Many-Body Synchronization in a Classical Hamiltonian System

Reyhaneh Khasseh,[1,2] Rosario Fazio,[2,3] Stefano Ruffo,[4,5] and Angelo Russomanno[2,6]
[1]*Department of Physics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan 45137-66731, Iran*
[2]*Abdus Salam ICTP, Strada Costiera 11, I-34151 Trieste, Italy*
[3]*Dipartimento di Fisica, Universita di Napoli "Federico II," Monte S. Angelo, I-80126 Napoli, Italy*
[4]*SISSA, Via Bonomea 265, I-34136 and INFN Trieste, Italy*
[5]*Istituto dei Sistemi Complessi—CNR, Via dei Taurini 19, I-00185 Roma, Italy*
[6]*NEST, Scuola Normale Superiore & Istituto Nanoscienze-CNR, I-56126 Pisa, Italy*

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We study synchronization between periodically driven, interacting classical spins undergoing a Hamiltonian dynamics. In the thermodynamic limit there is a transition between a regime where all the spins oscillate synchronously for an infinite time with a period twice the driving period (synchronized regime) and a regime where the oscillations die after a finite transient (chaotic regime). We emphasize the peculiarity of our result, having been synchronization observed so far only in driven-dissipative systems. We discuss how our findings can be interpreted as a period-doubling time crystal and we show that synchronization can appear both for an overall regular and overall chaotic dynamics.

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Since its discovery by Huygens, the phenomenon of synchronization [1–4] has emerged in the most diverse contexts. Examples of systems undergoing synchronized motion range from coupled mechanical oscillators to chemical reactions, from modulated lasers to neuronal networks or circadian rhythms in living organisms, just to mention only few of them. The essence of synchronization can be very simply stated. Classical nonlinear systems may asymptotically approach self-sustained oscillations; a tiny coupling between these systems can induce their oscillations to be locked in phase space.

All known systems undergoing synchronized dynamics are driven and dissipative. It is therefore natural to ask if synchronization can occur in a Hamiltonian classical system. This is the problem we will address in this Letter. As we will see below, this question, besides having a direct impact on our understanding of dynamical systems, has important connections to chaos and the foundations of statistical mechanics.

In the case of a finite number of coupled classical Hamiltonian systems whose dynamics is generically chaotic, synchronization can be ruled out. For more than two degrees of freedom, even small integrability breaking leads eventually to instability of the motion and chaos. In the many-body case, this fact leads to thermalization (at infinite temperature in the driven case) [5–7]. This picture can drastically change if an *infinite* number of coupled time-dependent classical Hamiltonian systems is considered. In this Letter we will show that synchronization is possible in this case. To the best of our knowledge, synchronization in Hamiltonian systems has not been considered before.

This result has nontrivial connections to the foundations of statistical mechanics. Usually, in the thermodynamic limit, any integrability breaking of short-range interacting classical Hamiltonian systems leads to an essentially fully chaotic behavior and hence to thermalization [8,9]. Nevertheless, this is not the whole story [6] and there may be important cases where this scenario does not apply. In many-body quantum systems, ergodicity can be broken in an extended region of coupling parameters due to interference effects; relevant examples are many-body localization (see [10] for a review) and many-body dynamical localization [11–13]. The challenge is to get a similar stabilization in classical Hamiltonian systems, where the quantum interference provides no help. Here we provide an example in a driven context.

In this Letter we consider a system of coupled classical spins undergoing a periodic pulsed driving. The key point of our analysis will be considering long-range interacting Hamiltonian systems. Here the dynamics can be equivalent to one single collective degree of freedom weakly coupled to other modes, and the dynamics can be regular [14,15] in the thermodynamic limit [16]. The effect of periodic kicking on the regular or chaotic dynamics of classical Hamiltonian systems has been already widely investigated, see, e.g., [6,9,25–28]. Here we make a step forward and analyze the synchronization behavior.

If uncoupled, the dynamics of the spins is regular and they show entrainment with the external driving: the magnetization oscillates with a period double of that of the driving field. Once the spins are coupled through the driving (see Fig. 1), they show synchronized period-doubling oscillations for a time, which scales with the system size and tends to infinity in the thermodynamic limit. Therefore synchronization is an emergent phenomenon, occurring only in the

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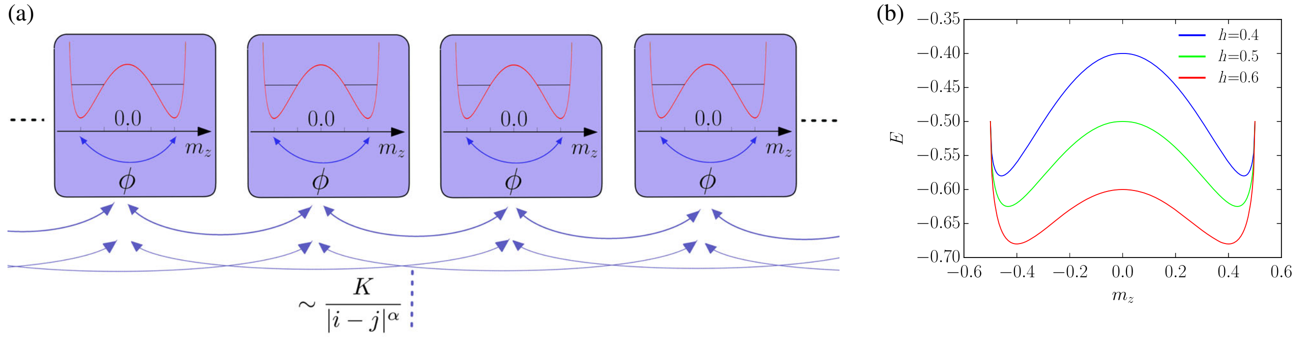

FIG. 1. (a) Long-range coupled period-doubling classical spin systems. The blue boxes symbolize the single oscillators: without interactions they are entrained with the single-particle kicking of angle ϕ, swapping the two symmetry sectors at each kick. The long arrows indicate the interacting part of the kicking which decays as a power law in the distance, with exponent α. If α is small enough there is still synchronization in the thermodynamic limit and appears as period-doubling oscillations in the average z magnetization. (b) Symmetry breaking in the phase space for a single oscillator (the plots are for $m^y = 0$). Around each of the two degenerate minima and for energies smaller than the broken-symmetry edge E^* (the central maximum at $m^z = 0$) there are trajectories that break the $\mathbb{Z}_2$ symmetry.

thermodynamic limit of an infinite interacting system, much like a spontaneous symmetry breaking (as the one occurring in the Kuramoto model [29,30]). We remark that the spins are both entrained with the external driving and synchronized with each other.

Being a form of spatiotemporal order in the thermodynamic limit, robust in a full region of the parameter space and for many initial conditions, occurring as a period doubling with respect to the driving, synchronization can be interpreted as a spontaneous breaking of the discrete time-translation symmetry (from the symmetry group $\mathbb{Z}$ to $2\mathbb{Z}$). Indeed, we can see this dynamics as a classical Hamiltonian period doubling inspired by Floquet quantum time crystals (see [31,32]). Our result is an example of spontaneous time-translation symmetry breaking in a classical Hamiltonian system. Until now, the only known examples of classical time crystal are driven-dissipative systems [33,34]. We remark that other forms of synchronization could be possible where the nontrivial response of the system has the same period of the driving: in this case there would be synchronization without time-translation symmetry breaking.

The Hamiltonian governing the N classical spins $\vec{m}_i$ is given by $\mathcal{H}(t) = \sum_{j=1}^N \mathcal{H}^{(0)}(t) + \mathcal{V}(t)$. The noninteracting part has the form

$$\mathcal{H}^{(0)}(t) = \sum_{j=1}^{N} [-2J(m_j^z)^2 - 2h_j m_j^x + \phi\delta_\tau(t) m_j^x], \quad (1)$$

and the kicked long-range interaction term is

$$\mathcal{V}(t) = -K\delta_\tau(t) \sum_{i,j\neq i} \frac{1}{D_{i,j}^{(\alpha)}} m_i^x m_j^x, \quad (2)$$

where, as in [26], we define $\delta_\tau(t) \equiv \sum_n \delta(t - n\tau)$ to characterize the periodic kicks of period τ and J, h_j, ϕ, and K are tunable parameters. Throughout all the Letter we consider periodic boundary conditions; we have defined $D_{i,j}^{(\alpha)} \equiv \kappa(\alpha)[\min\{|i-j|, N - |i-j|\}]^\alpha$ in order to implement them with the same prescription of [35,36] ($D_{i,j}^{(\alpha)} \sim \kappa(\alpha)|i-j|^\alpha$ when $|i-j| \ll N$). The quantity $\kappa(\alpha)$ is needed in order to make the interaction part of the Hamiltonian extensive [37]: $\kappa(\alpha) \equiv N^{1-\alpha}$ if $\alpha < 1$ and unit for $\alpha > 1$, in the marginal case $\alpha = 1$ equals $\log N$.

The dynamics of this Hamiltonian is obtained using the Poisson-bracket rules of the classical-spin components $\{m_i^\mu, m_j^\nu\} = \epsilon^{\mu\nu\rho}\delta_{ij} m_j^\rho$ where the Greek indices can take values in x, y, z, the Latin ones in $1, \ldots, N$ and $\epsilon^{\mu\nu\rho}$ is the Ricci fully antisymmetric tensor. With these Poisson-bracket rules it is easy to write down the Hamilton equations $\dot{m}_j^\nu(t) = -\{m_j^\nu(t), \mathcal{H}(t)\}$. Between two kicks they are a set of N decoupled systems of three differential equations. Across a kick they can be explicitly integrated, and they give rise for each j to a rotation around the x axis with angle depending on the values of the $\{m_l^x\}$.

Let us start from the case with no interactions ($K \equiv 0$). In this case there is a range of parameters where each classical spin can show a period-doubling response to the driving [38]. When $h_j < J$, the Hamiltonian shows a $\mathbb{Z}_2$ symmetry breaking. The Hamiltonian is indeed symmetric under the π rotation around the x axis ($m_l^{y,z} \to -m_l^{y,z}$, $m_l^x \to m_l^x \, \forall \, l$) but the trajectories with energy smaller than a broken-symmetry edge [39] break this symmetry. These trajectories are doubly degenerate and appear in pairs transformed into each other by the symmetry operation [see Fig. 1(b)]. The system shows period doubling if it is prepared in a symmetry-breaking trajectory and the kicking with $K \equiv 0$ is used to swap between this trajectory and its symmetric partner. The kick produces a rotation of angle ϕ around the x axis. By considering $\phi \equiv \pi$ there are period-doubling oscillations of the z magnetizations m_j^z (perfect swapping of the symmetric trajectories). These oscillations are stable if ϕ is made slightly different from π, there being a continuum of symmetry breaking trajectories (see Ref. [38]).

The analysis of the interacting dynamics $K \neq 0$ is crucial to understand when the period doubling is stable in the thermodynamic limit. We will characterize the interacting dynamics by analyzing the average magnetization along the z axis, $m^z(t)$ [see Eq. (3)]. For any finite size we see period-doubling oscillations of $m^z(t)$. These oscillations mark the synchronization of the oscillators and are discrete rotations in time analogous to the continuous ones of the Kuramoto order parameter [29,30]. The period-doubling oscillations die out after a transient; in order to see how this transient scales with the system size, we define the order parameter for period doubling

$$O(n\tau) \equiv (-1)^n m^z(n\tau) = \frac{(-1)^n}{N} \sum_{j=1}^{N} m_j^z(n\tau), \qquad (3)$$

where $m^z(t)$ is the average z magnetization. $O(n\tau)$ remains nonvanishing, keeping its sign until there are period-doubling oscillations of the spins. For any finite size of the system, we numerically see that this quantity vanishes after a transient, reaching in this way the thermal $T = \infty$ value $O_{T=\infty} = m^z_{T=\infty} = 0$. (The $T = \infty$ thermal values are computed in the microcanonical ensemble for the Hamiltonian without kicking.) To study the scaling of the transient, we quantify its duration as $t_d/\tau = \sum_{n=1}^{t^*/\tau} n O(n\tau) / \sum_{n=1}^{t^*/\tau} O(n\tau)$. Here t^*/τ is the first value of n where $O(n\tau)$ vanishes. In order to have persistent synchronized period-doubling oscillations in the thermodynamic limit, t_d must diverge with the system size N.

We initialize the system in a state where the order parameter, $O(0)$, is positive. A uniform initial state is a very singular case: it is easy to show that for a uniform Hamiltonian the dynamics is equivalent to a single spin. The synchronization is trivial and corresponds to the entrainment of the single oscillator. A nontrivial situation arises in the case of a random initial state ($m_j^z(0) = \sqrt{1-\epsilon_j^2}$, $m_j^x(0) = \epsilon_j \cos\varphi_j$, $m_j^y(0) = \epsilon_j \sin\varphi_j$ with ϵ_j being a random variable uniformly distributed in the interval $[0, \epsilon]$ and φ_j uniformly distributed in $[0, 2\pi]$). We can also include disorder in the Hamiltonian by taking the h_j random and uniformly distributed in the interval $[h - \Delta h, h + \Delta h]$. In these random cases we average our results over $N_{\rm rand}$ randomness realizations and evaluate the error bars of any randomness-dependent quantity $\mathcal{S}$ as $\mathrm{std}(\mathcal{S})/\sqrt{N_{\rm rand}}$ where $\mathrm{std}(\mathcal{S})$ is the standard deviation of $\mathcal{S}$ over randomness realizations.

As we vary the parameters of the system we find two regimes. In the synchronized regime the decay time of the order parameter scales as a power law, $t_d \sim N^b$, and there is synchronization in the thermodynamic limit. In the thermalizing regime, on the opposite, there is not such a scaling and consequently no synchronization. Some examples of these two different scalings are shown in Fig. 2(a). Here we have considered the case of a kicking different from the perfect-swapping one (we take $\phi = 0.99\pi$ instead of $\phi = \pi$): synchronization persists also for this imperfect kicking, marking thereby the robustness of this

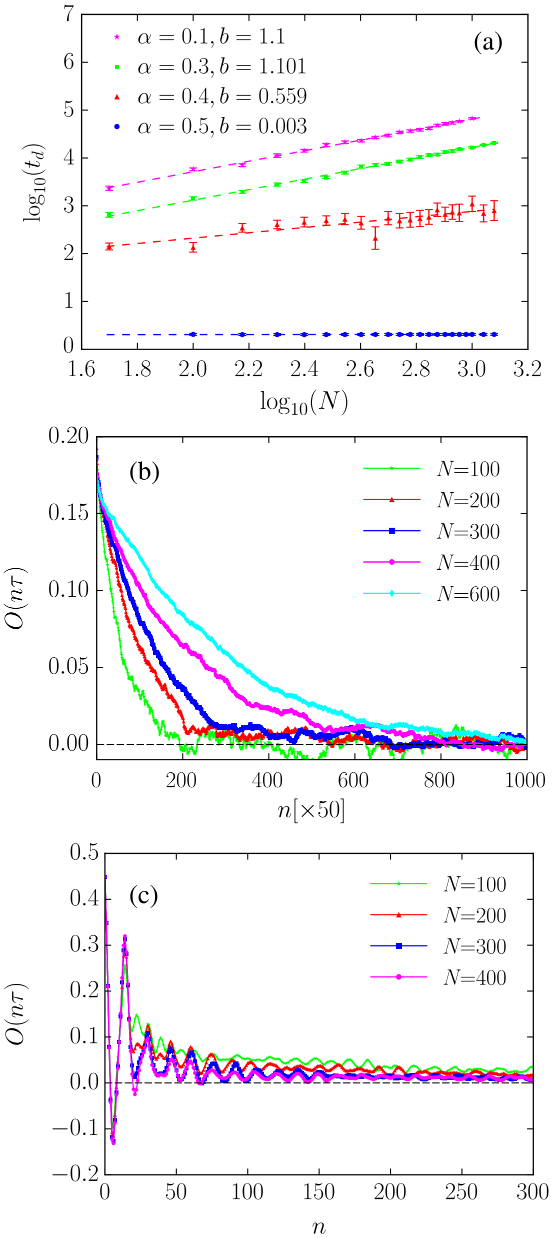

FIG. 2. (a) Scaling of t_d with N. Notice two possible regimes, one in which there is the power-law scaling $t_d \sim N^b$, and another where there is no scaling. (b) Some examples of evolution of the order parameter for different values of N and parameters inside the synchronized region ($\alpha = 0.3$). (c) The same for parameters in the thermalizing regime ($\alpha = 0.5$). Numerical parameters: $h = 0.32$, $\Delta h = 0$, $\phi = 0.99\pi$, $K = 0.3$, $\tau = 0.6$, $\epsilon = 0.05$, $N_{\rm rand} = 28$.

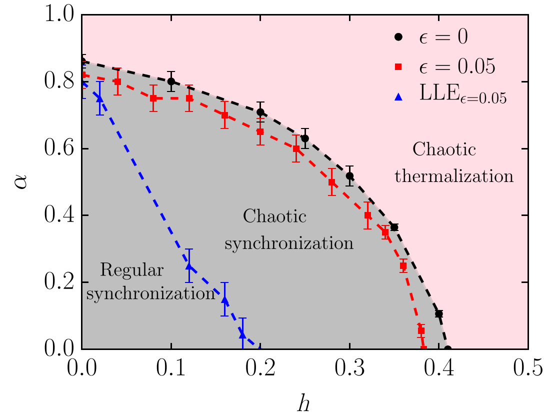

FIG. 3. Regions in the parameter space. The red and black curves separate synchronization from thermalization at infinite size for different ϵ. The blue curve separates regular behavior ($\mathrm{LLE}_{\epsilon=0.05} \to 0$ in the thermodynamic limit) from chaotic one. Notice the existence of an intermediate chaotic but nonthermalizing region where $\mathrm{LLE}_{\epsilon=0.05} > 0$ in the thermodynamic limit and there is also synchronization. (Numerical parameters: $K = 0.3$, $\tau = 0.6$, $\phi \equiv 0.99\pi$, $\Delta h = 0$, $N_{\mathrm{rand}} = 28$.)

phenomenon. Notice that well inside the synchronized regime the scaling exponent b is very near to 1 and consistent with a linear scaling. In order to show the markedly different behavior in the two regimes, in Fig. 2(b) we provide some examples of evolution of the order parameter for different sizes in a case where there is synchronization and in Fig. 2(c) we do the same for a case where the system thermalizes. Using the scaling properties of t_d, we can clearly distinguish in the thermodynamic limit $N \to \infty$ the synchronized regime from the thermalizing one and we can map a diagram of the dynamical regimes. We plot this diagram in Fig. 3 for uniform ($\epsilon = 0$, trivial) and random ($\epsilon \neq 0$, nontrivial) initial conditions.

We remark that synchronization is robust and survives the randomness in the initial state. To better show this fact, in Fig. 4(a) we plot α^* (the critical value separating synchronized from chaotic and ergodic) vs the randomness amplitude ϵ for different values of h. Synchronization is also robust if disorder is added to the model, as it occurs for example in the Kuramoto model [1,29,30]. We have checked this, adding disorder to h_j. The results are shown in Fig. 4(b) where we plot the value α^* as a function of the disorder strength Δh.

Let us now move to consider the regularity or chaoticity properties of the dynamics. The largest Lyapunov exponent (LLE) gives a measure of how much nearby trajectories diverge exponentially and is thereby a measure of chaos [40]. It is defined as $\mathrm{LLE} = \lim_{d(0)\to 0}\lim_{t\to\infty}(1/t)\ln[d(t)/d(0)]$ [$d(t)$ is the distance between trajectories at time t]. We compute the LLE using the orbit separation method (see [40,41]). We consider its average over the random initial conditions distribution introduced above: $\mathrm{LLE}_\epsilon \equiv \langle \mathrm{LLE} \rangle_{\varphi_j \in [0,2\pi], \epsilon_j \in [0,\epsilon]}$. In this way we fix the same distribution

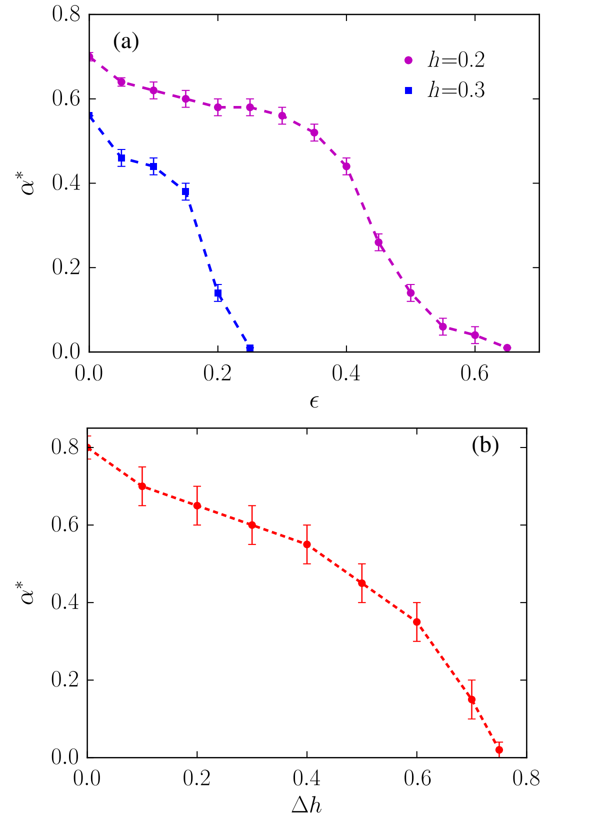

FIG. 4. Transition point α^* vs the initial-state randomness ϵ [panel (a)] and the randomness in the field h_j [panel (b)]. Numerical parameters: $K = 0.3$, $\tau = 0.6$, $\phi = 0.99\pi$, $N_{\mathrm{rand}} = 20$; for (a) $\Delta h = 0$, for (b) $h = 0.1$.

of the random initial conditions and here we can compare the regularity or chaoticity properties of the dynamics with the synchronization properties.

For N finite we find that LLE_ϵ is always larger than 0, as expected for a nonlinear, nonintegrable system, but we can notice two different behaviors in the limit $N \to \infty$ (in the numerics we have fixed $\epsilon = 0.05$). There is a regime where LLE_ϵ stays finite in the limit $N \to \infty$ and another regime where our numerics suggests that it scales to 0 as a power law when $N \to \infty$: $\mathrm{LLE}_\epsilon \sim N^{-\gamma_\epsilon}$ with $\gamma_\epsilon > 0$ (as it occurs for the full LLE in the Kuramoto model [42]). We show some examples in the Supplemental Material [43]. We can mark the boundary between the two regimes and plot it as a blue curve in Fig. 3. We see that the regular region of vanishing LLE_ϵ is smaller than the synchronized region. This suggests that there are three regions in the parameter space for the considered ϵ. *Regular synchronization*: there is synchronization and the $\mathrm{LLE}_\epsilon \to 0$ in the thermodynamic limit. In this case the $N \to \infty$ dynamics is essentially regular in the region of phase space corresponding to the considered random initial conditions. *Chaotic thermalization*: here $\mathrm{LLE}_\epsilon > 0$ and there is no synchronization. The dynamics here is essentially chaotic. *Chaotic synchronization*: There is chaos in the considered region of phase space ($\mathrm{LLE}_\epsilon > 0$) but forms of orderlike synchronization can emerge, in analogy with a related phenomenon of a driven-dissipative system [44]. We remark that the regularity or chaoticity and synchronization properties of the dynamics depend on the region of phase space we consider (given by the value of ϵ). We can see this in Fig. 4(a) where synchronization disappears beyond a threshold in ϵ.

In conclusion we have found a form of synchronization of a set of classical Hamiltonian oscillators that are driven and long-range interacting. Synchronization corresponds to collective period-doubling oscillations lasting for a time, which scales as a power law with the system size. The synchronization is robust to randomness in the Hamiltonian and the initial state and is connected to the time-crystal phenomena. Perspectives of future research include the analysis of quantum effects; indeed there are examples of quantum spins with long-range interactions that do not synchronize [45]. It is interesting to understand if this phenomenon can be interpreted classically or if quantum effects are crucial. It is also important to consider the role of thermal noise. The situation is very well known for noisy dissipative models with short range interactions [46,47]: noise generically destroys period n tupling for $n > 2$. Noisy dissipative long-range systems have yet to be explored from this perspective. In our specific model we think that thermal noise would spoil synchronization, but this might not be a general feature for long-range systems, especially moving towards the thermodynamic limit.

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