)}$ functions for the SG case were computed in the central part of the 128-SSs. More precisely, we used the region $x \in [-150; 150]$ where the ensemble-average wave field intensity of the N-SS is practically uniform and very close to unity, that allows us to mitigate the edge effects—see details in the Supplemental Material [49].

In this Letter we have demonstrated that the spectral (Fourier) and statistical properties of the asymptotic state of the MI precisely coincide with those of some specific SG. This SG can be constructed with exact N-soliton solutions of the 1D-NLSE having large values of N and one specific distribution of IST eigenvalues coinciding with the semi-classical distribution (4) for the discrete spectrum of the box potential [2]. As could be expected, the long-term statistical state of MI corresponds to a full stochastization of IST (soliton) phases.

Our numerical investigations reveal that spectral (Fourier) and other statistical properties (PDF and $g^{(2)}$) of SG made of stochastic N-soliton bound states are very sensitive to the exact distribution of the IST eigenvalues. An extensive study of all possible types of distributions is beyond the scope of this Letter. However, it is important to note that, for all the distributions that we have tried [except for Eq. (4)], the statistical properties of the SG do not coincide with the ones of the MI and/or are strongly nonhomogeneous in space. For example, if the eigenvalues are uniformly distributed in the interval $[0:i]$, the average intensity $\langle|\psi(x,t)|^2\rangle$ of the SG strongly depends on the position x (see Supplemental Material [49]). Thus, our investigations strongly suggest that the distribution (4) plays a very peculiar role in SG theory because (i) it models MI and (ii) it enables building a dense homogeneous SG, having constant statistical properties over a significant part of the space, by using a finite number of solitons.

We believe that the work reported in this Letter opens a new promising direction in the theory of integrable turbulence by establishing a link between the MI and SG dynamics. In particular, as is known, the nonlinear stage of the MI induced by small harmonic perturbations is characterized by the generation of Akhmediev breathers [28–31]. The picture is more complicated if the initial perturbation is a random noise, leading to IT with various breather structures appearing only locally [15,41,42,52]. Breathers also appear locally in some multisoliton interactions [53,54]. Our approach, based on the well-known multisoliton solutions, can shed light on the possible connection between soliton and breather gases [32,36,46].

Note that the mechanism underlying the MI induced by spatially homogeneous noise studied here is *a priori* different from the MI induced by localized perturbations which are studied within the framework of IST with nonzero boundary conditions [51,55–60]. In this work, we have demonstrated that the IST formalism for wave fields with decaying boundary conditions can be successfully applied to accurately describe the asymptotic stationary state of the MI computed numerically by using periodic boundary conditions [15]. An important mathematical problem for future studies is to explain the link between spatially periodic and localized descriptions of the MI in terms of the IST theory.

Finally, note that our model can be generalized to a broad class of IT problems when the (random) wave field is strongly nonlinear so that the impact of the nonsolitonic content can be neglected in the asymptotic state ($t \to \infty$). For such a case, our work suggests a general strategy for studying the asymptotic state: build N-soliton solutions with the distribution $\varphi(\lambda)$ of IST eigenvalues characterizing the field and random phases of the norming constants.

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