# Flux Tube *S*-Matrix Bootstrap

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We bootstrap the $S$ matrix of massless particles in unitary, relativistic two dimensional quantum field theories. We find that the low energy expansion of such $S$ matrices is strongly constrained by the existence of a UV completion. In the context of flux tube (FT) physics, this allows us to constrain several terms in the $S$ matrix low energy expansion or—equivalently—on Wilson coefficients of several irrelevant operators showing up in the FT effective action. These bounds have direct implications for other physical quantities; for instance, they allow us to further bound the ground state energy as well as the level splitting of degenerate energy levels of large FTs. We find that the $S$ matrices living at the boundary of the allowed space exhibit an intricate pattern of resonances with one sharper resonance whose quantum numbers, mass, and width are precisely those of the world-sheet axion proposed by Athenodorou, Bringoltz, and Teper and Dubovsky, Flauger, and Gorbenko. The general method proposed here should be extendable to massless $S$ matrices in higher dimensions and should lead to new quantitative bounds on irrelevant operators in theories of Goldstones and, also, in gauge and gravity theories.

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*Introduction.*—Unraveling the dual string description of the Yang-Mills theory is an long-standing problem. A first step toward achieving this goal is solving for the spectrum of long strings or confining flux tubes (FTs) of pure glue. At low energies, the massless FT excitations (or branons) decouple from the massive short strings (or glueballs) (If the number of colors $N_c$ tends to infinity, then the FTs decouple from the glueballs at any energy.) and can be described by a two dimensional world-sheet theory which can be formulated in terms of an effective Lagrangian or in terms of the branon $S$ matrix.

The FT's effective Lagrangian density is built out of derivatives of the fields $X^\mu$ describing the embedding of the world sheet in spacetime. At low energies, it is dominated by the square root of the induced metric determinant $h = \det h_{\alpha\beta} = \det \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$, i.e., the Nambu-Goto Lagrangian. Any interaction consistent with the bulk $D$-dimensional Poincaré symmetry is also permitted. Thus, the action is written in terms of curvature invariants [1–6]

$$A = \int d^2\sigma \sqrt{-h}[\ell_s^{-2} + \mathcal{R} + K^2 + \ell_s^2 K^4 + \ldots], \quad (1)$$

where $\mathcal{R}(h_{\alpha\beta})$ is the Ricci scalar, $K^\mu_{\alpha\beta} = \nabla_\alpha \partial_\beta X^\mu$ is the extrinsic curvature tensor, and implicit are Wilson coefficients multiplying any of these structures in the effective Lagrangian. The parameter $\ell_s$ is called the string length. In static gauge, $X^\mu(\sigma) = (\sigma^\alpha, X^i)$, where $i = 1, \ldots, D-2$ are the transverse excitations of the FT.

Ricci is a total derivative and $K^2$ vanishes on shell, so the first two terms in the effective field theory expansion can be dropped. Therefore, the low energy dynamics of (1) is tightly constrained by the nonlinearly realized target

Poincaré symmetry. This is known as low energy universality [1,2,7]. The leading deviations from the Nambu-Goto predictions for physical observables arise from $K^4$ operators in (1), namely, effects of $O(\partial^8 X^4)$. There are two $K^4$ operators and, thus, two coefficients $\alpha_3$ and $\beta_3$, which do depend on the specific underlying confining theory.

We will constrain them in this Letter and, thus, bound interesting physical quantities which depend on them. To constrain these parameters, we turn to the on-shell approach to the FT world-sheet theory pioneered by [1] which is based on the branon $S$ matrix. The $2 \to 2$ scattering amplitudes can be decomposed into channels, i.e., irreducible representations of the symmetry group $O(D-2)$. The low energy expansion of the phase shifts in each channel can be written as (see Setup for details)

$$\begin{aligned} 2\delta_{\rm sym} &= \frac{s}{4} + \alpha_2 s^2 + \alpha_3 s^3 + O(s^4), \\ 2\delta_{\rm anti} &= \frac{s}{4} - \alpha_2 s^2 + (\alpha_3 + 2\beta_3) s^3 + O(s^4), \\ 2\delta_{\rm sing} &= \frac{s}{4} - (D-3)\alpha_2 s^2 + (\alpha_3 - (D-2)\beta_3) s^3 + O(s^4), \end{aligned} \quad (2)$$

where $\alpha_2 = [(D-26)/(384\pi)]$, and $s$ is the square of the center of mass energy. Here and below, we set $\ell_s = 1$. The low energy universality mentioned above is manifest here up to $O(s^2)$ included [1]. The nonuniversal $K^4$ terms in (1) contribute at $O(s^3)$ and are encoded in the parameters $\alpha_3$ and $\beta_3$ in (2). To this order, the phase shifts are real because inelastic processes like $2 \to 4$ give rise to imaginary contributions of $O(s^6)$.

Below, we will see that, by requiring a consistent UV completion of the branon $S$ matrix, we can put bounds on its low energy expansion and, thus, bound the effective field theory parameters. This immediately leads to many interesting bounds on various low energy physical observables. One such interesting observable is the finite volume energy spectrum which we can compute in perturbation theory from the action (1) above. For example, for the ground state, we will find

$$E_0(R) = \sqrt{R^2 - \frac{\pi}{3}(D-2)} + \frac{\delta(D)}{R^7} + O(1/R^9), \quad (3)$$

where $R$ is the length of the FT loop and

$$\delta(D) = \frac{32\pi^6(2-D)[(D-2)\alpha_3 + (D-4)\beta_3]}{225}. \quad (4)$$

More speculatively, we will also study the boundary of the allowed $S$-matrix space and find a remarkable numerical coincidence: on that boundary lies an $S$ matrix exhibiting a resonance with the quantum numbers, mass, and width exactly as predicted in [8,9] and dubbed as the QCD world-sheet axion there. Amusingly, at that same point, the $S$ matrix we obtain also contains three further heavier excitations which we call the dilaton, the symmetron, and the axion*. Given the remarkable numerical coincidence with respect to the QCD axion, it is tempting to speculate that they should be present in the QCD FT.

Lattice Monte Carlo simulations of the pure Yang-Mills theory provide precious information on the dynamics of confining FTs. The measurements of the low energy spectrum support the outlined picture of universality at large radius—see [10] for a review—and should, hopefully, be sensitive to the nonuniversal corrections soon, e.g., [11] for $D = 3$. They also favor the existence of the conjectured axion excitation [8,9,12,13]; it would be very interesting to look for other more massive excitations.

*2D massless S-matrix bootstrap.*—Massless excitations in 2D can be left ($L$) or right ($R$) movers. In this section, we study the $L$-$R$ scattering amplitude of branons.

Setup: A long FT in $D$ dimensions breaks the target Poincaré symmetry to $ISO(1,1) \times O(D-2)$. (We assume that the $D$-dimensional theory and the FT preserve parity.) This leads to $D-2$ Goldstone bosons or branons. Now, consider the $2 \to 2$ scattering amplitude of these branons,

$$S_{ab}^{cd}(s) = \sigma_1(s)\delta_{ab}\delta^{cd} + \sigma_2(s)\delta_a^c\delta_b^d + \sigma_3(s)\delta_a^d\delta_b^c, \quad (5)$$

where the indices run over the $D-2$ transverse directions, and $s$ is the square of the center-of-mass energy. Crossing symmetry leads to

$$\sigma_2(-s) = \sigma_2(s), \qquad \sigma_3(-s) = \sigma_1(s). \quad (6)$$

The amplitude (5) can also be decomposed in partial waves of $O(D-2)$, namely, the singlet, the antisymmetric tensor, and the symmetric traceless tensor (see [14,15] for details),

$$\begin{aligned} S_{\rm sing} &= e^{2i\delta_{\rm sing}} = (D-2)\sigma_1 + \sigma_2 + \sigma_3, \\ S_{\rm anti} &= e^{2i\delta_{\rm anti}} = \sigma_2 - \sigma_3, \\ S_{\rm sym} &= e^{2i\delta_{\rm sym}} = \sigma_2 + \sigma_3, \end{aligned} \quad (7)$$

where $\delta_{\rm rep}$ may have an imaginary part due to particle production. In this basis, unitarity is simply

$$|S_{\rm rep}(s)|^2 \leq 1, \quad \forall\ s > 0. \quad (8)$$

The amplitudes $\sigma_i(s)$ are analytic functions of $s$ in the upper and the lower half plane related by real analyticity

$$\sigma_i(s^*) = [\sigma_i(s)]^*. \quad (9)$$

Therefore, it is enough to know the amplitudes in the upper half plane, where Eqs. (6) and (9) lead to

$$\sigma_2(-s^*) = [\sigma_2(s)]^*, \qquad \sigma_3(-s^*) = [\sigma_1(s)]^*. \quad (10)$$

The Nambu-Goto Lagrangian leads to the low energy expansion of the phase shifts as $2\delta_{\rm rep} = (s/4) + O(s^2)$. In principle, higher order terms may also include nonanalytic terms of the form $s^p(\log s)^k$ with $p > k > 0$. Furthermore, we know that $\mathrm{Im}\delta_{\mathrm{rep}} = O(s^6)$ because particle production starts with $|\mathcal{M}_{2\to4}|^2 \sim l_s^{12}$ [16]. [In $D = 3$, particle production starts at $\mathcal{O}(s^8)$]. Using just these facts and (10), we can derive the low energy expansion (2) with $\alpha_2$, $\alpha_3$, and $\beta_3$ as real parameters. In the context of the FT theory, $\alpha_2 = [(D-26)/(384\pi)]$ is universal and $\alpha_3$ and $\beta_3$ are nonuniversal coefficients related to the two independent $K^4$ terms in (1). In the Supplemental Material [17], we push this expansion up to $O(s^6)$ and find perfect agreement with the $O(s^4)$ results of [18].

$D = 3$ Flux Tubes: To start with, we focus on the $D = 3$ target space. In this case, only $\delta_{\mathrm{sing}} \equiv \delta$ is meaningful, and the amplitude $S = e^{2i\delta}$ obeys $S(-s^*) = [S(s)]^*$ for $s$ in the upper half plane. Furthermore, it was shown in [19] that $\mathrm{Im}\delta = O(s^8)$. This implies

$$2\delta(s) = \frac{s}{4} + \gamma_3 s^3 + \gamma_5 s^5 + \gamma_7 s^7 + i\gamma_8 s^8 + O(s^9), \tag{11}$$

where $\gamma_3$, $\gamma_5$, $\gamma_7$ are nonuniversal parameters. On the other hand, $\gamma_8$ is determined by the probability of particle production at leading order $P_{2\to n\geq 4} = 2\gamma_8 s^8 + O(s^9)$. As explained in [19], $\gamma_8 \propto \gamma_3^2$ is not an independent parameter.

The $S$ matrix $S(z)$ is a holomorphic function from the upper half plane $\mathbb{H}$ to the unit disc $\mathbb{D}$ because unitarity on the real axis along with the maximum modulus principle implies that $|S(z)| \leq 1$ in the full upper half plane. Next, we construct a new function

$$S^{(1)}(z|w) \equiv \frac{S(z) - S(w)}{1 - S(z)\overline{S(w)}} \Big/ \frac{z - w}{z - \bar{w}}, \tag{12}$$

where $w$ is any point in the upper half plane. It is easy to see that (as a holomorphic function of $z$) this function (a) has no singularities in the upper half plane and (b) is again bounded by 1 for $z$ on the real line. By the maximum modulus principle, it is bounded everywhere on the upper half plane: $|S^{(1)}(z|w)|_{\mathrm{Im}(z)\geq 0} \leq 1$. This is the so-called Schwarz-Pick theorem. Inserting (11) in the Schwarz-Pick combination (12) and expanding for small and imaginary $z$ and $w$, we find

$$S^{(1)}(z|w) = -1 + \left(\frac{1}{96} + 8\gamma_3\right)|zw| + \cdots \geq -1. \tag{13}$$

This leads to our first bound

$$\gamma_3 \geq -\frac{1}{768}. \tag{14}$$

The authors of [9,19] estimated $\gamma_3 \approx 3 \times 10^{-4}$ from lattice data for $SU(6)$ Yang-Mills theory [20]. Similarly, one can define $S^{(2)}(z|q,w)$ by replacing $S(z)$ by $S^{(1)}(z|q)$ in (12). Such Schwarz-Pick multipoint generalizations [21] can be used to derive (see Supplemental Material [17] for details, which includes Refs. [22,23])

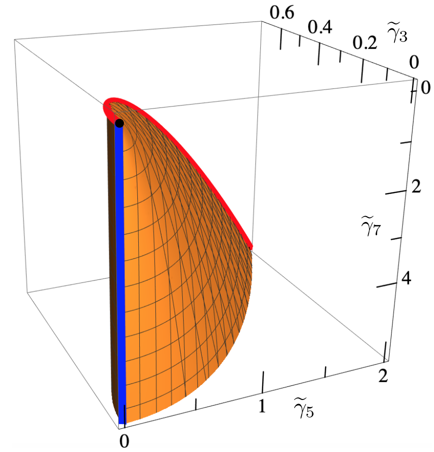


FIG. 1. Allowed $\{\tilde{\gamma}_3, \tilde{\gamma}_5, \tilde{\gamma}_7\}$ space for a $D = 3$ FT $S$ matrix.

$$\tilde{\gamma}_5 \geq 4\tilde{\gamma}_3^2 - \frac{1}{64}\tilde{\gamma}_{,3}\,,$$
$$\tilde{\gamma}_7 \geq \frac{\tilde{\gamma}_5^2}{\tilde{\gamma}_3} + \frac{1}{4096}\tilde{\gamma}_3 + \frac{1}{64}\tilde{\gamma}_5 - \frac{1}{16}\tilde{\gamma}_3^2, \tag{15}$$

where $\tilde{\gamma}_n = \gamma_n + (-1)^{(n+1)/2}[1/(n2^{3n-1})]$, see Fig. 1.

$D = 4$ Flux Tubes: In $D = 4$ dimensions, the branon $S$ matrix possesses an $O(2)$ symmetry. In addition, the crossing and unitary equations are invariant under $S_{\mathrm{sing}} \leftrightarrow S_{\mathrm{anti}}$ interchange corresponding to $\beta_3 \leftrightarrow -\beta_3$ in (2). Universality fixes the low energy expansion of the phase shifts up to order $s^2$ included. The leading nonuniversal behavior depends on the two coefficients $\alpha_3$ and $\beta_3$ introduced in (2).

Crossing mixes the various irreps, but the symmetric channel $S$ matrix is still bounded by 1 along all the real $s$ axes (By crossing $|S_{\mathrm{sym}}^{\mathrm{crossed}}| = \frac{1}{2}|S_{\mathrm{sing}} + S_{\mathrm{anti}}| \leq 1$, this actually holds for any $O(N = D-2)$ theory. [14]), so we can still apply the first Schwarz-Pick inequality in this channel. This leads to

$$\alpha_3 \geq -\frac{1}{768} + \frac{121}{9216\pi^2}. \tag{16}$$

However, this is not the full allowed $\{\alpha_3, \beta_3\}$ space as we have yet to explore all channels and their interrelations. To find the optimal bounds, we proceed numerically in the spirit of [24–26]. The numerical result of the optimization problem is shown in Fig. 2. When $\beta_3 = 0$, the numerical bound and the analytic one coincide. At this point, the $S$ matrix satisfies the Yang-Baxter equation; see Supplemental Material [17].

*Energy spectrum in finite volume.*—At very large $R$, we read off the string tension from the ground state energy $E_0(R) \simeq R/\ell_s^2$. Recall that the corrections up to $1/R^5$ to

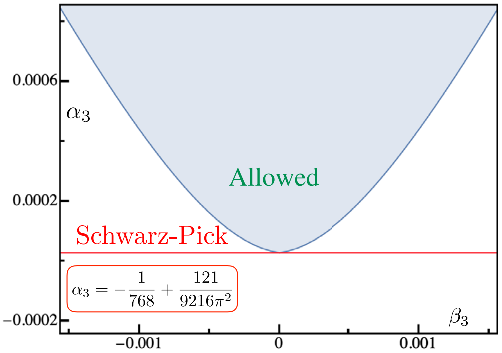


FIG. 2. Allowed region in the $\{\beta_3, \alpha_3\}$ parameter space of FT $S$ matrices in $D = 4$ as obtained by numerics.

this confining result are universal and given by the square root in (3). The sub-sub-sub-subleading term is not uniquely fixed by symmetry and is the subject of this section. Computing the nonuniversal correction in (3) is straightforward in perturbation theory, albeit increasingly complex as we move to higher orders in $1/R$. The leading correction $\delta(D)$ comes from the two $K^4$ possible interactions which we parametrize as

$$\mathcal{L}_{\text{nonuniv}} = \partial_a \partial_b X^i \partial_a \partial_b X^j \partial_c \partial_d X^k \partial_c \partial_d X^l \times [4\delta_{ik}\delta_{jl}(\alpha_3 + \beta_3) - 2\delta_{ij}\delta_{kl}(\alpha_3 + 3\beta_3)]. \quad (17)$$

Here, we parametrize the coefficient of the two invariant structures so as to match the eye pleasing expressions (2), as can be verified by a straightforward tree level computation. Thus, the leading order nonuniversal contribution to the vacuum energy density is

$$= f(D)\, [\partial_\mu \partial_\nu \partial_\rho \partial_\sigma \Delta_R(0)]^2 , \quad (18)$$

where $f(D) = 4(2 - D)[(D - 2)\alpha_3 + (D - 4)\beta_3]$. The derivative of the finite volume propagator is given by $\partial_\mu \Delta_R(x) = \sum_n \partial_\mu \Delta(x + n)$, where $\partial_\mu \Delta(x) = -x_\mu/(2\pi x^2)$ and $n_\mu = (0, nR)$ is a displacement vector in the winding direction. The zero mode $n_\mu = (0, 0)$ gives a short-distance divergence in the limit $x \to 0$ leading to (18). This is regulated by a local counterterm which, at this order, simply amounts to neglecting the zero mode. Thus, after a bit of algebra and excluding the zero mode, we are led to $[\partial_\mu \partial_\nu \partial_\rho \partial_\sigma \Delta_R(0)]^2_{\text{ren}} = 288/\pi^2 \sum_{n,m=1}^{\infty} 1/(R^8 n^4 m^4) = 8\pi^6/225R^8$, which gives the desired relation (4) between the first nonuniversal correction to the energy and the first nonuniversal low energy $S$-matrix parameters or Wilson coefficients. See Supplemental Material [17] for further details, which includes Ref. [27]. (In $D = 3$, physical quantities only depend on the combination $\alpha_3 - \beta_3 = \gamma_3$.)

Since we bounded the latter low energy parameters, see Figs. 1 and 2, we automatically obtain bounds on the Wilson coefficients and on the ground state energy. In three and four dimensions, for instance, we find the following bound on the deviation from the square root formula

$$\delta(3) = -\frac{32\pi^6\gamma_3}{225} \leq \frac{\pi^6}{5400}, \quad (19)$$

and

$$\delta(4) = -\frac{128\pi^6\alpha_3}{225} \leq \frac{\pi^6}{1350} - \frac{121\pi^4}{16200}. \quad (20)$$

Note that the right-hand side of (20) is negative so the square root formula must be corrected; the right-hand side of (19) is positive, in nice agreement with the fact that integrable $D = 3$ strings have precisely $E_0^{\text{int}} = \sqrt{R^2 - (\pi/3)}$. Note, also, that the four dimensional bound (20) is saturated when $\beta_3 = 0$ (see Fig. 2) which corresponds to the particular point where integrability is preserved.

In fact, if we exploit the low energy integrability of the theory, we can bypass the Lagrangian approach altogether and, by means of the so-called thermodynamic Bethe ansatz (TBA), compute (3) in terms of the $S$ matrix. We checked that, indeed, the TBA approach yields precisely the same results as Eqs. (19) and even (20) when $\beta_3 = 0$ which is the $D = 4$ integrable point.

It should be possible to relate, more generally, the various energy levels with the two-to-two $S$ matrix together with all higher point amplitudes of nonintegrable theories. The Lüscher corrections [28] provide the leading term, and generalized Lüscher corrections have recently been explored, e.g., in [29]. Perhaps the recent rederivation of the TBA in more diagrammatic terms, see, e.g., [30], can provide some insights for such a putative description, or, developing the approach of [31] for the FT, may turn out useful.

*Resonances.*—Given the bounds in Figs. 1 and 2, it is natural to ask which $S$ matrices lie on those boundaries. This is particularly relevant in 4D since lattice Monte Carlo simulations shows a rich phenomenology with the presence of a parity odd resonance [12,13] dubbed a QCD world-sheet axion in the $S$-matrix approach to the long FT [8,9]. In $D = 4$, we find that the $S$ matrices which saturate the bound have zeros, which physically correspond to resonances. Figure 3 describes the position of these resonances in the antisymmetric channel as we move along the boundary of Fig. 2.

Depending on whether we are to the right or left of the integrable $\beta_3 = 0$ point, there is one ($\beta_3 < 0$) or two zeros there ($\beta_3 > 0$). As we move along the boundary in the region $\beta_3 > 0$, the sharpest of these resonances passes spot on by the values of the world-sheet axion. The two dots correspond to estimates based on $SU(3)$ [8,9] and $SU(5)$ [32] lattice Monte Carlo simulations [12]. Because of these encouraging numerical coincidences, we will denote these two points along the boundary as the $SU(3)$ point and $SU(5)$ point. Remarkably, at these points, we find other

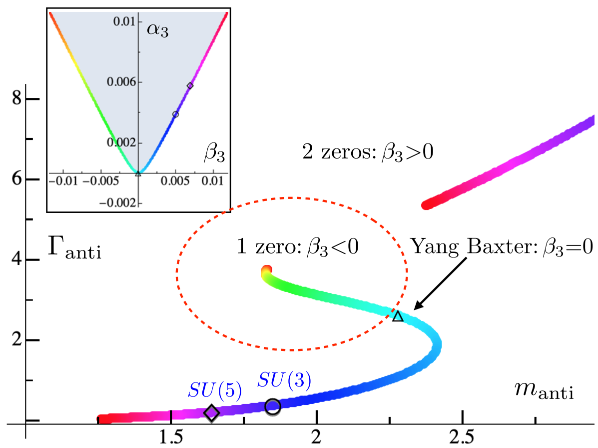


FIG. 3. As we move along the boundary, we encounter $S$ matrices whose resonance mass and width are in precise agreement with those predicted in [9,32] as extracted from $SU(3)$ (right point) and $SU(5)$ (left point) lattice data.

zeros in the $S$ matrices. One broader resonance shows up in the antisymmetric channel along with a resonance in each of the other two channels. The spectrum, measured as $s_0 = (m + i\Gamma/2)^2$ at the position $S_{\rm rep}(s_0) = 0$, for the $SU(3)$ and $SU(5)$ points is given by

| Spectrum $[m, \Gamma]$ | $SU(3)$ | $SU(5)$ |
|---|---|---|
| Axion | [1.85, 0.39] | [1.64, 0.22] |
| Axion* | [3.25, 8.84] | [2.83, 7.02] |
| Symmetron | [2.36, 4.99] | [2.34, 4.54] |
| Dilaton | [1.88, 3.37] | [1.84, 3.52] |

Even though these values should obviously be taken as benchmark values only, could these resonances be further excitations present in the Yang-Mills long FTs? Of course, these explorations must be taken with a grain of salt since there is *a priori* no strong reason for the real FT to be close to the boundary.

*Discussion.*—The physics of FTs is very rich. At low energy, we have universality which very powerfully constrains all physical observables. Here, we bounded the first nonuniversal corrections by means of the $S$-matrix bootstrap. In short, the existence of a properly UV completed branon $S$ matrix constrains the possible space of $S$ matrices and their low energy expansion and, therefore, the space of Wilson coefficients in effective field theory language.

To our knowledge, this is the first time that the low energy Wilson coefficients are bounded optimally using the full consistency conditions of four particle scattering amplitudes—see, e.g., (15).

Clearly, our bounds also apply to any stringlike defect with the same symmetry breaking pattern as FTs [33]. Moreover, the very same logic could be applied more broadly to bound other systems with spontaneous symmetry breaking in two dimensions such as those arising from broken supersymmetry and even in higher dimensions. It would be fascinating to follow our approach to bound the leading irrelevant operators showing up in pion physics or, more ambitiously, in gauge and gravity theories, and to compare with existing bounds based on the analytic properties of the forward amplitude [34–36]. We look forward to pursuing this further.

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