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Repeated Interactions and Quantum Stochastic Thermodynamics at Strong Coupling

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The thermodynamic framework of repeated interactions is generalized to an arbitrary open quantum system in contact with a heat bath. Based on these findings, the theory is then extended to arbitrary measurements performed on the system. This constitutes a direct experimentally testable framework in strong coupling quantum thermodynamics. By construction, it provides many quantum stochastic processes and quantum causal models with a consistent thermodynamic interpretation. The setting can be further used, for instance, to rigorously investigate the interplay between non-Markovianity and nonequilibrium thermodynamics.

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Introduction.—Formulating the laws of quantum thermodynamics forces us to rethink many assumptions, which are traditionally taken for granted. In particular, small systems are dominated by fluctuations and, in general, they do not interact weakly with a Markovian heat bath. Also the desire to monitor and manipulate quantum systems adds another layer of complexity due to the nontrivial effect of quantum measurements.

In this Letter, we present a unified thermodynamic framework, which overcomes the assumption of a weakly coupled Markovian heat bath and which allows us to include nonequilibrium resources and quantum measurements. These nonequilibrium resources are a set of small, externally prepared systems—called "units" in the following—which are sequentially put into contact with the system under study. This setup is known as the "repeated interaction framework" or "collisional model" and it has recently attracted much attention in quantum thermodynamics [1–13]. However, the coupling to an additional external heat bath (typically present in an experiment) was mostly ignored, and a weakly coupled Markovian one was only treated in Refs. [1,7,11]. Based on recent progress in strong coupling thermodynamics [14,15], we will show that even the assumption of a weakly coupled macroscopic heat bath can be completely overcome.

Afterward, following the operational approach to quantum stochastic thermodynamics [16,17], we will show how to explicitly take into account measurements into the thermodynamic description. This constitutes a crucial step in strong coupling quantum thermodynamics, where different strategies were used to arrive at many interesting conclusions [18–41]. However, all strategies rely on a formalism without any explicit measurements, thus making them hard to test and compare [42]. In contrast, our theory is. in principle. immediately testable in a lab as it only requires one to measure the system. Finally, we rigorously connect our thermodynamic framework to the field of quantum non-Markovianity.

Setting.—We start by considering a system S coupled to a bath B described by the Hamiltonian $H_{SB}(\lambda_t) = H_S(\lambda_t) + V_{SB} + H_B$, where λ_t denotes an externally specified driving protocol (e.g., a laser field) and V_{SB} denotes the system-bath interaction Hamiltonian. To this setup we add the framework of repeated interactions specified by the following global Hamiltonian:

$$H_{\mathrm{tot}}(\lambda_t) = H_{SB}(\lambda_t) + \sum_{k=0}^{n} V_{SU(k)}(\lambda_t). \tag{1}$$

Here, $V_{SU(k)}(\lambda_t)$ describes the time-dependent coupling between the system and unit $U(k)$, $k \in \{0, \ldots, n\}$, which is designed in such a way that, at most, one unit interacts with the system at a given time. Specifically, if we denote the interaction interval between the system and the kth unit by $I_k \equiv [t_k, t_{k+1})$, then $V_{SU(k)}(\lambda_t) = 0$ for all $t \notin I_k$. Within I_k the time dependence as specified by λ_t is arbitrary. Furthermore, we temporarily assume the bare unit Hamiltonian to be degenerate, i.e., $H_{U(k)} \sim 1_{U(k)}$. The problem is completely specified by fixing the global initial state, which is assumed to be of the form

$$\rho_{\mathrm{tot}}(t_0^-) = \pi_{SB}(\lambda_0) \otimes \rho_{U(0)} \otimes \ldots \otimes \rho_{U(n)}. \tag{2}$$

Here and in general we use the notation $t^\pm$ to denote the time $t \pm \epsilon$ in the limit where $\epsilon > 0$ becomes immeasurably small. Furthermore, $\pi_X = e^{-\beta H_X}/\mathcal{Z}_X$ denotes the equilibrium Gibbs state of some system X at inverse temperature β (perhaps depending on the value of some driving protocol). Finally, the initial state of the units is arbitrary but uncorrelated. A sketch of the present setup is shown in

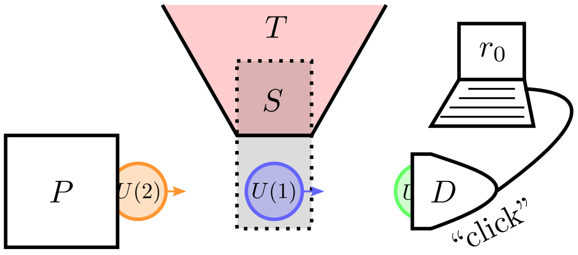

FIG. 1. A system S is in contact with a bath initialized at temperature T (we do not imply that the bath must be kept at a well-defined temperature at later times). In a preparation apparatus P units $U(k)$ are sequentially produced, which interact with the system when they enter the shaded gray area and which are afterward detected in D giving rise to the measurement outcome r_k (properly introduced later on in the text). Note that each unit can be different in principle. We also remark that the description of the units does not need to be taken literally in the sense that they are merely "ancillas" required for a consistent thermodynamic description of a non-Markovian quantum stochastic process, as introduced later on.

Fig. 1. We remark that various extensions are possible, as discussed at the end of this Letter.

Below we will need the notion of the "Hamiltonian of mean force," an old concept [43] (see also Refs. [14,15,18]), which is defined via the reduced equilibrium state of a bipartite system XB. Specifically,

$$\pi_X^* \equiv \mathrm{tr}_B\{\pi_{XB}\} \equiv \frac{e^{-\beta H_X^*}}{\mathcal{Z}_X^*}, \qquad \mathcal{Z}_X^* \equiv \frac{\mathcal{Z}_{XB}}{\mathcal{Z}_B}. \tag{3}$$

Note that $\pi_X^* \neq \pi_X$ in general. In addition, H_X^* depends on the inverse temperature β and (possibly) a control parameter. Notice that the Hamiltonian of mean force for the system and all units simplifies, as at any given time at most one unit is physically coupled to the system, e.g., for $t \in I_k$, $H^*_{SU(\mathbf{n})}(\lambda_t) = H^*_{SU(k)}(\lambda_t)$. Here and in general we use $U(\mathbf{n})$ to denote the entire sequence of units from $U(0)$ to $U(n)$.

The average rate of injected work (the power) has two contributions. For $t \in I_k$, we define

$$\dot{W}_S(t) = \left\langle \frac{\partial H_S(\lambda_t)}{\partial t} \right\rangle(t), \tag{4}$$

$$\dot{W}_{SU(k)}(t) = \left\langle \frac{\partial V_{SU(k)}(\lambda_t)}{\partial t} \right\rangle(t), \tag{5}$$

where $\langle \ldots \rangle(t)$ denotes a quantum statistical average at time t. It follows that the total mechanical work performed on the system up to time t is

$$\begin{aligned} W(t) &= \int_{t_0^-}^{t} ds \left[\dot{W}_S(s) + \sum_k \dot{W}_{SU(k)}(s) \right] \\ &= \langle H_{\mathrm{tot}}(\lambda_t) \rangle(t) - \langle H_{\mathrm{tot}}(\lambda_0^-) \rangle(t_0^-). \end{aligned} \tag{6}$$

Note that this definition of average mechanical work is widely accepted even in the strong coupling regime [15,19,20,23,24,26,29,34,35,37,39] as it is directly related to the change in internal energy of the universe (i.e., the system, the bath, and all units all together).

Strong coupling repeated interactions framework.—We start by introducing the basic concept of a nonequilibrium free energy adapted to the strong coupling regime [14,15],

$$F_X(t) \equiv \mathrm{tr}_X\{\rho_X(t)[H_X^*(\lambda_t) + \beta^{-1} \ln \rho_X(t)]\}. \tag{7}$$

In the weak coupling limit, where $H_X^*(\lambda_t) \approx H_X(\lambda_t)$, this definition reduces to the conventional one. The slight modification allows us to express the second law even at strong coupling and even in the presence of the system-unit interactions in the conventional way ($k_B \equiv 1$),

$$\Sigma(t) \equiv \beta[W(t) - \Delta F_{SU(\mathbf{n})}(t)] \geq 0. \tag{8}$$

Here, $\Sigma(t)$ denotes the entropy production and $\Delta F_{SU(\mathbf{n})}(t) \equiv F_{SU(\mathbf{n})}(t) - F_{SU(\mathbf{n})}(t_0^-)$. Positivity of the second law follows by confirming that

$$\Sigma(t) = D[\rho_{\mathrm{tot}}(t) \| \pi_{\mathrm{tot}}(\lambda_t)] - D[\rho_{SU(\mathbf{n})}(t) \| \pi^*_{SU(\mathbf{n})}(\lambda_t)], \tag{9}$$

where $D[\rho\|\sigma] = \mathrm{tr}\{\rho(\ln\rho - \ln\sigma)\} \geq 0$ is the quantum relative entropy. Hence, $\Sigma(t)$ is positive by monotonicity of relative entropy [44,45]. The derivation uses only Eqs. (2) and (3) and the unitary dynamics, which implies for the von Neumann entropy $S[\rho_{\mathrm{tot}}(0)] \equiv -\mathrm{tr}\{\rho_{\mathrm{tot}}(0) \ln \rho_{\mathrm{tot}}(0)\} = S[\rho_{\mathrm{tot}}(t)]$. It is lengthy, but straightforward, and hence not displayed here.

Equation (8) corresponds to the second law if we regard the system and *all* units as one big system and explicitly keep their correlations in the description. In practice, it often turns out that keeping the information about all units and all their correlations is superfluous (compare also with the discussion in Ref. [7]). Thus, at time t_{n+1}^- after the nth interaction but before the $(n+1)$th interaction, where the system is decoupled from all units, the following second law is practically more meaningful:

$$\begin{aligned} \Sigma_S(t_{n+1}^-) &= \beta\left(W(t_{n+1}^-) - \Delta F_S(t_{n+1}^-) + T \sum_{k=0}^{n} \Delta S[\rho_{U(k)}] \right) \\ &\geq \Sigma(t_{n+1}^-) \geq 0. \end{aligned} \tag{10}$$

Here, we added a subscript S to indicate that this is the entropy production from the system point of view ignoring superfluous information about the units. To arrive at

Eq. (10), we used subadditivity of entropy and $H_{U(k)} \sim 1_{U(k)}$. In contrast to Eq. (8), it contains only the change in the marginal von Neumann entropy of the units. The entropy production per interaction interval I_n is then given by

$$\begin{aligned}\Sigma_S(t_{n+1}^-) - \Sigma_S(t_n^-) \\ &= \beta[W(t_{n+1}^-) - W(t_n^-) - F_S(t_{n+1}^-) + F_S(t_n^-)] \\ &\quad + \{S[\rho_{U(n)}(t_{n+1}^-)] - S[\rho_{U(n)}(t_n^-)]\}. \qquad (11)\end{aligned}$$

It is the strong coupling generalization of the second law in the repeated interaction framework, see Eq. (49) in Ref. [7]. Interestingly, in contrast to the Markovian weak coupling situation, we cannot ensure the positivity of this expression. This is similar to the classical case [15] and we will connect it to the notion of non-Markovianity later on. But first we will advance conceptually by introducing explicit measurements in the description.

Quantum stochastic thermodynamics at strong coupling.—We consider the case where the experimenter measures the state of the unit *after* the interaction with the system, as indicated in Fig. 1 [i.e., $V_{SU(n)}(\lambda_t) = 0$ at the time t of the measurement]. By doing so, we can gather valuable information about the state of the system. In a moment, we will also show that this allows us to implement arbitrary generalized measurements on the system and that the resulting theory can be fruitfully linked to the study of quantum stochastic processes and quantum causal models.

Mathematically, we denote the measurement result of the kth unit by r_k and associate a positive operator P_{r_k} to it, which fulfills the normalization condition $\sum_{r_k} P_{r_k}^2 = 1_{U(k)}$. The state of the unit then changes according to the map $\mathcal{P}_{r_k}\rho_{U(k)} \equiv P_{r_k}\rho_{U(k)}P_{r_k} \equiv \tilde{\rho}_{U(k)}(r_k)$. Notice that $\tilde{\rho}_{U(k)}(r_k)$ is a subnormalized state with the probability $p(r_k) = \mathrm{tr}_{U(k)}\{\tilde{\rho}_{U(k)}(r_k)\}$ as its norm. After multiple units were subjected to their respective measurements, giving results $\mathbf{r}_n \equiv (r_n, \ldots, r_1, r_0)$, the global subnormalized state reads

$$\tilde{\rho}_{\mathrm{tot}}(\mathbf{r}_n, t_{n+1}^-) = \left(\prod_{k=0}^{n} \mathcal{P}_{r_k}\mathcal{U}_{k+1,k}\right)\rho_{\mathrm{tot}}(t_0^-). \qquad (12)$$

Here, $\mathcal{U}_{k+1,k}$ describes the global unitary evolution from t_k^- (shortly before the kth unit starts interacting with the system) to t_{k+1}^-. Because the measurement always acts after the interaction, we can also write

$$\tilde{\rho}_{\mathrm{tot}}(\mathbf{r}_n, t_{n+1}^-) = \left(\prod_{k=0}^{n} \mathcal{P}_{r_k}\right)\rho_{\mathrm{tot}}(t_{n+1}^-), \qquad (13)$$

where $\rho_{\mathrm{tot}}(t_{n+1}^-) = \mathcal{U}_{n+1,0}\rho_{\mathrm{tot}}(t_0^-)$ is the global time-evolved state without any measurements. This allows us to confirm the useful relation

$$\sum_{\mathbf{r}_n} \tilde{\rho}_{SB}(\mathbf{r}_n, t_{n+1}^-) = \rho_{SB}(t_{n+1}^-); \qquad (14)$$

i.e., the average system-bath state does not change due to the measurements. This is not true for the units.

Inspired by Ref. [16], we now introduce the following thermodynamic definitions along a single trajectory characterized by the measurement results $\mathbf{r}_n$. First, for $t \in I_n$, the stochastic power $\dot{w}_S(\mathbf{r}_n, t)$ and $\dot{w}_{SU(n)}(\mathbf{r}_{n-1}, t)$ are simply obtained from Eqs. (4) and (5) by replacing the average over the unconditional state $\rho_{SU(\mathbf{n})}(t)$ with an average over the conditional state $\rho_{SU(\mathbf{n})}(\mathbf{r}_n, t)$. Notice that $\dot{w}_{SU(n)}(\mathbf{r}_{n-1}, t)$ does not depend on the last measurement outcome r_n because, by construction, the measurement $\mathcal{P}_{r_n}$ acts after the nth unit has interacted with the system. Therefore, together with Eq. (14) we immediately obtain the relations $\sum_{\mathbf{r}_n} p(\mathbf{r}_n)\dot{w}_S(\mathbf{r}_n, t) = \dot{W}_S(t)$ and $\sum_{\mathbf{r}_n} p(\mathbf{r}_n)\dot{w}_{SU(n)}(\mathbf{r}_{n-1}, t) = \dot{W}_{SU(n)}(t)$. Second, we have to generalize the nonequilibrium free energy to the stochastic case, which becomes

$$\begin{aligned} f_{SU(\mathbf{n})}(\mathbf{r}_n, t) \equiv{}& \langle H_{SU(\mathbf{n})}^*(\lambda_t)\rangle(\mathbf{r}_n, t) \\ &+ T\{-S[\rho_{SU(\mathbf{n})}(\mathbf{r}_n, t)] + \ln p(\mathbf{r}_n)\}, \qquad (15)\end{aligned}$$

where $\langle \ldots \rangle(\mathbf{r}_n, t)$ denotes an average with respect to the conditional state. An essential difference compared to definition (7) is the appearance of the stochastic entropy $-\ln p(\mathbf{r}_n)$ associated with the measurement results obtained with probability $p(\mathbf{r}_n) = \mathrm{tr}\{\tilde{\rho}_{\mathrm{tot}}(\mathbf{r}_n, t)\}$. A similar but not identical construction is used in classical stochastic thermodynamics [46] (compare with the discussion in Refs. [16,17]). In contrast to the stochastic work, we have, in general, $\sum_{\mathbf{r}_n} p(\mathbf{r}_n) f_{SU(\mathbf{n})}(\mathbf{r}_n, t) \neq F_{SU(\mathbf{n})}(t)$. Finally, we introduce the stochastic entropy production

$$\sigma(\mathbf{r}_n, t) \equiv \beta[w(\mathbf{r}_n, t) - \Delta f_{SU(\mathbf{n})}(\mathbf{r}_n, t)]. \qquad (16)$$

As in classical stochastic thermodynamics, it can be negative along a single trajectory [47,48]. However, we will now prove that, on average, $\sum_{\mathbf{r}_n} p(\mathbf{r}_n)\sigma(\mathbf{r}_n, t) \geq 0$, which demonstrates the thermodynamic consistency of our strong coupling quantum stochastic framework.

As a consequence of Eq. (14) and our previously derived second law (8), we confirm that

$$\begin{aligned}\sum_{\mathbf{r}_n} p(\mathbf{r}_n)\sigma(\mathbf{r}_n, t) - \Sigma(t) \\ &= \sum_{\mathbf{r}_n} p(\mathbf{r}_n)\{S[\rho_{SU(\mathbf{n})}(\mathbf{r}_n, t)] - \ln p(\mathbf{r}_n)\} - S[\rho_{SU(\mathbf{n})}(t)]. \\ &\qquad (17)\end{aligned}$$

This quantifies the change in informational entropy of all constituents (system, units, and the classical memory) due to the big joint measurement (13). Its positivity follows from the lemma in Ref. [16], which simply combines Theorem 11 of Ref. [49] and Theorem 11.10 of Ref. [50] and which can be interpreted as the second law for a quantum measurement. Hence, we conclude

$$\sum_{\mathbf{r}_n} p(\mathbf{r}_n)\sigma(\mathbf{r}_n, t) \geq \Sigma(t) \geq 0. \tag{18}$$

As before, the (averaged) stochastic entropy production $\sigma(\mathbf{r}_n, t)$ contains the information about all the correlations in the units, which is typically not needed. Using subadditivity of entropy, it is again possible to arrive at expressions similar to Eq. (10). Note that, depending on the experimental situation, one could decide to not only discard information about the unit correlations, but also about the measurement results $\mathbf{r}_n$.

This concludes the formal part of the Letter, where we have introduced a consistent notion of work, nonequilibrium free energy, and entropy production along a single run of an experiment regardless of any details of the system-bath coupling. It is instructive to connect the present picture to the theory of quantum causal models and quantum stochastic processes. If we consider the limit of an instantaneous system-unit interaction, ideally described by a coupling of the form $V_{SU(k)}(\lambda_t) = v_k\delta(t - t_k)$, we can write the system-bath dynamics as

$$\tilde{\rho}_{SB}(\mathbf{r}_n, t_{n+1}^-) = \left(\prod_{k=0}^{n} \mathcal{U}_{k+1,k}^{SB}\mathcal{A}_{r_k}\right)\rho_{SB}(t_0^-). \tag{19}$$

Here, $\mathcal{U}_{k+1,k}^{SB}$ is the unitary time evolution generated by $H_{SB}(\lambda_t)$ and the completely positive map $\mathcal{A}_{r_k}$ is defined via its action $\mathcal{A}_{r_k}\rho_S = \mathrm{tr}_U\{\mathcal{P}_{r_k}[e^{-iv_k/\hbar}\rho_S\rho_{U(k)}e^{iv_k/\hbar}]\}$ on an arbitrary system state ρ_S. In this context, $\mathcal{A}_{r_k}$ is also known as an "instrument" describing the most general state transformation possible in quantum mechanics [51,52]. The application of a set of instruments $\mathcal{A}_{r_0}, \ldots, \mathcal{A}_{r_n}$ to an open quantum system *defines* a general quantum stochastic process (or quantum causal model), which can be formally represented by a "quantum comb" or "process tensor" [53–58]. The sole difference compared to the most general case is that we do *not* allow for real-time feedback control; i.e., the instruments $\mathcal{A}_{r_k}$ are not allowed to depend on the previous results $\mathbf{r}_{k-1}$, otherwise Eq. (14) would no longer be true. Whether the present framework can be extended to arbitrary real-time feedback control as in the Markovian case [16,17] remains an open question.

Thermodynamic signatures of non-Markovianity.—We now turn toward an important application linking the field of quantum thermodynamics and quantum non-Markovianity [59,60] in a rigorous way. As recognized below Eq. (11), at strong coupling we cannot ensure that the entropy production is positive in every time interval $[t_\ell, t_k)$ for $t_\ell > t_k > t_0$. There have been repeated claims in the literature that negative entropy production rates indicate non-Markovianity [61–66]. Doubts were raised in Refs. [15], since the definitions for entropy production rates used in Refs. [61–66] do not yield an overall positive entropy production when integrated from the initial time t_0 to any final time $t > t_0$. Moreover, they can be even negative for Markovian dynamics [15].

On the other hand, for a suitable notion of entropy production based on the Hamiltonian of mean force, progress was achieved for classical dynamics [15]. Therein, the situation of a strongly coupled system prepared in an arbitrary nonequilibrium state was considered (no repeated interactions were present). Then, it was shown that Markovian dynamics necessarily imply a positive entropy production rate if the system is *undriven* (i.e., $\lambda_t = \text{constant}$). In our quantum formalism, the system is initially in equilibrium such that its state does not change when undriven and left on its own. However, we can link the present picture to the classical case by realizing that we can use the very first unit $U(0)$ to prepare the system in an arbitrary nonequilibrium state via a short control operation. This preparation procedure has a thermodynamic cost captured by the always positive entropy production $\Sigma_S(t) = \beta W_{SU}(t) - \beta\Delta F_S(t) - \beta\Delta S[\rho_{U(0)}] \geq 0$ [compare with Eq. (10)]. After the system-unit interaction, the system is left on its own and the entropy production in between any two times $t_2 > t_1 > t_0$ reads

$$\begin{aligned}\Sigma_S(t_2) - \Sigma_S(t_1) &= -\beta[F_S(t_2) - F_S(t_1)]\\ &= D[\rho_S(t_1)\|\pi_S^*] - D[\rho_S(t_2)\|\pi_S^*].\end{aligned} \tag{20}$$

This quantifies the dissipation associated with the relaxation dynamics of the system. Equation (20) is positive if the dynamics are Markovian *and* if π_S^* is a steady state of the dynamics at any time t. Interestingly, the latter point can be shown rigorously based on the definition of Markovianity from Ref. [67], which is adapted to the situation of a general quantum stochastic process as used here. This is proven in the Supplemental Material [68] (which contains Ref. [69]). Thus, $\Sigma_S(t_2) - \Sigma_S(t_1) \geq 0$ for a Markov process in complete analogy to the classical result [15]. This opens up the door to investigate the interplay between entropy production and non-Markovianity in a mathematically and thermodynamically rigorous sense for quantum systems.

Further applications.—The ability to analyze general non-Markovian quantum processes from a thermodynamic perspective will find applications in various areas. One example is sequential quantum metrology [70]. Specifically, a particularly intriguing parameter to estimate is the temperature of a system. Much progress has been achieved to understand it from the perspective of metrology [71], but the thermodynamic costs of thermometry have not yet been explored. With the recent progress in the design of optimal quantum probes [72] and strong coupling thermometry [73], the present Letter opens up the possibility to thermodynamically analyze many scenarios in metrology and thermometry. Furthermore, recent progress shows how to unambiguously detect quantum features in quantum stochastic processes [74–76]. Since quantum thermodynamics is still in search of clear observable quantum effects induced by coherence [77], the present Letter will allow us to rigorously address such questions. Furthermore, originally used to understand Nobel-prize-winning experiments [78,79] in quantum optics from a thermodynamic perspective [16], the present framework can be used to explore more general cases where the system is not a high-quality cavity and has substantial losses or is coupled to other cavities. Also the units do not have to be identical, which opens up the possibility to, e.g., thermodynamically analyze single-photon distillation experiments [80], where an atom in a cavity is first probed by a weak coherent pulse (unit 1) followed by a measurement (modeled by unit 2) to herald the photon distillation. Quite generally, the present setup is even relevant for experiments in the Markovian regime, if detailed control about all system parts is not possible. Finally, the present framework can be combined with the traditional picture of scattering theory and, following Ref. [7], it allows us to investigate Maxwell's demon and Landauer's principle at strong coupling.

Extensions.—As detailed in the Supplemental Material [68], the present framework can be extended into various directions: the Hamiltonian of the units does not need to be degenerate, the initial state of the units can be correlated, and the general identities (8), (10), and (18) still hold in the case where the system-bath coupling is time dependent. In the last case, however, the theory is no longer "operational" in the sense that explicit knowledge about the state of the bath is necessary, which is hardly accessible. Similarly, we show in the Supplemental Material [68] how to treat multiple heat baths. Unfortunately, also in that case the state of the system and units does not suffice to have access to all thermodynamic quantities [81].

Concluding remarks.—The present contribution establishes a consistent thermodynamic framework—even along a single trajectory recorded in an experiment—for a system in contact with an arbitrary bath and additionally subjected to arbitrary nonequilibrium resources interacting one by one with the system. This pushes the applicability of nonequilibrium thermodynamics far beyond its traditional scope. Furthermore, the present work also demonstrates how quantum stochastic thermodynamics departs from its classical version in the strong coupling regime [14,15,82–84]. While the basic concepts at the unmeasured level are similar, any possible measurement strategy has a nontrivial influence on the description in the quantum regime, even on average. For instance, in general, there is a strict inequality on the left-hand side of Eq. (18). This is not a deficiency of our theory, but a *necessary* ingredient, which can be already recognized at the level of the work statistics [85]. Quantum stochastic thermodynamics is more than a mere extension of its classical counterpart. The present operational approach is, however, flexible enough to reproduce the unmeasured picture: it is recovered by choosing the trivial but legitimate measurement operator $P_{r_k} = 1_{U(k)}$, i.e., the identity. Then, the stochastic entropy production $\sigma(\mathbf{r}_n)$ reduces to Σ.

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