Multifractal Scalings Across the Many-Body Localization Transition

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In contrast with Anderson localization where a genuine localization is observed in real space, the many-body localization (MBL) problem is much less understood in Hilbert space, the support of the eigenstates. In this Letter, using exact diagonalization techniques we address the ergodicity properties in the underlying $\mathcal{N}$-dimensional complex networks spanned by various computational bases for up to $L = 24$ spin-1/2 particles (i.e., Hilbert space of size $\mathcal{N} \simeq 2.7 \times 10^6$). We report fully ergodic eigenstates in the delocalized phase (irrespective of the computational basis), while the MBL regime features a generically (basis-dependent) multifractal behavior, delocalized but nonergodic. The MBL transition is signaled by a nonuniversal jump of the multifractal dimensions.

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Introduction.—Anderson localization (AL) [1,2] is a fundamental phenomenon where transport is hindered by disorder in quantum systems composed of free particles. When interactions between the particles are added, localization tends to be prohibited and thermalization is favored. The eigenstate thermalization hypothesis (ETH) [3,4], which provides a theoretical understanding of how thermal equilibrium is encoded at the level of individual many-body eigenfunctions, applies generically to most interacting systems (with the important exception of integrable models). It is thus quite remarkable that disordered quantum interacting systems can still show an absence of transport and thermalization in the form of many-body localization (MBL) [5,6], a topic that has attracted considerable interest recently (see reviews Refs. [7–10]). Besides arrested transport, MBL features several striking properties, e.g., anomalously small (area-law) entanglement [11], emergent integrability and Poisson spectral statistics [12,13], and very slow (logarithmic) entanglement spreading [14–16], some of which expose the differences between MBL and AL. Numerical simulations of 1D lattice models [17,18], analytical rigorous proofs [19], and effective localized-bits (l-bits) models [12,13,20,21] have been instrumental in understanding these properties of the MBL phase.

However, several important questions remain open regarding the MBL-ETH transition and the nature of eigenstates. This contrasts with the much-better understood case of AL where the spatial extension of *single-particle* orbitals is known to display *multifractality* at criticality on finite-dimensional lattices, a behavior which characterizes the transition and its underlying field-theoretical description [2]. In the MBL context on the other hand, the behavior of *many-body* eigenstates in the Hilbert space is not firmly established [18,22–27]. We aim at addressing three open issues: (i) the possible existence of an intermediate non-ergodic delocalized phase in the ETH phase [23,24,26]; (ii) the wave functions properties in the MBL regime sometimes described as Fock-space localized [26,28], while perturbative [29,30] and numerical [18] results indicate multifractality; and (iii) a consistent description of the critical regime.

The central quantities of interest in our work are the participation entropies (PE) S_q, derived from the qth moments of a wave function $|\Psi\rangle$ expressed in a given basis:

$$|\Psi\rangle = \sum_{\alpha=1}^{\mathcal{N}} \psi_\alpha |\alpha\rangle \quad \text{and} \quad S_q = \frac{1}{1-q} \ln\left(\sum_{\alpha=1}^{\mathcal{N}} |\psi_\alpha|^{2q}\right). \tag{1}$$

In the Shannon limit $S_1 = -\sum_\alpha |\psi_\alpha|^2 \ln|\psi_\alpha|^2$, while the case $q = 2$ recovers the usual inverse participation (IPR) [31] with $S_2 = -\ln(\text{IPR})$. Let us first describe the possible leading asymptotic behaviors for S_q with the support size $\mathcal{N}$. For a perfectly *delocalized* state $S_q = \ln\mathcal{N}$. In contrast, if a state is *localized* on a finite set, one gets $S_q = \text{const}$, as observed for AL. In an intermediate situation, wave functions are extended but nonergodic, with $S_q = D_q \ln\mathcal{N}$ ($D_q < 1$ are q-dependent multifractal dimensions).

In this Letter, we inspect *many-body* eigenstates in the Hilbert space by studying D_q across the ETH-MBL transition for model Eq. (2). Building on large-scale numerics for the most relevant basis sets $\{|\alpha\rangle\}$, we provide a complete scaling analysis describing both phases and the critical point, which captures finite-size effects. We conclude for full ergodicity ($D_q = 1$) for ETH, generic multifractality in the entire MBL regime, and a nonuniversal jump of D_q at the transition, as sketched in Fig. 1 [32].

MBL as a complex network Anderson problem.—We focus on the random-field XXZ $S = \frac{1}{2}$ chain model

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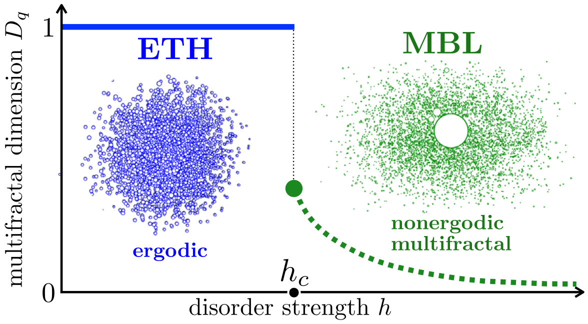

FIG. 1. Schematic picture of the multifractal properties of eigenstates in Fock and spin configuration bases across the MBL transition for Eq. (2). Two typical eigenstates of Eq. (2) on a small $L = 14$ system for ETH ($h = 0.5$, blue) and MBL ($h = 10$, green) regimes are graphically represented, with circle sizes proportional to $|\psi_\alpha|^2$ in the spin configuration basis.

$$\mathcal{H} = \sum_{i=1}^{L} \left[\Delta S_i^z S_{i+1}^z - h_i S_i^z + \frac{1}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) \right], \quad (2)$$

with periodic boundary conditions and h_i randomly drawn from a uniform distribution $[-h, h]$. Equation (2) is equivalent [33] to interacting spinless fermions in a random potential. This system has been intensively studied [11,14,17,18,22,36,37] and its phase diagram is well known for the case $\Delta = 1$ with a critical disorder estimated to be $h_c = 3.7(2)$ in the middle of the many-body spectrum $\{E\}$ such that $\epsilon = (E - E_{\rm min})/(E_{\rm max} - E_{\rm min}) = 0.5$. This Hamiltonian can be recast as a single particle Anderson problem of the general form

$$\mathcal{H} = \sum_\alpha \mu_\alpha |\alpha\rangle\langle\alpha| + \sum_{\langle\alpha\beta\rangle} t_{\alpha\beta} |\alpha\rangle\langle\beta|, \quad (3)$$

in a given basis $\{|\alpha\rangle\}$. Of course, the localization properties measured by the PE depend crucially on the choice of $\{|\alpha\rangle\}$. We focus on two bases: spin configurations $\{|\alpha\rangle\}_S$ and Fock basis $\{|\alpha\rangle\}_F$, which we argue are the most relevant for the model Eq. (2): (i) both bases diagonalize $\mathcal{H}$ in specific limits where localization is well understood (the noninteracting limit $\Delta = 0$ for $\{|\alpha\rangle\}_F$, the limit $h \to \infty$ for $\{|\alpha\rangle\}_S$) and are thus used as a starting point for the l-bits construction or efficient numerical simulations of MBL, (ii) they implement the U(1) conservation rule (particle number or magnetization conservation) of the model, and (iii) $\mathcal{H}$ is *sparse* in both bases: an ingredient which eases numerical diagonalization and may favor ergodicity breaking.

Spin configuration basis: The basis $\{|\alpha\rangle\}_S$ uses the local projection of S_i^z, i.e., $|\alpha\rangle = |\uparrow\downarrow\uparrow, \ldots\rangle$. We restrict the study to the zero magnetization sector $\sum_i S_i^z = 0$ (half-filling for fermions) of dimension $\mathcal{N} = \binom{L}{L/2} \simeq 2^L/\sqrt{L}$. In the spin configuration basis, Eq. (2) becomes a hopping problem Eq. (3) with disordered on site energies $\mu_\alpha = \langle\alpha| \sum_i \Delta S_i^z S_{i+1}^z - h_i S_i^z |\alpha\rangle$, and constant hopping terms $t_{\alpha\beta} = 1/2$ [allowing tunneling between neighboring states $\langle\alpha\beta\rangle$ connected by spin-flip terms of Eq. (2)]. As the site-dependent connectivity $\bar{z} \approx L/2$ grows faster with system size L than the average on site disorder strength $\sigma_\mu \sim h\sqrt{L}$, general arguments would prohibit genuine AL in this complex network (note however the strong correlations of the potential between neighboring sites).

Fock basis: $\{|\alpha\rangle\}_F$ are many-body states built from noninteracting localized orbitals that diagonalize the free-fermion part $\mathcal{H} - \Delta \sum_i S_i^z S_{i+1}^z$ of Eq. (2). On site potentials μ_α are the sum of the noninteracting orbital energies corrected by the Hartree-Fock term, and off-diagonal hoppings $t_{\alpha\beta}$ are built from the interaction terms [11,33,38]. Viewing MBL as an Anderson problem on graphs defined by these Fock states has been promoted in very early works [5,6,39]. Nevertheless, the hopping problem expressed in $\{|\alpha\rangle\}_F$ is qualitatively different than from in $\{|\alpha\rangle\}_S$. While diagonal terms have similar behaviors, there is a much larger number of nonzero matrix elements $z_{\alpha\beta}$ between Fock states, which is constant over the graph: $z_{\alpha\beta} = \frac{1}{4}\{(L/2)[(L/2) - 1]\}^2 + (L^2/4)$ (in the $\sum_i S_i^z = 0$ sector). Moreover, hoppings $t_{\alpha\beta}$ are not constant but random, both in sign and magnitude [33].

Participation entropies: large-scale numerics.—Eigenstates of Eq. (2) are extracted from full and shift-invert subset [40] diagonalization, focusing on high-energy states ($\epsilon = 0.5$) on various system sizes, ranging from $L = 8$ spins (with $\mathcal{N} = 70$) up to $L = 24$ ($\mathcal{N} = 2705432$). Disorder average is performed over many independent samples, typically tens of thousand for the smallest sizes $L \leq 16$, and several hundreds for the largest samples $18 \leq L \leq 24$ [41]. We fix $\Delta = 1$.

We first discuss the scaling with Hilbert space size $\mathcal{N}$ of the disorder-average PE $\overline{S_q}$ defined by Eq. (1) and shown in Fig. 2 for both Fock and configuration spaces for a few representative values of the disorder. In the ETH phase [Figs. 2(a) and 2(b)], low disorder data follow a purely ergodic scaling of the form $S_q = \ln \mathcal{N} + b_q$ with $b_q < 0$ for both Fock and configuration spaces. Upon increasing h a curvature develops, indicating that an asymptotic scaling regime might be eventually reached for larger $\mathcal{N}$, as exemplified by $h = 2.6$ and $h = 3.2$ data. In the MBL regime [Figs. 2(c) and 2(d)] with $q = 1$, 2 at $h = 5$, both bases exhibit a delocalized behavior $S_q = D_q \ln \mathcal{N} + b_q$ with a q- and basis-dependent multifractal dimension $D_q < 1$ and a correction $b_q > 0$.

Very interestingly, b_q changes sign (negative for ETH and positive for MBL) as seen in Fig. 3 where a crossing point in $\overline{S_q}/\ln \mathcal{N}$ appears in the vicinity of the critical disorder strength $h_c \sim 3.8$. Crossings are equally observed for Fock and configuration spaces in Figs. 3(a), 3(d), and 3(e). This effect is also clearly visible from the distributions $P(S_q/\ln \mathcal{N})$ shown in Figs. 3(b) and 3(c)

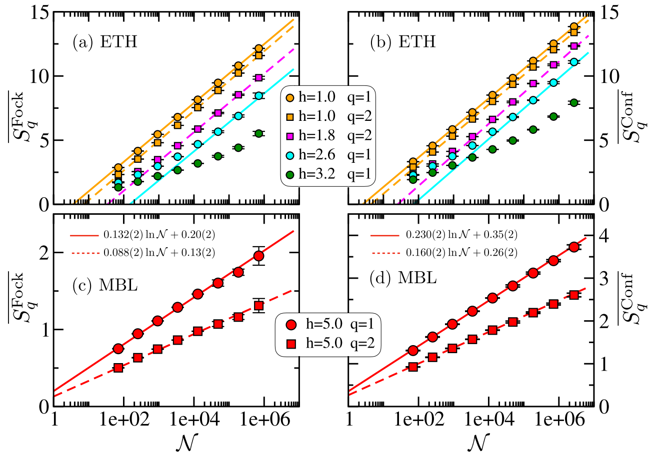

FIG. 2. Finite size scaling of the disorder average PEs $\overline{S_q}$ shown for both Fock [left: (a), (c)] and configuration spaces [right: (b), (d)] and a few representative values of disorder strength h in both ETH (a), (b) and MBL (c), (d) regimes for $q = 1, 2$. Lines are of the form $D_q \ln \mathcal{N} + b_q$ with $D_q = 1$ and $b_q < 0$ in the ETH regime (guides to the eye), while $D_q < 1$ and $b_q > 0$ in the MBL regime (fits).

where we observe strikingly distinct behaviors on both sides of the transition, with qualitatively different finite size effects and opposite skewness. As $S_q/\ln \mathcal{N}$ is restricted to the interval [0, 1], the distributions at small disorder slowly converge (from below) towards unity with system size. Conversely, in the MBL regime, rare events tails extend above the average while distributions shrink with system size (with a variance vanishing as a power law [33]), thus leading to positive finite size corrections. As argued below, the negative constant correction b_q in the ETH phase can be physically related to a nonergodicity volume $\Lambda_q = \exp(-b_q)$. In contrast, the positive value of b_q in the MBL regime seems to be rooted in rare events showing up in positive skewness distributions.

Scaling analysis.—We turn to a finite size scaling analysis of PE across the MBL transition. Figure 4 shows results for $q = 2$ in the two basis where very good collapses can be obtained using the following ansatz [42], inspired by a recent investigation [43] of the Anderson problem on random graphs:

$$\overline{S_q}(\mathcal{N}) - \overline{S_{q,c}}(\mathcal{N}) = \begin{cases} \mathcal{G}_{\mathrm{vol}}\left(\frac{\mathcal{N}}{\Lambda_q}\right) & \text{if } h < h_c \\ \mathcal{G}_{\mathrm{lin}}\left(\frac{\ln \mathcal{N}}{\xi_q}\right) & \text{if } h > h_c. \end{cases} \tag{4}$$

Note that we first subtracted $\overline{S_{q,c}}(\mathcal{N})$ the critical PE at $h_c = 3.8$ [44]. In the ETH regime (top panels), a volumic scaling $\mathcal{G}_{\mathrm{vol}}(\mathcal{N}/\Lambda)$ yields an excellent description of the crossover with Λ interpreted as a nonergodicity volume [43]. When the Hilbert space dimension $\mathcal{N} \gg \Lambda$, full ergodicity is recovered with $S_q = \ln \mathcal{N} - (1 - D_{q,c}) \ln \Lambda$, $D_{q,c}$ being the multifractal dimension at the transition, while the critical behavior $S_q \to S_{q,c}$ is recovered below the nonergodicity volume $\mathcal{N} \ll \Lambda$. This volumic scaling is found to be universal as it occurs for both Fock and spin basis, with a nonergodicity volume diverging very fast as the transition is approached: $\ln \Lambda \sim (h_c - h)^{-a}$ [insets in Fig. 4(a)],with $a \approx 0.45$ [45]. An analogous volume law with $a = 0.5$ was reported [43,46] for the Anderson problem on random regular graphs, in agreement with analytical predictions in infinite dimensions [47–49].

Contrasting with ETH, data in the MBL regime rather follow a linear scaling function $\mathcal{G}_{\mathrm{lin}}(\ln \mathcal{N}/\xi)$ with a universal form

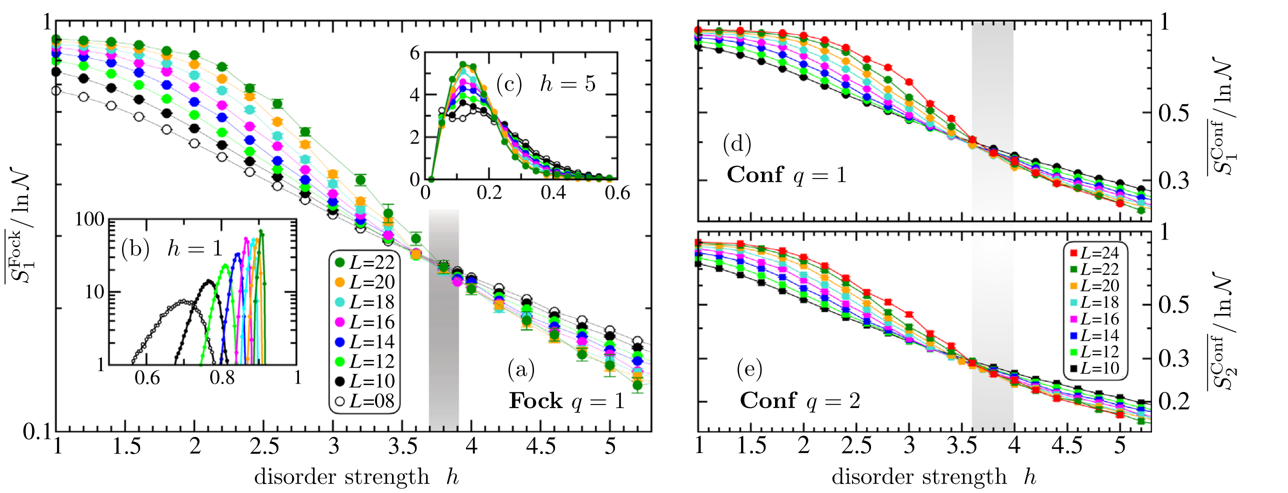

FIG. 3. Scaled PE $\overline{S_q}/\ln \mathcal{N}$ plotted against disorder h in both Fock (a) and configuration spaces (d), (e). Crossing points signal a sign change of the subleading correction b_q in the PE scalings $D_q \ln \mathcal{N} + b_q$, which occurs in the vicinity of the ETH-MBL transition point $h_c \sim 3.8$ (gray shaded region). Insets (b), (c) show histograms $P(S_1^{\mathrm{Fock}}/\ln \mathcal{N})$ of PE in the Fock basis.

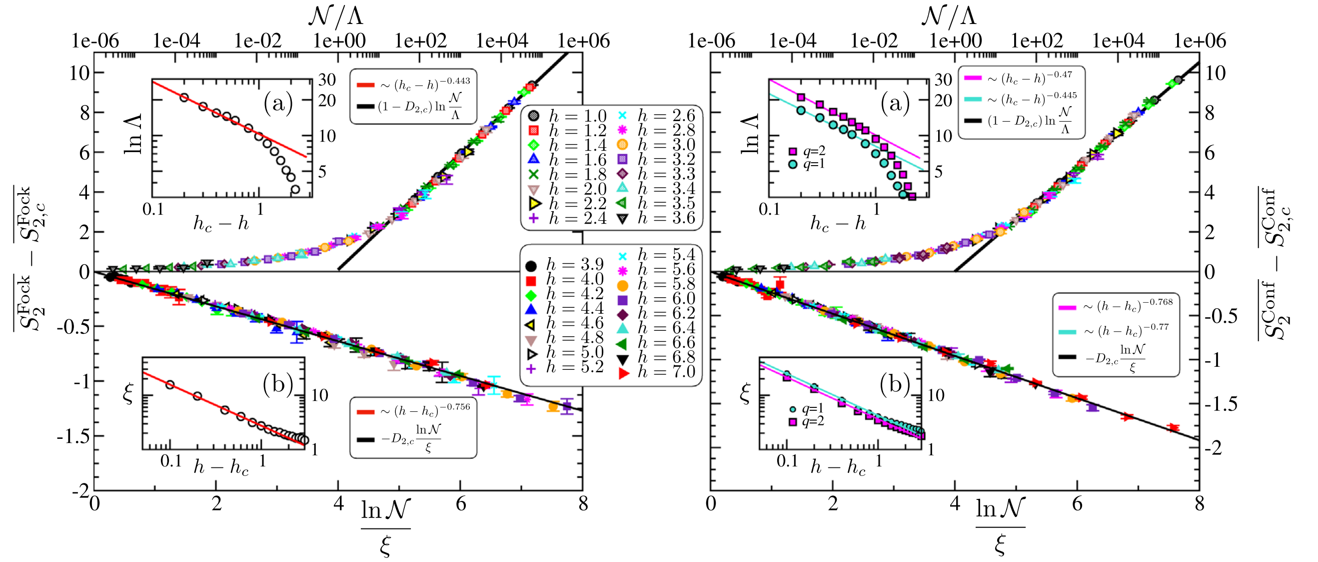

FIG. 4. Scaling curves for the PE following Eq. (4) across the full ETH-MBL regimes with varying disorder strengths (colored symbols). Data for $q=2$ are displayed for both Fock (left) and configuration (right) spaces, after subtracting the critical PE data $\overline{S_{2,c}}$ taken at $h=3.8$. For $h<h_c$ (top panels) a volumic scaling $\mathcal{G}_{\rm vol}(\mathcal{N}/\Lambda)$ gives the best data collapse. The nonergodicity volumes Λ plotted in insets (a) show similar (universal) divergences $\ln\Lambda \sim (h_c-h)^{-a}$ with $a\sim 0.45$ for Fock ($q=2$) and configuration ($q=1$ and $q=2$) spaces, as indicated on the plot. Bottom panels show data collapses in the MBL regime for $h>h_c$ following a linear scaling $\mathcal{G}_{\rm lin}(\ln\mathcal{N}/\xi)$. The length scale ξ plotted in insets (b) diverges as $(h-h_c)^{-a'}$ with $a'\sim 0.76$. Note that small deviations to scaling Eq. (5) are expected due to nonuniversal subleading constant corrections $b_q>0$, but are difficult to observe in this strong disorder regime.

$$S_{q,\rm MBL} - S_{q,c} = -D_{q,c}\frac{\ln\mathcal{N}}{\xi}, \tag{5}$$

again yielding a very good data collapse (bottom panels of Fig. 4). The length scale ξ, extracted from the best collapse, is shown in panels (b) of Fig. 4 as a function of the distance to criticality $h-h_c$, where again $h_c=3.8$ has been fixed. In Eq. (5), ξ lies in the range $(1,+\infty)$ which guarantees $S_{q,\rm MBL}$ to remain positive in the limit $\ln\mathcal{N}\gg\xi$ where the leading term follows $S_q = D_{q,c}(1-1/\xi)\ln\mathcal{N}$. In the other limit $\ln\mathcal{N}\ll\xi$, one retrieves the critical scaling. Close to criticality, we observe a divergence of the length scale $\xi \propto (h-h_c)^{-a'}$, with $a'\approx 0.76$.

It is crucial to make a distinction between random graphs which display AL at large disorder (see for instance Refs. [43,46,50]), and the present many-body problem where instead, multifractality is present in the entire MBL phase $\forall\, h\geq h_c$. We clearly observe [33] $D_{q,\rm MBL}(h\gg h_c)\propto 1/h$, thus vanishing only in the infinite disorder limit, in agreement with recent results [30]. This behavior can be accounted for by simple strong disorder arguments [33]. Numerical data further suggest that this form persists almost down to the critical point such that $D_{q,\rm MBL}(h)\simeq D_{q,c}(h_c/h)$.

Discussion.—Finite-size effects might affect the estimates of MBL critical point and exponents [51,52]. However our scaling analysis carefully accounts for them (Fig. 4) and strongly supports the following scenario across the ETH-MBL transition for the standard model Eq. (2). (i) Eigenstates are generically ergodic, with no multifractality over the entire ETH phase ($D_q=1$) thus removing the possibility for an intermediate nonergodic state with $D_q<1$ in the model Eq. (2). (ii) The entire MBL regime harbors delocalized nonergodic eigenstates with disorder and basis dependent multifractal dimensions, only vanishing in the infinite disorder limit. (iii) The ETH-MBL transition point belongs to the same nonergodic regime, with a jump of the multifractal dimension $D_q<1$ at the transition, in agreement with recent predictions for a nonthermal critical point [53–57]. ETH-MBL criticality displays two radically different scalings across the transition for eigenstates extension through the Hilbert space: a volumic law $\mathcal{G}_{\rm vol}(\mathcal{N}/\Lambda)$ for ETH with an emergent nonergodicity volume Λ, diverging at the transition, and a linear scaling $\mathcal{G}_{\rm lin}(\ln\mathcal{N}/\xi)$ in the MBL regime. This is clearly compatible with a second-order critical point, but with an unsymmetrical nature, as also recently highlighted in Refs. [53,54,56,58]. To which extent ξ is related to a localization length in real space remains to be understood. We do not see signs of a putative second MBL phase [57,59] in our multifractal analysis which is not directly comparable with the (real-space) MBL multifractality proposed by a renormalization group study [58].

Our results on multifractality of the MBL phase have the following consequences on other approaches for the MBL problem: (i) they give the correct scaling behavior for variational computational approaches to eigenstates based on the two basis at hand, (ii) put constraints for putative theories of the critical point (in the same spirit as the multifractal spectrum does for AL [2]), and (iii) provide a new phenomenology, clearly different from the Anderson problem on random graphs or Cayley trees [5,39,43,46,50,60–66], thus challenging possible analogies.

Other perspectives are also opened up by the present Letter, such as connecting with the anomalous slow dynamics observed [25,67–74] in a large part of the ETH phase, e.g., by considering multifractal properties of operators [75]. We note that while rare-regions effects [76] can induce such subdiffusive transport, much less dramatic consequences are expected for eigenstates (reflecting "infinite-time" physics) [77]. Another extension will consist in testing for universal behavior by considering other models with a MBL transition or disturbing the computational basis [78]. Finally, our results pave the way for a better control of decimation approaches in configuration [79] or Fock space [38], as well as for a recently proposed Hilbert space percolation approach [80,81] to the MBL transition.

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from a much higher connectivity. We have instead chosen to perform rotations from configuration to Fock basis, still computationally costly but allowing us to get PE data up to $L = 22$ in the Fock space.
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