Anomalous Spin Diffusion in One-Dimensional Antiferromagnets

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(Received 26 March 2019; published 28 October 2019)

The problem of characterizing low-temperature spin dynamics in antiferromagnetic spin chains has so far remained elusive. Here we reinvestigate it by focusing on isotropic antiferromagnetic chains whose low-energy effective field theory is governed by the quantum nonlinear sigma model. Employing an exact nonperturbative theoretical approach, we analyze the low-temperature behavior in the vicinity of nonmagnetized states and obtain exact expressions for the spin diffusion constant and the NMR relaxation rate, which we compare with previous theoretical results in the literature. Surprisingly, in $SU(2)$-invariant spin chains in the vicinity of half filling we find a crossover from the semiclassical regime to a strongly interacting quantum regime characterized by zero spin Drude weight and diverging spin conductivity, indicating superdiffusive spin dynamics. The dynamical exponent of spin fluctuations is argued to belong to the Kardar-Parisi-Zhang universality class. Furthermore, by employing numerical time-dependent density matrix renormalization group simulations, we find robust evidence that the anomalous spin transport persists also at high temperatures, irrespective of the spectral gap and integrability of the model.

DOI: 10.1103/PhysRevLett.123.186601

One-dimensional isotropic antiferromagnets reveal several remarkable aspects, which made them a subject of very intense experimental and theoretical investigations in the past. One of the most profound features is a fundamental distinction between spin systems with odd and integer spin. In one dimension, the latter exhibit a dynamically generated gapped spectrum while the former is characterized by gapless excitations with fractional statistics [1–3].

In the context of nonequilibrium physics, the main focus has been to explain the peculiar properties of the spin relaxation dynamics of the Haldane-gapped spin chain compounds. In spite of various theoretical approaches, ranging from field-theoretical techniques such as the form-factor expansions [4,5], to the semiclassical approximations [6–10], the status of the topic remained controversial, with a number of conflicting statements concerning the spin Drude weight, spin diffusion constant, and the nuclear magnetic resonance (NMR) rate.

Recent years have brought many theoretical advancements in the domain of nonequilibrium phenomena in exactly solvable interacting systems. One of the key achievements among them is the formalism of the generalized hydrodynamics [11,12], see also Refs. [13–25], which offers an efficient and universal language to tackle various nonequilibrium problems. Among others, it enables us to obtain closed-form analytic expressions for transport coefficients, such as Drude weights [26–29] (see also Ref. [30]) and, more recently, diffusion constants in interacting quantum systems [31–34]. This powerful toolbox puts us in a position to address a number of perennial issues that fall outside of the scope of previous approaches.

In this work, we revisit and resolve the problem of spin transport in antiferromagnetic spin chains at low temperatures in the half-filled sector, investigated previously in Refs. [4,35,36]. Here we focus our attention to two physically relevant quantities, the spin diffusion constant and the nuclear spin relaxation rate. We concentrate entirely on locally interacting quantum spin-S chains with $SU(2)$-symmetric Hamiltonians where our findings markedly differ from previous predictions. We demonstrate that in the experimentally relevant regime $h/T \ll 1$, where T is the temperature and h the external magnetic field, the spin dynamics is dominated by collective magnonic bound-state excitations as described by the full many-body scattering matrix of the underlying effective field theory. This has several far-reaching physical consequences, most prominently the *divergent* spin (charge) diffusion constant and spin conductivity at any finite temperature, which signals *superdiffusive* spin transport, with time-dependent dc conductivity growing as $t^{1/3}$ at large times. This anomalous feature was initially observed numerically in an *integrable* isotropic Heisenberg model [37,38], and established rigorously in Ref. [39]. A recent numerical study in the same model [40] gives strong evidence that the spin relaxation dynamics falls into the Kardar-Parisi-Zhang (KPZ) universality class, otherwise better known from the physics of growing interfaces [41–43].

By performing exact nonperturbative calculations, we argue that this type of anomalous spin transport is a distinguished feature of spin or charge transport at low temperatures even in generic one-dimensional *nonintegrable* isotropic antiferromagnetic compounds and regardless of whether the low-lying theory is gapped or gapless. Moreover, our numerical time-dependent density matrix renormalization group (tDMRG) simulations give evidence that the anomalous spin relaxation also persists at higher temperatures. This indicates that non-Abelian global symmetry of spin interaction can have a profound consequence on the nature of spin transport on sub-ballistic timescales irrespective of integrability.

Spin diffusion constant from integrability.—Let $\hat{H}$ be a spin-chain Hamiltonian with the conserved total magnetization $\hat{S}^z = \sum_i \hat{s}_i^z$. The linear-response spin diffusion constant $\mathfrak{D}$ is computed as the spatiotemporal integrated spin current autocorrelation function [44,45],

$$\mathfrak{D}(T,h) = \frac{1}{T\chi_h(T,h)} \int_0^\infty dt (\langle \hat{J}(t)\hat{j}_0(0)\rangle_{T,h} - \mathcal{D}), \quad (1)$$

where $\hat{J} = \sum_i \hat{j}_i$ is the total spin current with density $\hat{j}_i$ at site i, $\langle \bullet \rangle_{T,h}$ corresponds to the equilibrium average with respect to the grand-canonical Gibbs ensemble $\hat{\varrho}_{\mathrm{GC}}(T,h) \simeq \exp[-(\hat{H} - h\hat{S}^z)/T]$, while $\chi_h(T,h) = -\partial^2 f(T,h)/\partial h^2$ is the static spin susceptibility, where $f(T,h) = -T \log \mathrm{Tr}[\hat{\varrho}_{\mathrm{GC}}(T,h)]$, and $\mathcal{D}(T,h)$ is the spin Drude weight which has been subtracted in order to ensure that $\mathfrak{D}(T,h)$ is well defined. The spin Drude weight is defined as the large-time limit of the spatially integrated current-current correlator in Eq. (1), and is generically finite in integrable systems. However, in a nonmagnetized sector (i.e., at half filling $h=0$) which is of our interest here, $\mathcal{D}(T,0) = 0$ essentially due to particle-hole symmetry of local conservation laws [27,46,47]. This is in agreement with the prediction of semiclassical theory [8].

The task of computing the exact diffusion constants in integrable models remains, on the other hand, a challenging open question. Very recently, an exact explicit expression for the diffusion matrix in a general equilibrium state has been derived in Ref. [33] using the thermal form factor expansion and in Ref. [34] within the kinetic theory approach. In this work, we employ the general formula for the exact spin diffusion constant obtained in Refs. [33,34]. Here we specifically examine the vicinity of the *half filled* equilibrium states where, remarkably, we found that the formula further simplifies and, in fact, exactly coincides with the curvature of the zero-frequency noise (or Drude self-weight) [28,52],

$$\mathcal{D}^{\mathrm{self}}(T,h) = 2\int_0^\infty dt \langle \hat{j}_0(t)\hat{j}_0(0)\rangle_{T,h}, \quad (2)$$

with respect to the magnetization $\nu(T,h) \equiv 4T\langle \hat{S}^z \rangle_{T,h}$,

$$\mathfrak{D} \equiv \mathfrak{D}(T,0) = \frac{\partial^2 \mathcal{D}^{\mathrm{self}}(T,\nu)}{\partial \nu^2}\bigg|_{\nu=0}. \quad (3)$$

The obtained expression can alternatively be viewed as the optimized diffusion-lower derived in Ref. [53]. We note that Eq. (3) remains valid also for small h, up to corrections of the order $\mathcal{O}(h^2)$. The spin diffusion constant can accordingly be expressed in terms of equilibrium state functions via the hydrodynamic mode resolution

$$\mathfrak{D} = \sum_s \mathfrak{D}_s, \quad (4)$$

with $\mathfrak{D}_s = \int \{[dp_s(\theta)]/(2\pi)\} n_s(\theta)[1-n_s(\theta)] \times |v_s^{\mathrm{eff}}(\theta)| \partial_\nu^2 (m_s^{\mathrm{dr}})^2|_{\nu=0}$. Here the integer label s runs over all distinct quasiparticle species [19,33], $n_s(\theta)$ correspond to their (thermal) Fermi occupation functions, $p_s(\theta)$ are their effective (i.e., dressed) momenta parametrized by rapidity variable θ, $v_s^{\mathrm{eff}}(\theta) = \partial\varepsilon_s(\theta)/\partial p_s(\theta)$ are the effective (group) velocities, and finally m_s^{dr} the dressed magnetization (spin) with respect to a thermal background, see the Supplemental Material [47], which follows a paramagnetic Curie's law in nearly half filled thermal states $m_s^{\mathrm{dr}} = h\mu_s^{\mathrm{dr}} + O(h^3)$. We will now apply this formula to models with different particle contents and in the low temperature regime.

Nonintegrable isotropic antiferromagnetic chains.—We now consider the low-temperature spin dynamics in *generic* antiferromagnetic spin chains with isotropic spin interactions. For definiteness, we focus on the $SU(2)$-symmetric Heisenberg spin-S chains $\hat{H}_S = J\sum_i \hat{\mathbf{s}}_i \cdot \hat{\mathbf{s}}_{i+1}$, with $\hat{\mathbf{s}} \cdot \hat{\mathbf{s}} = S(S+1)$. In the large-$S$ limit, the effective low-energy action which describes the evolution of the staggered and ferromagnetic fluctuations $\hat{\mathbf{s}}_i \approx S(-1)^i \hat{\mathbf{n}} + \hat{\mathbf{m}}$ yields a non-Abelian quantum field theory known as the $O(3)$ nonlinear sigma model (NLSM) [1,2,54]. In dimensionless units $v = 2JS \to 1$ and coupling parameter $g = 2/S$, the Hamiltonian reads

$$\hat{H}_\Sigma^{(\Theta)} = \frac{v}{2}\int dx \left[g\left(\hat{\mathbf{m}} + \frac{\Theta}{4\pi}\partial_x \hat{\mathbf{n}}\right)^2 + \frac{1}{g}(\partial_x \hat{\mathbf{n}})^2 \right], \quad (5)$$

where ferromagnetic magnetization $\hat{\mathbf{m}} = \hat{\mathbf{n}} \times \hat{\mathbf{p}}$ generates spatial rotations of the unit vector field $\hat{\mathbf{n}} = (\hat{n}^x, \hat{n}^y, \hat{n}^z)$, with the canonically conjugate momentum $\hat{\mathbf{p}} = (1/g)\partial_t \hat{\mathbf{n}} + (\Theta/4\pi)\hat{\mathbf{n}} \times \partial_x \hat{\mathbf{n}}$ and $\Theta = 2\pi S$ is the topological angle. For $\Theta \in \{0, \pi\}$ the $O(3)$ NLSM model is an *integrable* QFT with a completely factorizable scattering matrix [55,56]. Specifically, at $\Theta = 0$ the model yields the effective low-energy theory for the staggered ($k \approx \pi$) and the ferromagnetic ($k \approx 0$) fluctuations in the Haldane-gapped integer spin chains. The $k \to 0$ component of the spin-lattice magnetization corresponds to the conserved Noether charge $\hat{\mathbf{m}}$, obeying continuity equation $\partial_t \hat{\mathbf{m}} + \partial_x[\hat{\mathbf{n}} \times (1/g)\partial_x \hat{\mathbf{n}}] = 0$. The elementary excitations are a *massive* triplet of bosons with a relativistic dispersion $e(k) = \sqrt{k^2 + \mathfrak{m}^2}$, with $\mathfrak{m}$ being a dynamically generated mass $\mathfrak{m} \sim \Lambda e^{-\pi S}$ whose magnitude is determined by the underlying spin-S lattice model at momentum scale Λ. While the NLSM has no physical bound states in the spectrum, the scattering is nondiagonal and governed by a nontrivial exchange of spin degrees of freedom (d.o.f.). At $\Theta = \pi$, the $O(3)$ NLSM describes the low-energy continuum theory of the half-integer spin chains with *massless* elementary excitations [3,54].

Low-temperature spin transport.—Hydrodynamic description of transport is based on the notion of quasiparticles. The physical excitations of the $O(3)$ NLSM are spinful bosons which interact via a nontrivial spin exchange. This is conventionally understood in terms of interacting spin waves (magnons) which are regarded as additional auxiliary quasiparticles and are characterized by internal quantum numbers $s > 0$ corresponding to a quantized amount of bare spin they carry. The elementary bosonic excitation is ascribed $s = 0$.

In the low-temperature limit and small h, with ratio $h/T \gg 1$ large, the contributions of spin-carrying auxiliary quasiparticles become suppressed, and a dilute gas of spinful bosons serves as a good approximation. In this regime we accordingly recover the prediction of the semiclassical theory (cf. Ref. [47])

$$\mathfrak{D}_\Sigma \simeq \mathfrak{D}_0 = \mathfrak{D}_{\rm cl}(T, h), \qquad h/T \gg 1, \tag{6}$$

where $\mathfrak{D}_{\rm cl}(T, h) = (e^{\mathfrak{m}/T}/\mathfrak{m})/[1 + 2\cosh(h/T)]$, see Ref. [6]. In contrast, the behavior of the spin diffusion constant in the regime $h/T \ll 1$ is fundamentally different and the subleading corrections attributed to internal magnonic excitations can no longer be neglected. Even worse, their net contribution to the diffusion constant diverges at small field as $\sim 1/|h|$. The correct expression for the spin diffusion constant is then given by Eq. (3),

$$\mathfrak{D}_\Sigma = \sum_{s \geq 0} \mathfrak{D}_s \sim \frac{e^{\mathfrak{m}/T}}{3\mathfrak{m}|h|} + \mathcal{O}(h^0), \qquad h/T \ll 1. \tag{7}$$

In particular the spin dc conductivity [47] reads $\sigma(T, h) = \mathfrak{D}(T, h)\chi_h(T, h) = \kappa(T)|h|^{-1} + \mathcal{O}(h^0)$, with $\kappa(T) \sim T^{-1/2}$ at small T. Then one can check that $\kappa(T) > 0$ for any T, see Ref. [47], implying that spin transport in the NLSM at half filling $h = 0$ and $T > 0$ is *superdiffusive*. For half-integer gapless spin chains we can repeat the same logic for the NLSM with the topological angle $\Theta = \pi$, and once again find a diverging spin conductivity. This leads us to conclude that the presence or absence of the spectral gap plays no essential role for this observed superdiffusive spin dynamics in isotropic antiferromagnetic chains.

Spin transport at intermediate and high temperatures.—Characterizing spin dynamics at intermediate and high temperatures in physical spin chains goes beyond a simple effective QFT description and thus poses a more challenging task. Here we rely on tDMRG simulations. In Fig. 1 we display the time-dependent spin dc conductivity $\sigma(t) = (1/T)\int_0^t dt' \langle \hat{J}(t')\hat{j}_0(0)\rangle_{T,h=0}$ at half filling and for various temperatures. The latter can be deduced from the growth rate of the spin current following a quench from an initial bi-partitioned state with a tiny magnetization imbalance δs^z, namely, $\sigma(t) = \lim_{\delta s^z \to 0}\langle \sum_x \hat{j}_x(t)\rangle_{T,\delta s^z}/\delta s^z$, which is simpler from the numerics standpoint. While very low temperatures cannot be reached by this numerical technique, at higher temperatures we find a clear signature of superdiffusion, characterized by time-dependent conductivity $\sigma(t) \sim t^{1/3}$ at large times, see Fig. 1 as well as Ref. [47], both for the gapless spin-1/2 and the gapped (nonintegrable) spin-1 *XXX* chain. In our simulations, we have employed the finite-temperature time-dependent density matrix renormalization group algorithm [57,58], using a fixed discarded weight and the maximum bond dimension of 4000 for spin 1/2 and 2000 for spin 1, with system size large compare to the causality light cone at the largest simulation time.

Comparison with previous results.—To further elaborate on the physical implications of our findings, we now discuss our theoretical predictions in a broader context and clarify the pitfalls of the previous approaches.

Semiclassical approach.—It is instructive to first shortly summarize the semiclassical approach to the low-T quantum transport developed in Refs. [6,7] (see also Ref. [59–61]). Using that in the regime $T, h \ll \mathfrak{m}$ the

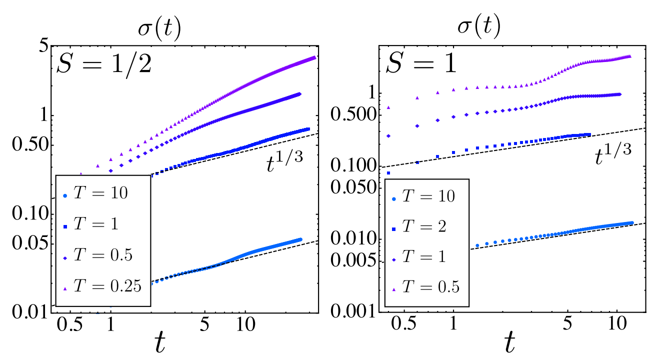

FIG. 1. Time-dependent spin conductivity (in units of exchange coupling J) for the isotropic *gapless* Heisenberg spin $S = 1/2$ (left) and the spin $S = 1$ (right) *(nonintegrable) gapped* chain at half filling $h = 0$, displayed for several different temperatures (increasing from top to bottom) computed using tDMRG simulations. Both cases exhibit an algebraic law $\sigma(t) \sim t^{1/3}$, indicating that the spin superdiffusion is unrelated to the spectral gap and integrability of the model.

mean collision time (i.e., the inverse density) becomes exponentially large ($\sim T^{-1}e^{\mathfrak{m}/T}$), it has been argued that on large spatiotemporal scales (compared to inverse temperature $t \gg T^{-1}$ and the thermal de Broglie wavelength $x \gg \lambda_T$) the spin dynamics essentially becomes "universal" and can be accurately described in terms of classical trajectories. By accordingly keeping only the zero-momentum part of the full quantum scattering matrix in the gapped $O(3)$ NLSM ($\Theta = 0$), Ref. [6] predicts a large but *finite* spin diffusion constant $\mathfrak{D}_{\rm cl} \sim e^{\mathfrak{m}/T}/3\mathfrak{m}$, valid in the regime $h \ll T \ll \mathfrak{m}$, which corresponds to the contribution of massive physical excitations corresponding to $s = 0$, see Eq. (6). It is important to keep in mind however that the semiclassical scattering theory effectively interchanges the noncommuting $T \to 0$ and $t \to \infty$ limits and, as a consequence, it is blind to the coherent contributions of the internal magnonic d.o.f. [terms with $s > 0$ in Eq. (4)]. It turns out that they are crucial to correctly determine the nature of spin transport in the regime $h/T \ll 1$.

Normal spin diffusion at finite temperatures is, on the other hand, restored upon adding interaction *anisotropy*. To clarify this aspect, we briefly consider the *XXZ* spin-1/2 chain with anisotropy Δ, assuming $\Delta > 1$, where the quasiparticles pertain to compounds of s bound magnons [47]. In the low-temperature limit and small h, with $h/T \gg 1$ large, the bound-state contributions ($s > 1$) are *suppressed* and from Eq. (3) we find (cf. Ref. [47]) $\mathfrak{D}_{XXZ} \simeq \mathfrak{A}e^{\mathfrak{m}/T}$ where $\mathfrak{A} = \mathfrak{c}^2/(\mathfrak{n}\mathfrak{m})$, $\mathfrak{n} = 2$ is the number of low-energy d.o.f. with the low-momentum dispersion law $\varepsilon_1(k) \approx \mathfrak{m} + (\mathfrak{c}k)^2/2\mathfrak{m}$, where $\mathfrak{m}$ denotes the spectral gap, with $\mathfrak{m} = \frac{1}{2}\sinh(\eta) \times \sum_{k\in\mathbb{Z}}(-1)^k/\cosh(k\eta)$, $\eta = \cosh^{-1}\Delta$. The obtained result agrees with the semiclassical result of Ref. [7] and it provides the first direct confirmation of the semiclassical approximation in an *anisotropic* chain.

Dressed versus bare form factors.—Form-factor expansions established themselves as a powerful theoretical tool for studying integrable QFTs [62–66]. In the form-factor formalism one traditionally operates with the trivial (bare) Fock vacuum as the reference state. In contrast, a more general expansion with respect to, e.g., a thermal background is a more delicate and technical subject which has not been fully developed yet [67–69]. In context of low-temperature transport, many previous works [4,5,36,70] thus employed a series expansion with respect to the bare vacuum, with the reasoning that the spectral gap renders a summation over multiparticle excitations quickly convergent. Based on this, it has been further advocated that the ground-state dynamical structure factor experiences a small thermal broadening at finite T, which for $T \ll \mathfrak{m}$ matches the diffusive (Lorentzian) peak predicted by the semiclassical approach. Strictly speaking, however, such a *dilute* gas picture only adequately describes physics at zero temperature. The computation of equilibrium correlation functions instead necessitates an expansion based on *dressed* (instead of bare) form factors of local densities, and these are given by matrix elements of particle-hole excitations on top of a finite-density thermal background [33,71–74]. Considering the longitudinal magnetization component $\hat{s}^z$, the matrix element between a thermal state $|\hat{\varrho}_{T,h}\rangle$ and an excited state with a single particle-hole excitation of "type s," with momenta $\Delta k_s = k_s(\theta_s^+) - k_s(\theta_s^-)$, reads

$$\langle \hat{\varrho}_{T,h}|\hat{s}^z_x|\hat{\varrho}_{T,h};\theta_s^+,\theta_s^-\rangle = e^{ix\Delta k_s}m_s^{\rm dr} + \mathcal{O}(\Delta k_s). \tag{8}$$

The dressed magnetization $m_s^{\rm dr}$ of a quasiparticle of type s immersed in a finite-density thermal background can be radically different from the bare value $m_s^{\rm bare} = s$. This effect is particularly pronounced in the vicinity of half-filled thermal equilibria, where the effective magnetization exhibits a crossover from paramagnetic $m_s^{\rm dr} \sim s^2 h$ ($s \ll |h|^{-1}$) to the bare $m_s^{\rm dr} \sim s$ ($s \gg |h|^{-1}$) regime. We note that the vanishing of the spin Drude weight as $h \to 0$ can be seen as a consequence of the paramagnetic behavior of the dressed form factors (8), which are key building blocks in the approach of Ref. [33].

Furthermore, we wish to point out that nonperturbative effects attributed to the quasiparticle dressing also have a profound influence on the NMR spin relaxation rate $1/T_1$ [75–79]. Motivated by the preceding studies, see, e.g., Refs. [4,76], we here specialize to the experimentally relevant regime $h \ll T \ll \mathfrak{m}$, disregarding for simplicity possible effects of the single-ion anisotropy or interchain couplings. The *zero-momentum* contribution to the low-temperature dependence of the intraband relaxation rate T_1^{-1} of the longitudinal spin component is expressible in terms of the dressed form factors (8) as $T_1^{-1} = 2|A^{xz}|^2\sum_s\int dp_s(\theta)[1-n_s(\theta)]n_s(\theta)r_s(\theta)$, where A^{xz} denotes the hyperfine couplings and $r_s(\theta) = (m_s^{\rm dr})^2/[\sqrt{\varepsilon_s''(0)}\sqrt{\varepsilon_s''(0)\theta^2 + \omega_N}]$ with the NMR frequency $\omega_N = h$ (in units $\mu_N = 1$). By taking the $h \to 0$ limit *after* first performing the summation over the entire quasiparticle spectrum $s > 0$, we find

$$\frac{1}{T_1} \sim e^{-(3/2)\mathfrak{m}/T}|h|^{-1/2}. \tag{9}$$

This scaling plays nicely with the experimental study on the $S = 1$ compound [80] and, somewhat surprisingly, is in qualitative agreement with the semiclassical results $T_1^{-1} \sim T\chi_h|\mathfrak{D}_{\rm cl}h|^{-1/2}$ found in Refs. [6,7]. The key difference, however, is that within our method the activation rate $(3/2)\mathfrak{m}/T$ comes from the contributions of the internal magnonic d.o.f. In contrast, the previous calculation from Ref. [76] based on the free spinful bosons and the bare form-factor expansion carried out in Ref. [4] yields the incorrect behavior $T_1^{-1} \sim e^{-\mathfrak{m}/T}\log h$.

KPZ universality.—The unexpected divergent spin conductivity, observed in both the $SU(2)$ symmetric spin chains and the $O(3)$ NLSM, is rooted in anomalous properties of thermally dressed quasiparticles that carry large bare spin s (see also Ref. [34]). Recalling that the spin diffusion constant (3) is an infinite sum over individual quasiparticle contributions, one can readily notice that for the isotropic interspin interactions the summand saturates at large s, $\lim_{s\to\infty}\mathfrak{D}_s = \mathfrak{D}_\infty > 0$, thus rendering the spin diffusion constant infinite. Furthermore, thermal fluctuations of the local spin $\delta\langle\hat{s}^z_x\rangle = \langle\hat{s}^z_x\rangle - \langle\hat{s}^z\rangle_{T,h}$ can be directly linked to fluctuations of "giant quasiparticles" via [47] $\delta\langle\hat{s}^z\rangle = T\chi_h(T,h)\lim_{s\to\infty}\{\delta n_s/[sn_s(n_s-1)]\}$, with δn_s denoting local fluctuations of the Fermi occupation functions. Saturation at a finite asymptotic value $\mathfrak{D}_\infty$ may correspondingly be interpreted as a self-interacting term in the dynamics of fluctuations $\delta\langle\hat{s}^z\rangle$, in analogy to the Burgers equation $\partial_t\delta\langle\hat{s}^z_x(t)\rangle = \partial_x\{\mathfrak{D}_{\rm reg}\partial_x\delta\langle\hat{s}^z_x(t)\rangle + \lambda[\delta\langle\hat{s}^z_x(t)\rangle]^2 + \ldots\}$; here $\mathfrak{D}_{\rm reg} < \infty$ is the "regularized" diffusion constant which accounts for the finite contributions of "light" quasiparticles and $\lambda = \lambda(\mathfrak{D}_\infty)$ is the nonlinearity (self-interaction) coefficient such that $\lim_{\mathfrak{D}_\infty\to 0}\lambda(\mathfrak{D}_\infty) = 0$. This provides a phenomenological model which underlies the KPZ universality class with dynamical exponent $z = 3/2$ [34,81], i.e., $\langle\hat{s}^z_x(t)\hat{s}^z_0\rangle_{T,h=0} \sim t^{-1/z}$, consistently with the observed divergent time-dependent conductivity $\sigma(t) \sim t^{1/3}$, see Fig. 1, and in agreement to that observed in the integrable spin-$1/2$ Heisenberg chain [40].

Conclusions.—We have outlined a theoretical framework for studying low-temperature spin dynamics in gapped and gapless one-dimensional isotropic antiferromagnets based on the effective low-energy quantum field theory. In the vicinity of half filling, we found a crossover from the semiclassical regime $h/T \gg 1$ to the strongly correlated regime $h/T \ll 1$. In the $h \to 0$ limit, we analytically established a divergent spin diffusion constant and conjectured a superdiffusive behavior with fluctuations in the KPZ universality class. The phenomenon is seen in both half-integer and integer spin chain, which rules out the importance of the spectral gap. Instead, the anomalous behavior can be attributed to the effective self-interaction of thermally dressed interacting magnonic waves. Presently, we exclude the conventional interpretation based on mode-coupling theory within the phenomenological framework of the classical nonlinear fluctuating hydrodynamics [81–83] due to the vanishing diagonal terms of the Hessian in the current derivative expansion.

Our findings have direct applications in inelastic neutron scattering spectroscopy and quantum transport experiments [80,84–87], while they also open new venues for further theoretical research on the microscopic mechanisms that underlie the observed anomalous spin transport in the isotropic antiferromagnetic chains. Perhaps the most striking observation is that the phenomenon remains present even at high temperatures. While this could be a footprint of the low-lying sigma model physics, it may as well be due to an emergent classical hydrodynamical description. For instance, the isotropic classical Landau-Lifshitz field theory is also known to exhibit superdiffusive spin transport both in equilibrium [88] and far from equilibrium [89]. We leave these exciting questions to future studies.

We thank D. Bernard, B. Doyon, R. Konik, M. Kormos, T. Prosen, and S. Sachdev for comments on the manuscript and useful related discussions. J. D. N. is supported by the Research Foundation Flanders (FWO). E. I. is supported by VENI Grant No. 680-47-454 by the Netherlands Organisation for Scientific Research (NWO). C. K. acknowledges support by the Deutsche Forschungsgemeinschaft through the Emmy Noether program (KA 3360/2-1).

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