Universality Classes of Stabilizer Code Hamiltonians

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Stabilizer code quantum Hamiltonians have been introduced with the intention of physically realizing a quantum memory because of their resilience to decoherence. In order to analyze their finite temperature thermodynamics, we show how to generically solve their partition function using duality techniques. By unveiling each model's universality class and effective dimension, insights may be gained on their finite temperature dynamics and robustness. Our technique is demonstrated in particular on the 4D toric code and Haah's code; we find that the former falls into the 4D Ising universality class, whereas Haah's code exhibits dimensional reduction and falls into the 1D Ising universality class.

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The stabilizer formalism is a powerful mathematical framework for designing quantum error correcting codes [1–6]. Kitaev proposed to turn a stabilizer code into an interacting many-body system by associating a coding space to the ground state subspace of a stabilizer code Hamiltonian, a linear combination of elements of the stabilizer group [7,8]. Stabilizer code Hamiltonians display a gapped spectrum with a topologically quantum ordered ground manifold and where errors, typified by finite energy excitations, become energetically unfavorable at zero temperature. These systems are natural candidates for physical realization of a robust quantum memory, a q RAM or q hard drive, because of their inherent resilience to decoherence. Since Kitaev's original proposal, several stabilizer code models have been advanced in various spatial dimensions D, including the most recent *fracton* models [9–14].

It was emphasized long ago that the effect of temperature on these memories cannot be ignored [15–17]. Finite temperature decoherence times may be affected by effective *dimensional reduction* [15,18,19]. Specifically, the spectral degeneracy of stabilizer code models is associated with symmetries that may involve a macroscopic fraction of degrees of freedom, the so-called d-dimensional gaugelike symmetries [20,21], $0 \leq d \leq D$, later on dubbed "subsystem" symmetries [22–25]. Duality transformations [26–28] may unveil the lower dimensional *classical* theory isomorphic to the stabilizer code model. Such dual theories exhibit nonanalyticities (and critical exponents associated with continuous transitions [29]) of identical character, and therefore belong to the same universality class. Understanding the universality classes and dynamics of stabilizer models may aid in the design of robust quantum memories.

The primary goal of this Letter is to show how duality techniques can be utilized to exactly determine the partition function of stabilizer code Hamiltonians. In particular, we demonstrate how the stabilizer algebra encodes any nonanalyticities (or lack thereof) in the thermodynamic free energy of the corresponding stabilizer Hamiltonian, via the scaling of constraints on the stabilizer algebra with system size. The Abelian nature of the stabilizer group allows for a particularly simple analysis: while the studied models are in principle constructed using a large number of entangled quantum spins, the resulting algebra will be shown to factor into independent Ising algebras. Consequently, the partition function of *any* CSS stabilizer code Hamiltonian [30] may be easily analyzed using various duality techniques. The effective dimensionality of the resulting classical models vary depending on the constraints. We find that $D = 2$ or 3 dimensional stabilizer models are often dual to classical Ising chains, implying the absence of phase transitions in many stabilizer models (see Table I).

In this Letter, we analyze the 4D toric code [8] and Haah's 3D cubic code [13,14,35]. These models represent two extremes of the dimensional reduction paradigm: we will show that the 4D toric code (4DTC) features no dimensional reduction and belongs to the 4D Ising universality class. By contrast, typical odd lattice size renditions of Haah's 3D cubic code lie in the 1D Ising universality class. The 4DTC therefore exhibits a finite temperature phase transition with critical exponents given exactly by those of mean field theory, while Haah's cubic code may be unstable to thermal fluctuations (a phenomenon known as *thermal fragility* [15]) and exhibit no finite temperature transitions.

Methodology.—We will investigate thermal properties of the above two stabilizer Hamiltonians by identifying their classical Ising duals. The models are defined on D-dimensional lattices $\Lambda = \mathbb{Z}_L^D$ of length L in each direction with vertices $v = (x, y, z, \ldots) \in \Lambda$. The lattices Λ are endowed with periodic boundary conditions, although any local finite temperature properties should not depend

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TABLE I. Universality classes of stabilizer code Hamiltonians. D is the spatial dimension of the lattice model. d is the dimension of the gaugelike symmetries. Dualities are defined as equivalence relations between partition functions: the 3DTC, for example, has a partition function proportional to the product of a 1D Ising and a 3D Ising partition function. While Chamon's *XXYYZZ* model is not a stabilizer code, it can also be shown by duality to exhibit dimensional reduction [19]. Additionally, while all listed models above are constructed using Pauli operators, very similar results may be obtained for non-Pauli models, such as those with $\mathbb{Z}_p$ clock operators or $U(1)$ operators. *: While the X-cube model's universality class does not depend on any choice of boundary conditions, the particular duality chosen holds for the case of cylindrical boundary conditions. **: The duality given below for Haah's code holds explicitly for those values of L for which the ground state degeneracy (GSD) is 4.

Model	D	d	Dual model	Universality Class
2D toric code [7,15]	2	1	Two decoupled 1D Ising chains	1D Ising
2D honeycomb toric code [19,31]	2	1	Two decoupled 1D Ising chains	1D Ising
Color codes [19,32]	2	1	Two decoupled 1D Ising chains	1D Ising
3D toric code [15,33]	3	0, 1	Decoupled 1D Ising and 3D Ising models	3D Ising
X-cube* [9,34]	3	1, 2	Decoupled L 1D Ising and $L-1$ 1D Ising gauge	1D Ising
Haah's code** [13,14,35]	3	2	Two decoupled 1D Ising chains	1D Ising
4D toric code [8,36]	4	2	Two decoupled 4D Ising models	4D Ising
Chamon's *XXYYZZ* [19,28,37]	3	1	Four decoupled 1D Ising chains	1D Ising

on the boundary conditions in the thermodynamic limit. We associate with the lattice N qubits, each with a local complex Hilbert space $\mathcal{H}_n = \mathbb{C}^2$; the global Hilbert space $\mathcal{H} = \otimes_n \mathcal{H}_n$ is of dimension 2^N. Each qubit belongs to a unit k cell of the lattice: $k = 0$, 1, and 2 represent qubits on the vertices, links, and plaquettes of the lattice, respectively. Our arguments are easily generalized to p qudits and $U(1)$ models [34,38].

Next, we define the operators A_r and B_s as:

$$A_r \equiv \prod_{n\in N_r} \sigma_n^x, \qquad 1 \le r \le R,$$
$$B_s \equiv \prod_{n\in N_s} \sigma_n^z, \qquad 1 \le s \le S, \tag{1}$$

where N_r and N_s are indexing sets used to generate R operators A_r and S operators B_s, respectively. We further require that each A_r and B_s commute. The Hamiltonian for this generic stabilizer model reads

$$H = -a\sum_{r=1}^{R} A_r - b\sum_{s=1}^{S} B_s, \tag{2}$$

with coupling constants $a, b > 0$. All operators in (2) commute and square to the identity $\mathbb{1}$ on $\mathcal{H}$. The partition function is then given by the following high-temperature $[\beta = 1/(k_B T)]$ series expansion:

$$\mathcal{Z} = \mathrm{Tr}e^{-\beta H} = \mathrm{Tr}\left(\prod_{r=1}^{R}(\mathbb{1}\mathsf{C}_a + A_r\mathsf{S}_a)\prod_{s=1}^{S}(\mathbb{1}\mathsf{C}_b + B_s\mathsf{S}_b)\right)$$
$$= 2^N \mathsf{C}_a^R \mathsf{C}_b^S \mathcal{T}_a \mathcal{T}_b. \tag{3}$$

Here, $\mathsf{C}_a \equiv \cosh(\beta a)$ and $\mathsf{S}_a \equiv \sinh(\beta a)$, with C_b and S_b similarly defined. In the above, $\mathcal{T}_a$ (and analogously $\mathcal{T}_b$) are given by

$$\prod_{r=1}^{R}[\mathbb{1} + A_r\mathsf{T}_a] = \mathcal{T}_a\mathbb{1} + \text{t.t.}, \quad \text{with} \quad \mathcal{T}_a = \sum_{P\in\mathcal{A}} \mathsf{T}_a^{|P|}, \tag{4}$$

with "t.t." denoting traceless terms and $\mathsf{T}_a \equiv \tanh(\beta a)$ [and $\mathsf{T}_b \equiv \tanh(\beta b)$]. $P \in \mathcal{A}$ ($\mathcal{B}$) denotes operators A_r (B_s) multiplying to $\mathbb{1}$,

$$\prod_{\ell\in P} A_\ell = \mathbb{1} \quad \forall\ P \in \mathcal{A}. \tag{5}$$

Each P corresponds to a *constraint* on the stabilizer algebra. The only terms contributing to the trace in (3) are those proportional to the identity ($2^N = \mathrm{Tr}[\mathbb{1}]$). The traceless terms (t.t.) in (4) and those corresponding to $\mathcal{T}_b$ cannot combine to yield the identity—by construction in (1), there are no nontrivial constraints between A_r and B_s operators. We have thus reduced the problem of solving each model's partition function to identifying which and how many constraints exist among A_r or B_s operators separately. From the partition function, we may then compute the thermodynamic free energy density,

$$f(\beta) = \lim_{L\to\infty} -\frac{1}{\beta L^D}\log\mathcal{Z}. \tag{6}$$

This means of describing the thermodynamics of spin models is particularly effective for stabilizer Hamiltonians. The algebra of a stabilizer Hamiltonian has three important properties: (i) each element of the stabilizer commutes with one another, (ii) each element of the stabilizer is usually composed of either entirely σ^x or σ^z operators

(these stabilizer codes are known as CSS codes [30], and most stabilizer code Hamiltonians fall into this category), and (iii) each element has eigenvalues ± 1. This implies that the stabilizer algebras we investigate factor into two classical Ising algebras. As a result, these stabilizer Hamiltonians are dual [26–28] to classical Ising-like Hamiltonians using bond-algebraic dualities [28].

4D toric code.—As befits its name, the 4DTC [8,39] is defined on a $D = 4$ dimensional lattice. Qubits are associated with all $(6L^4)$ plaquettes p. For each link ℓ, the operator A_ℓ is defined by

$$A_\ell \equiv \prod_{\ell \in \partial p} \sigma_p^x, \tag{7}$$

where the above product is over the six plaquettes p whose boundary ∂p contains the link ℓ. The operator B_c is defined for each three-dimensional cube as

$$B_c \equiv \prod_{p \in \partial c} \sigma_p^z. \tag{8}$$

The above product is over the six plaquettes contained in the cube c's boundary. The Hamiltonian H_{4DTC} and partition function $\mathcal{Z}_{\mathrm{4DTC}}$ are as defined in (2) and (3), respectively, and it is trivially verified that each A_ℓ and B_c commute. For an analysis of its gaugelike symmetries see [40].

We now show that the 4DTC is dual to two copies of the 4D nearest neighbor Ising (4DI) model defined by

$$H_{\mathrm{4DI}} = -J \sum_{\langle v, v' \rangle} s_v s_{v'}, \tag{9}$$

in the sense that the thermodynamic free energy density can be trivially written in terms of the 4D Ising model's free energy. In (9), s_v is a classical spin variable at each vertex $v \in \Lambda$ and the sum is over all nearest neighbor pairs v, v' in Λ. We express the partition function of this model via a *low* temperature series expansion. Starting from the ground state of $s_v = +1$ for all v, we consider excited states and expand in the number of higher energy "broken bonds"; a "broken bond" corresponds to $s_v s_{v'} = -1$ for a nearest neighbor pair v, v'. The partition function is then given by

$$\mathcal{Z}_{\mathrm{4DI}} = 2 e^{4L^4 \beta J} \sum_{\mathcal{C} \subset \Lambda} e^{-2\beta J \Delta_{\mathcal{C}}}. \tag{10}$$

Each $\mathcal{C}$ represents a set of flipped spins from the chosen ground state. We demand that $|\mathcal{C}| \leq L^4/2$, noting that each configuration $\mathcal{C}$ has a global spin-flip "symmetry partner" $\Lambda \backslash \mathcal{C}$ with the same bond structure as $\mathcal{C}$. The ground state energy is $-4L^4 J$, and $\Delta_{\mathcal{C}}$ is the number of "broken bonds" in configuration $\mathcal{C}$ with $2J$ being the energy penalty for "breaking a bond."

We begin investigating $\mathcal{T}_a$ by noting the constraint

$$\prod_{v \in \partial \ell} A_\ell = \mathbb{1}. \tag{11}$$

The above product is over the eight links containing the vertex v. This can be verified by noting that there are 24 plaquettes adjacent to v (four for each μ-ν plane), and each plaquette is included by exactly two links in (11). Higher order constraints may be found by taking products of (11) for some subset $\mathcal{C}$ of vertices, and eliminating any A_ℓ included in the product twice. Note that each A_ℓ for a given link $\ell = \{v, v'\}$ will be included in such a product if and only if $v \in \mathcal{C}$ or $v' \in \mathcal{C}$, so that $\mathcal{C}$ and $\Lambda \backslash \mathcal{C}$ yield the same constraint (see Fig. 1).

This set of constraints suggests the following duality: for each spin flip configuration $\mathcal{C}$ in the Ising low temperature expansion (10), we obtain a unique identity product in the 4DTC high temperature expansion. Moreover, each A_ℓ operator included in such a constraint must correspond exactly to a bad bond in $\mathcal{C}$. This shows that the series (10) is entirely contained within (4), with a global prefactor and the replacement $e^{-2\beta J} \to \mathsf{T}_a$:

$$\mathcal{T}_a = \frac{1}{2} \mathsf{T}_a^{2L^4} \mathcal{Z}_{\mathrm{4DI}} \left(\frac{1}{2J} \log \frac{1}{\mathsf{T}_a} \right) + \mathcal{O}(\mathsf{T}_a^{L^3}), \tag{12}$$

where the terms of order $\mathsf{T}_a^{L^3}$ or higher, $\mathcal{O}(\mathsf{T}_a^{L^3})$, arise due to the topology of the lattice, and are negligible to the thermodynamic free energy.

Turning to $\mathcal{T}_b$, we similarly note the constraint

$$\prod_{c \in \partial h} B_c = \mathbb{1}, \tag{13}$$

where the product is over the eight cubes contained in a minimal 4D hypercube h. By multiplying such constraints,

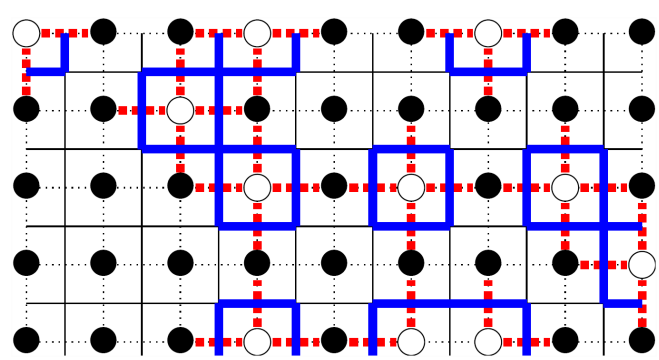

FIG. 1. A 2D cross section of a 4D lattice, with classical Ising spins at each site. Each red dotted link corresponds to a broken bond, and each blue loop is the 1D cross section of a 3D hypersurface domain wall. In the 4D lattice, the product of A_ℓ over all red links and the product of B_c over all blue cubes correspond to two independent constraints.

we can generalize (13) to include products over any closed three-dimensional hypersurface; the number of B_c operators in such products is the three-dimensional hypersurface area (Fig. 1). This set of constraints also suggests a duality to (10) via another lens. Here, instead of placing Ising spins at each vertex of the lattice, we imagine placing spins at the *center* of each hypercube h, creating another lattice Λ' a half-spacing off from Λ. Whereas the A_ℓ in (11) stem from broken bonds in the Ising model on the same lattice, each B_c operator in this duality represents a *domain wall* separating different spin orientations in an Ising model on Λ'. The hypersurface area of this domain wall is equal to the number of broken bonds in the Ising configuration. From this, we see that the same duality as (12) holds for $\mathcal{T}_b$, with T_a replaced with T_b. Indeed, once $\mathcal{Z}_{\mathrm{4DTC}}$ has been factored as in (3), this duality is known as a Wegner duality [27], or more particularly as a "lattice gerbe theory" duality [41].

One might reasonably worry that the above discussion is too cavalier: although it is clear that each pair of spin flip configurations $\mathcal{C}$ and $\Lambda\backslash\mathcal{C}$ generates a unique constraint via (11), how do we know that all terms of (4) below order $\mathsf{T}_a^{L^3}$ can be found this way? Additionally, how do we know that the subextensive contributions to $\mathcal{Z}_{\mathrm{4DTC}}$ are negligible in the free energy's thermodynamic limit? These questions, and analogous ones for $\mathcal{T}_b$, are addressed with a careful proof of the duality (12) in the Supplemental Material [40].

Haah's code.—Haah's code is defined as follows [12,13]: let Λ be a $D = 3$ lattice, where we associate *two* qubits with each vertex $v \in \Lambda$. Letting σ_v^μ and τ_v^μ label the first and second qubits, respectively, at each vertex, the operator A_v is then defined as in Fig. 2:

$$A_v \equiv \sigma_v^x \tau_v^x \tau_{v+e_x}^x \tau_{v+e_y}^x \tau_{v+e_z}^x \sigma_{v+e_x+e_y}^x \sigma_{v+e_x+e_z}^x \sigma_{v+e_y+e_z}^x. \tag{14}$$

The operator B_v is similarly defined as in Fig. 2:

$$B_v \equiv \tau_{v+e_x}^z \tau_{v+e_y}^z \tau_{v+e_z}^z \sigma_{v+e_x+e_y}^z \sigma_{v+e_x+e_z}^z \sigma_{v+e_y+e_z}^z \sigma_{v+e_x+e_y+e_z}^z \tau_{v+e_x+e_y+e_z}^z. \tag{15}$$

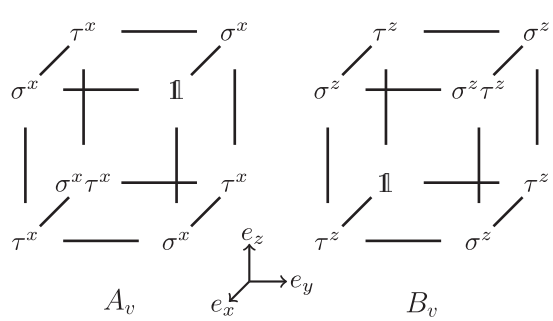

FIG. 2. Haah's code: the two operators of Eqs. (14) and (15).

As usual, $[A_v, B_{v'}] = 0$ for any two sites v and v', and the model's Hamiltonian H_{Haah} and partition function $\mathcal{Z}_{\mathrm{Haah}}$ are defined as in (2) and (3), respectively. For an analysis of its gaugelike symmetries see [40]. Note that the operators A_v and B_v are simply reflections of one another, so their constraints will be identical.

Haah's code features an intricate ground state degeneracy (GSD) [14]: unless L is a multiple of $4^p - 1$ for $p \geq 2$, $\mathrm{GSD} = 4$ for odd L [13]. We will restrict our attention to these models, as they are the most pertinent to quantum error correction: it has been argued [14,35] that, for these values of L, the model demonstrates long memory timescales at low temperatures.

While the nature of constraints in H_{Haah} varies wildly for different values of L, the number of constraints (including the trivial empty product) is always equal to $\log_2 \mathrm{GSD}$ [14]. Thus, when $\mathrm{GSD} = 4$, the two independent constraints are those present for all L,

$$\prod_{v\in\Lambda} A_v = \mathbb{1}, \qquad \prod_{v\in\Lambda} B_v = \mathbb{1}. \tag{16}$$

These relations can easily be verified by observing that the product of each of the eight corners of the cubic operators yields the identity. The partition function is then:

$$\mathcal{Z}_{\mathrm{Haah}} = 2^{2L^3}(\mathsf{C}_a^{L^3} + \mathsf{S}_a^{L^3})(\mathsf{C}_b^{L^3} + \mathsf{S}_b^{L^3}). \tag{17}$$

Alternatively, let s_i and t_i be classical Ising spins ($1 \leq i \leq L^3$). Then, within the bond-algebraic framework of dualities as isomorphisms between the terms ("bonds") appearing in dual Hamiltonians [28], the mapping

$$A_v \to s_i s_{i+1}, \qquad B_v \to t_i t_{i+1}, \qquad 1 \leq i \leq L^3 \tag{18}$$

with $L^3 + 1 \equiv 1$ similarly implies the duality of Haah's code to two periodic Ising chains. This duality suggests that the finite temperature dynamics of Haah's code are identical to those of finite temperature classical Ising chains, and may thus be unstable to thermal fluctuations (i.e., exhibit "thermal fragility" [15]).

We caution that the duality (18) only holds exactly for the case $\mathrm{GSD} = 4$: although the thermodynamic limit can be taken along an infinite sequence of L satisfying this condition [35], each variation of constraints at different L requires a new bond-algebraic duality. Nevertheless, the number of constraints never exceeds $2^{\mathcal{O}(L)}$ for any system size, and the free energy remains analytic at any finite temperature in any thermodynamic limit as a result [14]. This suggests that Haah's code may remain in the 1D Ising universality class along any thermodynamic limit.

Conclusions.—We showed how to generically analyze the partition function of a CSS stabilizer code Hamiltonian using duality techniques. We illustrated our strategy on the 4DTC and Haah's cubic code, two quintessential stabilizer

codes. Our results support the generally held belief that the 4DTC exhibits self-correcting properties at sufficiently low temperatures. While several works suggested that Haah's code may be partially self-correcting at finite temperatures [13,14,35,42], our results instead suggest that Haah's code may suffer the same thermal fragility as Ising chains. The dimensional reduction implied by generalized Elitzur's theorem [19,20] bounds general correlation functions on d-dimensional subsystems. This, however, does not imply that the thermodynamics is that of canonical d-dimensional systems [18]. Examples to that effect are afforded by (i) the 90° square lattice compass model [43], a system with $d = 1$ symmetries and 2D Ising universality class and (ii) the "XXZ honeycomb model" [44], a compass model with similar $d = 1$ symmetries, that is dual to the 2D quantum Ising lattice gauge theory. It should indeed be noted that even in the absence of exact dualities, the bounds implied by Elitzur's theorem (reducing various correlation functions to those in conventional d-dimensional systems) may suggest restrictions on memory times [15,19,20].

The 2D Ising model can serve as a self-correcting classical memory below its critical temperature, in the sense that the information stored is robust to magnetic or thermal fluctuations [45,46]. By contrast, the 1D Ising model suffers a finite memory timescale independent of system size at all nonzero temperatures due to the absence of an ordered phase [45,47]. Similarly, it is commonly accepted that the 2DTC suffers from relaxation times independent of system size, while the 4DTC is believed to function as a robust quantum memory below a critical temperature [15,16,36,48]. The relationship between these classical and quantum memories can be understood through duality: bond algebraic dualities suggest that the dynamics of the 2DTC on the torus are identical to that of two 1D Ising chains. In the Supplemental Material [40], we discuss how the topological degeneracy commonly associated with these models appears as a global prefactor in the dual partition function. A common concern regarding the use of dualities for analyzing dynamics is that local heat bath perturbations become generally nonlocal in the dual model. In the Supplemental Material [40], we show that a local coupling to a heat bath in the 2DTC can induce a local coupling in the dual Ising model as well.

This analogy highlights the utility of the duality techniques developed in this Letter: by determining a stabilizer code Hamiltonian's classical dual and corresponding universality class, one obtains all information regarding the model's critical phenomena without performing detailed numerical analyses (see Table I). Using the techniques explicitly demonstrated here and in previous works, we conjecture that all sufficiently generic stabilizer models—CSS and beyond—can be analyzed for thermodynamic and, in some cases, dynamical behaviors. A finite temperature phase transition and corresponding stable phase may be a crucial ingredient [5,15] for large autocorrelation times and robust quantum memories. As we will elaborate elsewhere, dualities similar to those invoked in the current work may find further applications in the study of entropic measures.

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