Solenoidal Scaling Laws for Compressible Mixing

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Mixing of passive scalars in compressible turbulence does not obey the same classical Reynolds number scaling as its incompressible counterpart. We first show from a large database of direct numerical simulations that even the solenoidal part of the velocity field fails to follow the classical incompressible scaling when the forcing includes a substantial dilatational component. Though the dilatational effects on the flow remain significant, our main results are that both the solenoidal energy spectrum and the passive scalar spectrum assume incompressible forms, and that the scalar gradient essentially aligns with the most compressive eigenvalue of the solenoidal part, provided that only the solenoidal components are consistently used for scaling. A slight refinement of this statement is also pointed out.

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A defining feature of turbulence is the ability to mix substances with orders of magnitude greater effectiveness than molecular mixing. The subject has been studied extensively [1] when the mixing agent is incompressible turbulence because it is a fundamentally important problem in its own right and a good paradigm for many practical circumstances. However, there are critically important applications from astrophysics to high-speed aerodynamics in which compressibility needs to be explicitly considered. Including compressibility renders inapplicable the Reynolds number scaling laws [2,3] that are used extensively in incompressible turbulence. This paper shows one successful way of incorporating compressibility explicitly. We show by three examples that the standard incompressible laws work in the compressible case by rescaling the appropriate variables.

The initial attempt to include compressibility was made through a suitably defined Mach number as an additional parameter. For the ideal case of homogeneous isotropic turbulence in a cubic box with periodic boundary conditions, this Mach number, $M_t = u'/\langle c\rangle$, where $u' = \langle u_i^2\rangle^{1/2}$, u_i being the velocity in the Cartesian direction i, and the angular brackets indicate a suitable average. However, as has been pointed out by Ni [4], M_t is not adequate when the velocity field has a strong dilatational component. (By construction, Helmholtz decomposition leads to a solenoidal part, $\mathbf{u_s}$, which represents vortical contribution and satisfies the incompressibility condition, and a dilatational part, $\mathbf{u_d}$, which represents the irrotational component.) Indeed, DNS data with different types of large scale forcing, such as pure solenoidal forcing [5–7], homogeneous shear forcing [8], dilatational forcing [9,10], and thermal forcing [11] have revealed that the dilatational flow field characteristics depend on the details of forcing, even for fixed M_t. Further progress has been made recently [12] by adding yet another parameter, namely, δ, which is the ratio of root-mean-square (rms) dilatational velocity to the solenoidal counterpart. The improved physical understanding that arises from Ref. [12] can be used to assess the scaling of the passive scalars in compressible turbulence.

Even though the inadequacy of classical scaling in compressible turbulence has been pointed out as just described, it has not been demonstrated using data for a variety of forcing schemes with a varying dilatational component for both velocity and scalar fields. We show this here conclusively by using the data from direct numerical simulations (DNS) of compressible Navier-Stokes equations in a periodic box yielding homogeneous and isotropic turbulence, and span the following conditions: the microscale Reynolds number $R_\lambda \equiv \langle\rho\rangle u'\lambda/\mu$, where $\langle\rho\rangle$ is the mean density, λ is the Taylor microscale and μ the mean dynamic viscosity, ranges from 38 to 165; the turbulent Mach number, M_t, varies between 0 and about 0.6; the Schmidt number $\mathrm{Sc} = \mu/[\langle\rho\rangle D]$, where D is the diffusivity of the scalar, is unity. A particularly important point is that the forcing at low wave numbers contains a strong dilatational component as well, with δ ranging from 0 to 7.5. Figure 1 shows the wide range of compressibility conditions covered for the scalar field in the parameter spaces of R_λ, δ, and M_t.

The first instance of the inadequacy of incompressible scaling is the energy spectrum which, according to Ref. [13], follows the relation $E(k) = C\langle\epsilon\rangle^{2/3}k^{-5/3}$ in the inertial range, where C is the Kolmogorov constant, k is the wave number, and $\langle\epsilon\rangle$ is the mean total energy dissipation. The energy spectrum has the property that

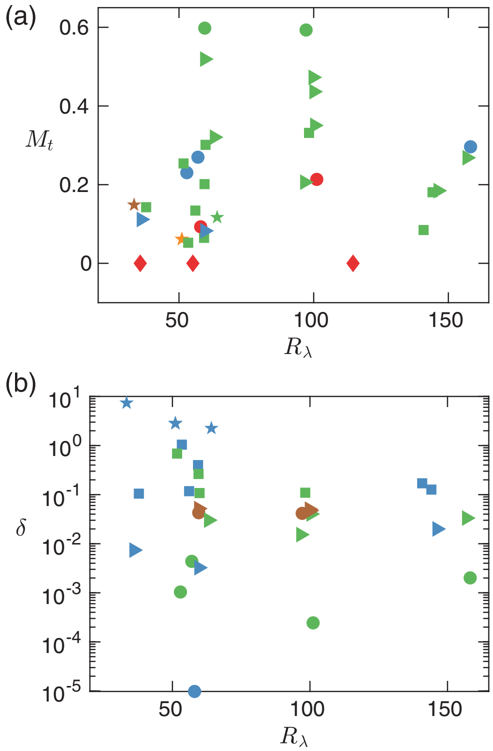

FIG. 1. Parameter space of simulations for the scalar field. (a) $M_t - R_\lambda$ plane: red, $\delta < 10^{-3}$; blue, $10^{-3} < \delta < 10^{-2}$; green, $10^{-2} < \delta < 10^{-1}$; orange, $10^{-1} < \delta < 1$; brown, $1 < \delta < 10$. (b) $\delta - R_\lambda$ plane: blue, $M_t < 0.2$; green, $0.2 < M_t < 0.4$; brown, $0.4 < \delta < 0.7$. Symbols in both figures correspond to different percentages of dilatational forcing, σ: diamonds, incompressible simulations; circles, $\sigma = 0$; triangles, $\sigma = 10$–30; squares, $\sigma = 30$–65; stars, $\sigma = 65$–100.

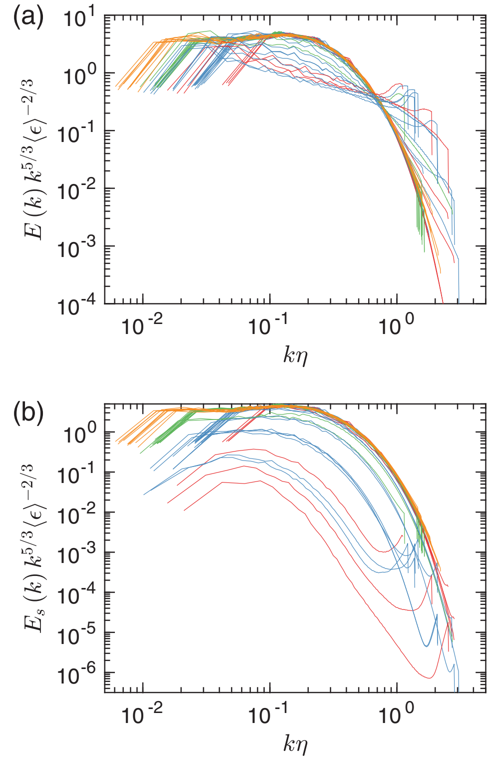

FIG. 2. (a) Kolmogorov-compensated total energy spectra using total energy dissipation and Kolmogorov length scale $[\eta = (\mu^3/\langle\rho\rangle^2\langle\epsilon\rangle)^{1/4}]$ [5]. (b) Kolmogorov-compensated solenoidal energy spectra using total dissipation and the Kolmogorov length scale based on it. Here and in all figures to follow except Fig. 7, different colors correspond to different Reynolds numbers. Red, $R_\lambda < 40$; blue, $40 < R_\lambda < 75$; green, $75 < R_\lambda < 115$; and orange, $115 < R_\lambda < 170$. The velocity data of this figure and in Fig. 5(a) include a larger set of conditions than shown for the scalar field in Fig. 1.

$\int_0^\infty E(k)dk = \langle u^2 \rangle/2; \langle u^2 \rangle = \langle u_i u_i \rangle$. In Fig. 2(a) we see that, unlike in incompressible turbulence, there is no collapse of spectral data when normalized according to Ref. [13]. This is not surprising: it has been pointed out already in theories [14–16] and simulations [5,7,9,10] that the dilatational component of energy can take on a wide range of behaviors and can depart from the classical Kolmogorov scaling.

As an improvement, it has been suggested [5,7] that the solenoidal part of the energy spectra ($\nabla \cdot \mathbf{u_s} = \mathbf{0}$) does scale according classical Kolmogorov scaling; the basis for this claim comes from solenoidally forced DNS. However, this result does not hold when the forcing has a strong dilatational component, as shown in Fig. 2(b), where the Kolmogorov-compensated solenoidal energy spectra $E_s(k)$, defined such that $\int_0^\infty E_s(k)dk = \langle u_s^2 \rangle/2$, does not scale when the forcing includes a dilatational part. The second instance of this inadequacy is the scalar spectrum. In incompressible turbulence, its behavior is reasonably well understood at the phenomenological level [1,17–23]. For unity Schmidt number, the appropriate normalization for the passive scalars is the Obukhov-Corrsin scaling $E_\phi(k) = C' \langle \epsilon_\phi \rangle \langle \epsilon \rangle^{-1/3} k^{-5/3}$, where E_ϕ is defined such that $\int_0^\infty E_\phi(k)dk = \langle \phi^2 \rangle/2$ and $\langle \epsilon_\phi \rangle$ is the mean scalar dissipation; C' is the Obukhov-Corrsin constant. In Fig. 3, we plot the Obukhov-Corrsin compensated scalar spectra for all cases. There is no collapse of the data, and so compressibility appears to have a first order effect on the scalar spectra.

As a third instance, consider the alignment of the scalar gradient with the directions of the eigenvectors of the strain field. In incompressible turbulence, the turbulent velocity field plays an important role in the stirring action of passive scalars where the different isosurfaces of the scalars are brought together [1,24,25]. This stirring action results in high scalar gradients across the flow field, ultimately enabling molecular diffusion to act. Batchelor's theory [19], initially proposed for large Schmidt numbers, shows that the scalar gradient aligns itself with the most compressive eigenvalue. DNS studies [26] have shown that this aspect of the theory is valid, perhaps surprisingly, even for Schmidt numbers of order unity; see also Vedula *et al.* [27].

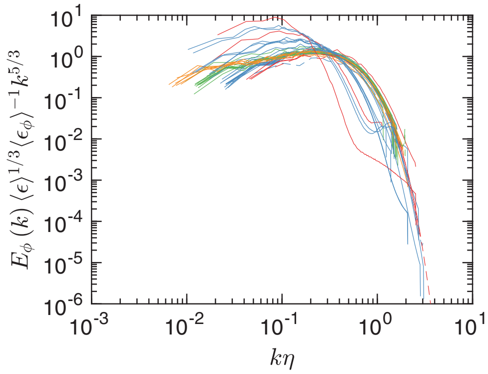

FIG. 3. The Obukhov-Corrsin compensated scalar spectra using total dissipation and Kolmogorov length scale based on the total energy dissipation. No scaling is observed. The dashed line is for the incompressible case.

Danish *et al.* [28] studied this alignment for decaying compressible turbulence and found that the topology and alignment were universal for a range of Reynolds and Mach numbers, but their studies were confined to a narrow range of initial M_t (0.50–0.70) and $R_\lambda(18-24)$. For the wider range of compressible turbulent states considered here, in terms of R_λ, M_t, and δ, Fig. 4 shows that the scalar gradient, $\nabla\phi = \partial\phi/\partial x_i$, does not align uniquely with the symmetric part of the velocity gradient tensor, S_{ij}, where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The eigenvectors of this tensor, called here e_α, e_β, and e_γ, correspond, respectively, to the maximum, intermediate, and minimum eigenvalues with $\alpha > \beta > \gamma$; incompressible turbulence is constrained by $\alpha + \beta + \gamma = 0$. The previous observations by Blaisdell *et al.* [29], and more recently by Ni [4], that contributions from the dilatational field to the scalar flux are negligible compared to the solenoidal part alone, also correspond to a narrow range of conditions.

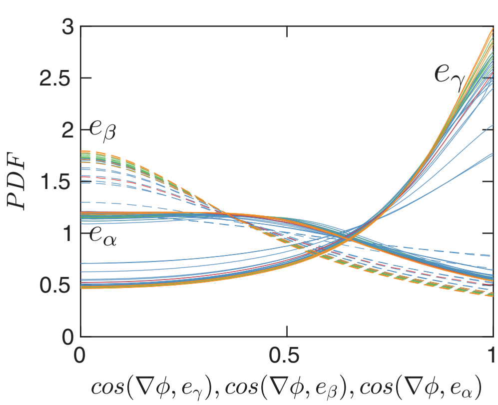

FIG. 4. Alignment of scalar gradient ($\nabla\phi$) with the eigendirections of the S_{ij}, i.e., e_γ, e_β, e_α, which correspond to the eigenvalues (γ, β, α) with $\gamma < \beta < \alpha$.

The discussion so far makes it clear that even the spectrum for just the solenoidal part of the velocity field does not satisfy the incompressible scaling laws if we consider forcing that includes a dilatational component (see Fig. 1). Existing work [5,7,30,31] that makes this claim concerns the velocity field under pure solenoidal forcing and decaying turbulence.

We now propose the following paradigm. Similar to the velocity field one can decompose the dissipation into solenoidal and dilatational contributions as $\langle\epsilon\rangle = \langle\epsilon_s\rangle + \langle\epsilon_d\rangle$, where $\langle\epsilon_s\rangle = \langle\mu\omega_i\omega_i\rangle$, $\boldsymbol{\omega}$ being the vorticity of the fluid motion, and $\langle\epsilon_d\rangle = (4/3)\langle\mu(\partial u_i/\partial x_i)^2\rangle$ are the solenoidal and dilatational parts, respectively. Indeed, under solenoidal forcing conditions when $\delta \ll 1$ and $\langle\epsilon_s\rangle \approx \langle\epsilon\rangle$, we do not expect significant departures in the scaling of the solenoidal energy spectra. However, under general conditions of mixed solenoidal-dilatational forcing where δ can vary by orders of magnitude, one may expect that the use of solenoidal variables in the compensation of the solenoidal spectra would yield better collapse. Indeed, Fig. 5(a) shows the excellent collapse of the Kolmogorov-compensated solenoidal energy spectra when both velocity and the dissipation pertain solely to the solenoidal variables. The solenoidal Kolmogorov length scale is defined [5] as $\eta_s = (\langle\mu\rangle^3/\langle\rho\rangle^2\langle\epsilon_s\rangle)^{1/4}$, which is larger than the total Kolmogorov scale since $\langle\epsilon\rangle > \langle\epsilon_s\rangle$.

In Fig. 5(b), we plot the Obukhov-Corrsin compensated scalar spectrum using just the solenoidal part of the velocity field. A robust collapse occurs for scalar spectra under a wide range of conditions and the spectra look similar to the incompressible case. This suggests that even at really high levels of dilatational content in the flow field, the interaction between the passive scalars and solenoidal velocity field is universal. A plausible reason is that the large scales of the passive scalar and the cascade process by which they are broken down to smaller scales is due mainly to the vortical (solenoidal) motions that are essentially independent of compressibility. Thus, when the classical scaling laws are applied to the solenoidal part, they obey the same incompressible turbulence models in highly compressible flows, even when the dilatational part is quite strong.

We now come to the orientation of the scalar gradient with respect to the velocity strain field. Following the observations above, we assess the effect of the solenoidal component of the tensor, S^s_{ij}. In particular, we examine the statistics of the normalized eigenvalues (β_s) [27] given by $\hat{\beta}_s = \sqrt{6}\beta_s/\sqrt{\alpha_s^2 + \beta_s^2 + \gamma_s^2}$, such that $-1 \le \hat{\beta}_s \le 1$.

In Fig. 6(a) is plotted the probability density function (PDF) of $\hat{\beta}_s$ for a wide range of compressibility conditions. Excellent collapse is observed [curve (i)], indicating that the ratio of the PDF of the solenoidal part of the eigenvalues is unaffected by compressibility. Similar universal behavior is observed for the ratio of β_s/γ_s shown as curve (ii) in the same figure. We also note that the maximum probable value of β_s/γ_s is approximately 0.28 which corresponds to the

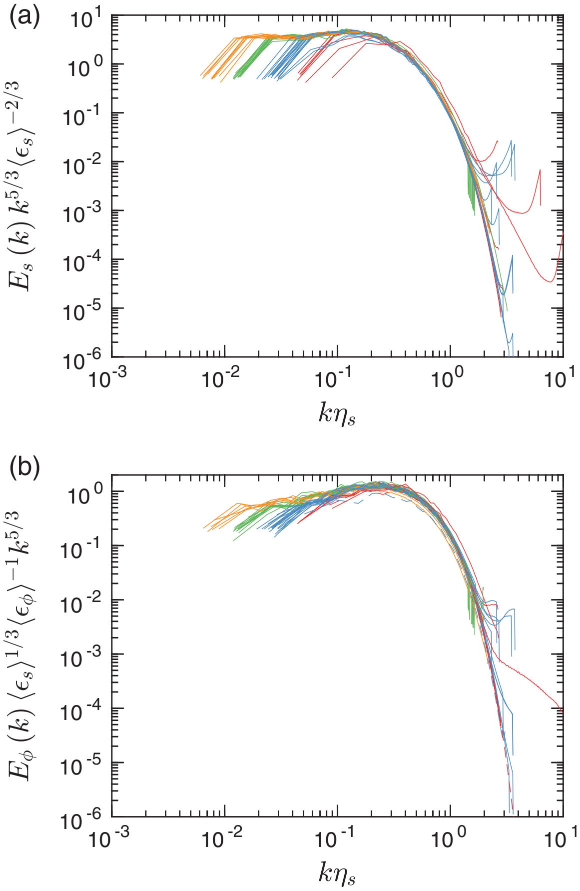

FIG. 5. Kolmogorov-compensated solenoidal energy spectra (a) and Obukhov-Corrsin compensated scalar spectra (b) using solenoidal dissipation $\langle\epsilon_s\rangle$ and solenoidal Kolmogorov length scale η_s. The dashed line in the bottom figure is for the incompressible case.

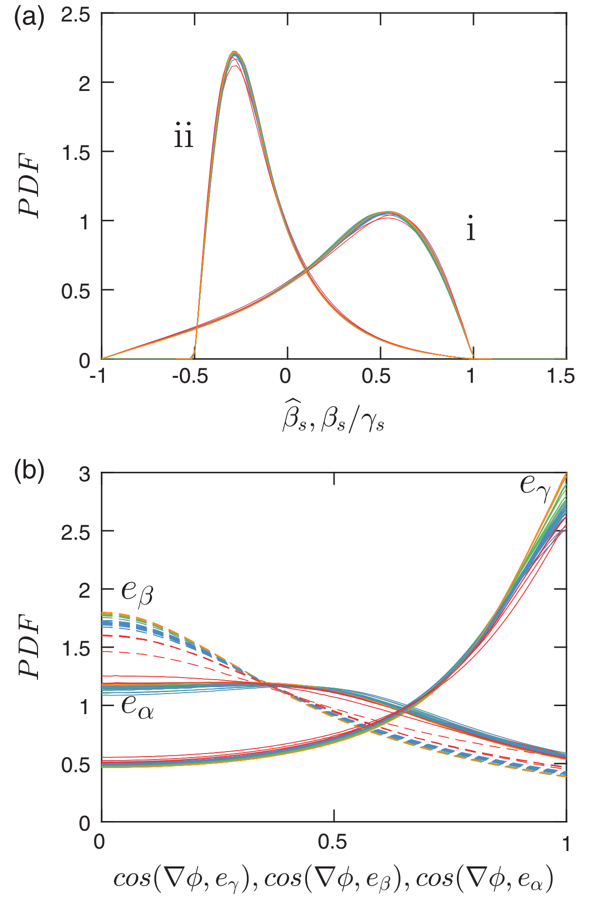

FIG. 6. (a) Normalized eigenvalues of the solenoidal symmetric velocity gradient tensor S^s_{ij}: (i) $\hat{\beta}_s = \sqrt{6}\beta_s/\sqrt{\alpha_s^2+\beta_s^2+\gamma_s^2}$; (ii) β_s/γ_s. (b) Alignment of scalar gradient $(\nabla\phi)$ with e_γ, e_β, e_α, the eigenvectors of S^s_{ij}.

ratio of $\gamma_s/\beta_s = 3.7$, close to the finding for incompressible turbulence [32], and consistent with results for solenoidal forcing [33]. This feature suggests that, while compressibility may change the solenoidal field itself, it does not alter its mixing capability and would remain as efficient as incompressible turbulence.

Figure 6(b) plots the alignment of the scalar gradient with the solenoidal frame of reference. Its behavior is very similar to that of incompressible turbulence [27], with a high probability for the scalar gradient to align with the most compressive direction. Similar observations were made by Foysi *et al.* [34] at $\mathrm{Re}_\lambda \approx 50$ and M_t between 0.05–0.63 for solenoidally forced simulations (compared to the larger DNS database and more general conditions of forcing discussed in the current work.) There are, however, some weak compressibility effects at the small scale. To understand them qualitatively, we show in Fig. 7 the PDF values for $\cos(\nabla\phi, e_\gamma) \in [0.995, 1]$—that is, when the two vectors are almost perfectly aligned—as a function of turbulent Mach number, M_t. The figure shows that R_λ is the major effect, though a weaker decreasing trend with M_t is also

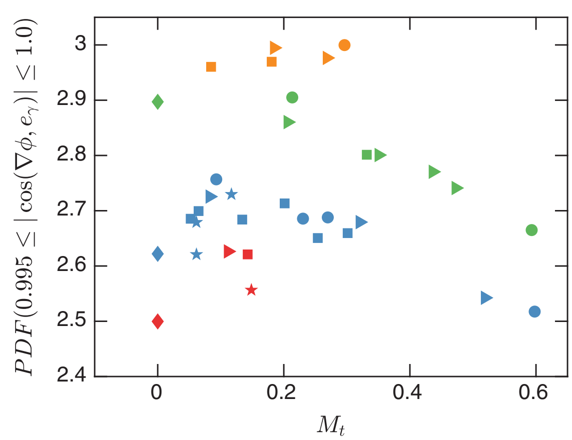

FIG. 7. (a) Probability that the scalar gradient $(\nabla\phi)$ aligns perfectly with the eigendirection e_γ corresponding to the most compressive eigenvalue. Symbols in the figure correspond to different percentages of dilatational forcing. σ: incompressible simulations (diamonds), $\sigma = 0$ (circles), 10–30 (triangles), 30–65 (squares), and 65–100 (stars).

seen. This suggests that the shocklets formed in compressible turbulence influence the dissipative motions modestly but not the inertial-like scales considered in the figures. The modest magnitude of the effect is in part due to the small volume occupied by shocklets (less than 3%, [5]) even at the most extreme dilatational conditions studied here. Furthermore, under this condition, η_s becomes larger than that for purely solenoidal forcing [seen as a shift to the left of the spectrum in Fig. 2(b) as δ increases], which results in a wider difference between the smallest scalar scales and the shocklets. This scale separation would also contribute to the weak interaction between the scalar and the dilatational velocity field at small scales.

In summary, using high fidelity DNS data, we have shown that the interaction between the passive scalar and solenoidal velocity field is universal under a wide range of compressibility conditions, if both the velocity field and the energy dissipation are taken from the solenoidal part of the velocity.

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