Device-Independent Tests of Structures of Measurement Incompatibility

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In contrast with classical physics, in quantum physics some sets of measurements are incompatible in the sense that they cannot be performed simultaneously. Among other applications, incompatibility allows for contextuality and Bell nonlocality. This makes it of crucial importance to develop tools for certifying whether a set of measurements respects a certain structure of incompatibility. Here we show that, for quantum or nonsignaling models, if the measurements employed in a Bell test satisfy a given type of compatibility, then the amount of violation of some specific Bell inequalities becomes limited. Then, we show that correlations arising from local measurements on two-qubit states violate these limits, which rules out in a device-independent way such structures of incompatibility. In particular, we prove that quantum correlations allow for a device-independent demonstration of genuine triplewise incompatibility. Finally, we translate these results into a semidevice-independent Einstein-Podolsky-Rosen-steering scenario.

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The fact that some pairs of quantum observables do not commute implies that they cannot be measured simultaneously as the corresponding operators do not share a common set of eigenvectors [1]. This incompatibility property of quantum measurements is used in several quantum information protocols such as quantum cryptography [2] and quantum state discrimination [3–5], and is also required in proofs of contextuality [6,7], Einstein-Podolsky-Rosen steering (EPR steering) [8,9], and Bell nonlocality [10].

It is thus of fundamental and practical importance to develop tools to experimentally certify that a set of measurements respects a given type of incompatibility, required for producing a specific type of quantum correlation. Moreover, it would be very useful to be able to achieve such a certification without needing to model the experimental procedures that generate the experimental statistics. This is precisely the aim of the paradigm of device-independent certification used, for instance, for certifying secure communication [11] and randomness [12]. This paradigm assumes that quantum theory (QT) is correct and that signaling between spacelike separated events is impossible. Then, it uses the violation of specifically tailored Bell inequalities [13] to certify a targeted property using only the experimental statistics.

The relation between Bell inequality violation and measurement incompatibility was first studied by Fine, who showed that, in the scenario where two parties are restricted to two dichotomic measurements, a Bell inequality can only be violated if the observers use incompatible measurements [14]. Later, Wolf *et al.* [15] showed that every pair of incompatible measurements can be used to violate the simplest Bell inequality, namely, the Clauser-Horne-Shimony-Holt (CHSH) inequality [16]. Moreover, methods for device-independent quantification of incompatibility have been proposed [17–19] and it is known that some sets of incompatible measurements cannot be used to violate Bell inequalities [20–22]. Finally, it is known that when more than two measurements are considered, different compatibility structures may appear [23,24].

In this Letter, we show how to test if a specific structure of incompatibility is required to generate the statistics observed in a Bell test. Our approach is based on the intuition that, if the measurements used in the Bell test satisfy a targeted structure of compatibility, then the amount of Bell violation becomes limited and, therefore, any violation beyond this limit rules out the presence of the targeted compatibility structure. We also show examples of such violations in the simplest scenario of local measurements applied to two-qubit systems. Thus, at least the simplest structures of incompatibility can be certified in a device-independent way.

Pairwise and n-wise incompatibility.—In quantum mechanics, a measurement M on a d-dimensional quantum system is described by a set of positive operators $M_a \geq 0$, where a labels the outcome of the measurement, acting on a d-dimensional complex Hilbert space $\mathbb{C}^d$ and satisfying the normalization condition $\sum_a M_a = \mathbb{1}$ ($\mathbb{1}$ is the identity operator). A set of n quantum measurements $\{M_x\}_{x=1}^n$ is said to be fully compatible if and only if there exists a set of measurement operators $\{E_\lambda\}$ ($E_\lambda \geq 0$ and $\sum_\lambda E_\lambda = \mathbb{1}$) such that

$$M_{a|x} = \sum_\lambda p(a|x,\lambda)E_\lambda, \quad \forall \;\; a, x, \tag{1}$$

where $p(a|x,\lambda) \geq 0$ and $\sum_a p(a|x,\lambda) = 1 \;\; \forall \;\; x, \lambda$ [25]. Otherwise, they are incompatible. Notice that a set of compatible measurements can be implemented simultaneously by employing the measurement $\{E_\lambda\}$ and post-processing the results according to the probability distribution $\{p(a|x,\lambda)\}$.

Given the previous definition, a set of measurements can present different structures of compatibility. For instance, a set of three measurements can be pairwise compatible but incompatible when all three measurements are considered [23]. In general the compatibility structure of a set of measurements can be represented by a hypergraph $\mathcal{C} = [C_1, C_2, \ldots, C_k]$, where each hyperedge C_i indicates a subset of measurements that are compatible. For instance, the structure $\mathcal{C}_{\text{pair}} = [\{1,2\},\{1,3\},\{2,3\}]$ indicates that the measurements 1,2, and 3 are pairwise compatible, but not triplewise compatible, while the structure $\mathcal{C}_{\text{3full}} = [\{1,2,3\}]$ indicates full triplewise compatibility (see Fig. 1 for more examples). In Supplemental Material [26], we show how the different kinds of measurement incompatibility can be tested by semidefinite programming.

Within this framework, we can also define genuine triplewise (or in general n-wise) incompatibility: A set of three measurements is genuinely triplewise incompatible when it cannot be written as a convex combination of measurements that are pairwise compatible on different partitions. Let us illustrate this concept with an example.

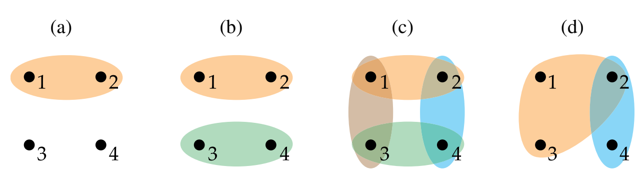

FIG. 1. Examples of incompatibility structures for four measurements. Each node represents a complete measurement (i.e., a complete set of positive-value-operator measure elements), and each hyperedge (represented by a region colored with the same color) contains measurements that are compatible. If a set of measurements are not contained in a hyperedge they are incompatible. The respective incompatibility structures are represented by the following hypergraphs: (a) $\mathcal{C}_{\text{A}} = [(1,2)]$; (b) $\mathcal{C}_{\text{B}} = [(1,2),(3,4)]$; (c) $\mathcal{C}_{\text{C}} = [(1,2),(1,3),(2,4),(3,4)]$; (d) $\mathcal{C}_{\text{D}} = [(1,2,3),(2,4)]$.

Consider a set of three noisy qubit Pauli measurements given by measurement operators

$$M^\eta_{a|x} \coloneqq \eta\Pi_{a|x} + (1-\eta)\frac{\mathbb{1}}{2}, \tag{2}$$

where $x = 1, 2, 3$ refers to each Pauli measurement (X, Y, Z), respectively, and $\Pi_{a|x}$ are their eigenprojectors. These measurements are fully compatible for $\eta \leq 1/\sqrt{3} \approx 0.58$ and pairwise compatible for $\eta \leq 1/\sqrt{2} \approx 0.71$ [23]. It turns out that, for $\eta \leq [(\sqrt{2}+1)/3] \approx 0.80$, the set can be written as a convex combination of other sets in which two measurements are compatible (see Fig. 2). Thus, although for $\eta > 1/\sqrt{2}$ the measurements are incompatible [i.e., they do not admit a decomposition like (1)], it is only for $\eta > [(\sqrt{2}+1)/3]$ that they are genuinely triplewise incompatible.

Device-independent test of structures of incompatibility.—We now turn to the question of certifying the different types of measurement incompatibility in a device-independent way, i.e., by analyzing the statistics of input and output relations of measurements. We consider a bipartite Bell scenario where two parties, Alice and Bob, share a bipartite state ρ onto which they perform measurements labeled by x and y with outcomes a and b, respectively. After many rounds of the experiment, Alice and Bob can determine the set of conditional probability distributions $\{p(ab|xy)\}$, which we call the observed behavior [46]. A behavior is local when it can be written as [10]

$$p(ab|xy) = \sum_\lambda p(\lambda)p_A(a|x,\lambda)p_B(b|y,\lambda), \quad \forall \;\; a, b, x, y, \tag{3}$$

where $p(\lambda)$, $p_A(a|x,\lambda)$, and $p(b|y,\lambda)$ are probability distributions. We denote the set of local behaviors by L.

If one of the parties, say Alice, performs a set of measurements that are fully compatible, the observed behavior is local regardless of the shared state and the measurements of Bob [14]. This can be explicitly seen by using the definition (1) as follows:

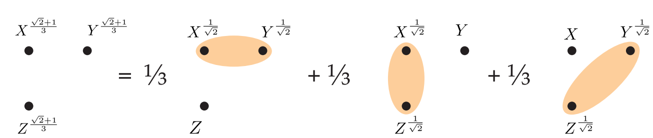

FIG. 2. The set of noisy Pauli measurements X^η, Y^η, Z^η defined by (2) for $\eta = [(\sqrt{2}+1)/3]$ can be written as a uniform convex combination of Pauli measurements that have a compatible pair (represented in a shaded area). Thus, one can implement these measurements by randomly implementing sets of measurements that are not triplewise incompatible.

$$\begin{aligned} p(ab|xy) &= \mathrm{tr}(M_{a|x} \otimes M_{b|y}\rho) \\ &= \sum_\lambda p_A(a|x,\lambda)\mathrm{tr}(E_\lambda \otimes M_{b|y}\rho) \\ &= \sum_\lambda p_A(a|x,\lambda) p_B(b,\lambda|y) \\ &= \sum_\lambda p(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda). \end{aligned} \quad (4)$$

It then follows that the observation of a nonlocal behavior (or equivalently the violation of a Bell inequality) certifies in a device-independent way that both parties used incompatible measurements.

Similarly, in the case that Alice performs a set of measurements that satisfy a more general compatibility structure $\mathcal{C}$, the observed behavior is local when restricted to the measurements in the hyperedges C_i of $\mathcal{C}$. For instance, let us consider the case of three measurements on Alice's side for the sake of simplicity. Let A represent a condition that the collected total behavior is guaranteed to satisfy; for instance, A can be the nonsignaling condition (denoted it by $p(ab|xy) \in NS$) or the requirement that the behavior has a quantum realization in terms of local measurements on a quantum state (denoted it by $p(ab|xy) \in Q$). For any behavior respecting the condition A, if Alice's measurements $x = 1$ and $x = 2$ are compatible, the probabilities of this behavior respect $p(ab|xy) = \sum_\lambda p(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda)$ for $x = 1$, 2 and any y. We denote by L_{12}^A this set of behaviors that respects the condition A and is Bell local for $x = 1$ and $x = 2$.

Notice that the observation that $\{p(ab|xy)\} \notin L_{12}^A$ allows us to conclude that the measurements 1 and 2 are incompatible. Analogously, we can define the sets L_{23}^A and L_{13}^A that correspond to the case where the other pairs of Alice's measurements are compatible. With these three sets representing pairwise compatibility, we can also define their convex hull $L_{2\mathrm{conv}}^A \coloneqq \mathrm{Conv}(L_{12}^A, L_{23}^A, L_{13}^A)$ and intersection $L_{2\cap}^A \coloneqq L_{12}^A \cap L_{23}^A \cap L_{13}^A$ (see Fig. 3). The observation that a behavior does not belong to these sets allows us to conclude the following: (i) If $\{p(ab|xy)\} \notin L$, then Alice's measurements are incompatible. (ii) If $\{p(ab|xy)\} \notin L_{ij}^A$, then the measurements $x = i$ and $x = j$ are incompatible. (iii) If $\{p(ab|xy)\} \notin L_{2\mathrm{conv}}^A$, then the measurements of Alice are genuinely triplewise incompatible. (iv) If $\{p(ab|xy)\} \notin L_{2\cap}^A$, then there is some pairwise incompatibility on Alice's measurements. Notice that we can also define similar sets with respect to Bob's measurements and consider sets generated by given compatibility structures in Alice's measurements and others in Bob's.

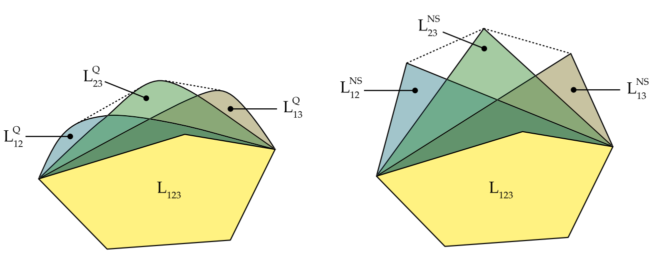

FIG. 3. Geometrical interpretation of sets of three pairwise and triplewise compatible measurements. Here L_{123} is the standard local set, where all local measurements are compatible. The set L_{ij}^{NS} consists of probabilities that are nonsignaling and are partially local with respect to i and j, i.e., are local when only the measurements i and j are considered on Alice's side. Analogously, L_{ij}^{Q} is a set of behaviors that are quantum and partially local with respect to measurements i and j.

In what follows, we show that using a set of measurements that satisfy a compatibility structure bounds the amount of violation of certain Bell inequalities. Thus, the observation of a value higher than this bound serves as a certificate that the measurements are incompatible with respect to this structure. To find these bounds, we need to solve the following optimization problem: given a Bell expression $S = \sum_{abxy} c_{abxy} p(ab|xy)$ and a compatibility structure $\mathcal{C}$,

$$\begin{aligned} &\text{maximize} \quad S \\ &\text{such that} \quad p(ab|xy) \in L_{\mathcal{C}} \\ &\qquad\qquad\quad p(ab|xy) \in Q, \end{aligned} \quad (5)$$

where $L_{\mathcal{C}}$ indicates the set of behaviors that are partially local according to the compatibility structure $\mathcal{C}$. Geometrically, this problem can be seen as a maximization of S with respect to a set of behaviors that are quantum and satisfy some partial locality [such as the sets L_{ij}^{Q} in Fig. 3(A)]. The last constraint in (5) imposes that the behavior is quantum (Q), i.e., that it has a quantum realization in terms of local measurements on a quantum state. In practice, since there is no tractable way of imposing that, we consider sets $Q_n \supseteq Q$ that outer approximate Q, Q_n being the n level of the Navascués-Pironio-Acín (NPA) hierarchy [47]. At each level n, the problem is a semidefinite program whose solution provides an upper bound to the desired bound and, hence, is still a valid bound for detecting incompatibility.

We emphasize that if Alice performs quantum measurements that are not genuinely triplewise incompatible, the resulting behavior is inside $L_{2\mathrm{conv}}^{Q}$; hence the set $L_{2\mathrm{conv}}^{Q}$ can be used for device-independent quantum genuine triplewise incompatibility certification. But since Bell locality does not necessarily imply measurement compatibility in general, the set $L_{2\mathrm{conv}}^{Q}$ may be larger than the set of quantum behaviors generated by imposing that Alice's measurements are not genuinely triplewise incompatible. We discuss this in Supplemental Material [26] where we show that the set of measurements generated by non-genuinely triplewise incompatible measurements is strictly smaller than $L_{2\mathrm{conv}}^{Q}$.

Nonsignaling device-independent witnesses of incompatibility structures.—It is also possible to test structures of measurement incompatibility not only in QT but in more general nonsignaling theories. For that, we just need to do a similar optimization, but now considering the set of nonsignaling behaviors rather than the set of quantum behaviors. This entails changing the last constraint in (5) to the set of linear constraints that defines the general nonsignaling set NS; i.e., the optimization problem is now

$$\begin{aligned} \text{maximize} \quad & S \\ \text{such that} \quad & p(ab|xy) \in L_{\mathcal{C}} \\ & p(ab|xy) \in \text{NS}, \end{aligned} \tag{6}$$

where the last constraint means that the behavior satisfies the nonsignaling conditions

$$\sum_a p(ab|x'y) = \sum_a p(ab|x''y) \quad \forall\ x', x'', \tag{7a}$$

$$\sum_a p(ab|xy') = \sum_a p(ab|xy'') \quad \forall\ y', y''. \tag{7b}$$

Geometrically, this means that the maximization is now running over a bigger set, since $\text{NS} \supseteq Q$ [see, e.g., Fig. 3(B)].

Notice that some of the sets in the problem (6), which we denote $L_{\mathcal{C}}^{\text{NS}}$, are easily characterized. In fact, in the case of dichotomic measurements, it can be straightforwardly shown that the set L_{ij}^{NS} is precisely characterized by the NS constraints plus all the CHSH inequalities involving A_i, A_j and any two measurements on Bob's side, independently of the number of observables Bob has. Similarly, the set $L_{2\cap}^{\text{NS}}$, obtained as the intersection of the sets for all ij, i.e., the union of the systems of inequalities, is described by the NS constraints and all CHSH-type inequalities between Alice's and Bob's observables.

Results.—We have run the above optimization problems for a variety of known bipartite Bell expressions S in scenarios where Alice has three choices of dichotomic measurements and Bob has three, four, or five choices of dichotomic measurements. After, we completely characterize the polytope $L_{2\text{conv}}^{NS}$ by explicitly obtaining all the inequalities representing its facets; with that one can easily decide when device-independent certification of genuine triplewise incompatibility is possible if both parties have three dichotomic observables. In order to help compare the values, we have set the local bounds of the Bell expressions to 0 and renormalized them such that their maximal nonsignaling bounds are 1. The results are given in Table I.

We first considered all tight Bell inequalities of these scenarios [8,48]. Using these inequalities we can test all possible incompatibility structures, including genuine triplewise incompatibility. We then looked at the chained Bell inequality with three inputs [49,50], which is not tight but can be generalized to multiple inputs. We also analyzed the elegant Bell inequality I_E [51] and the chained version of the CHSH inequality proposed in Ref. [52], which self-test orthogonal Pauli measurements on Alice's side. Although we find quantum violations for every incompatibility structure bound, we did not manage to find a quantum violation of the genuine triplewise incompatibility bounds for general nonsignaling theories.

In the case of three dichotomic measurements per party we were able to characterize the polytope $L_{2\text{conv}}^{NS}$ by explicitly obtaining all the inequalities representing its facets (see Supplemental Material [26]). Among all the inequalities found there is a single class of inequalities that can be violated by quantum systems, and this inequality is

TABLE I. Maximal value of some Bell expressions with respect to several constraints. The L and NS columns show the local (set to 0) and nonsignaling (set to 1) bounds, respectively. The column "Qubits," with values in italic, reports a lower bound for the maximal violation achieved with two-qubit states (see Supplemental Material for details [26]). The column Q_3 gives the maximal value given by the third level of the NPA hierarchy [47], and provides an upper bound on the maximal value that can be found within QT. From column $L_{\cap}^{Q_3}$ to column $L_{2\text{conv}}^{NS}$, we give the bounds found by solving (5) for different types of compatibility structures on Alice's side, where NS or Q_3 indicates whether the nonsignaling constraints or the third level of the NPA hierarchy was used, respectively. A violation of any of these bounds rules out the corresponding compatibility structure. We have depicted in bold the bounds that are smaller than the qubit bound, indicating that the compatibility structure can be ruled out in two-qubit experiments.

Ineq.	L	Qubits	Q_3	$L_{\cap}^{Q_3}$	$\min L_{ij}^{Q_3}$	$L_{2\text{conv}}^{Q_3}$	$L_{\cap}^{\text{NS}}$	$\min L_{ij}^{\text{NS}}$	$L_{2\text{conv}}^{\text{NS}}$	NS
I_{3322}	0	*0.2500*	0.2509	**0.2224**	**0.2359**	**0.2487**	0.3333	0.5000	0.6667	1
I_{3422}^1	0	*0.2761*	0.2761	**0.1998**	**0.1998**	0.2761	0.3333	0.5000	0.6667	1
I_{3422}^2	0	*0.2990*	0.2990	**0.2538**	**0.2769**	**0.2769**	0.3333	0.6667	0.6667	1
I_{3422}^3	0	*0.2910*	0.2910	**0.1893**	**0.2599**	**0.2616**	**0.2222**	0.6667	0.6667	1
I_{3522}	0	*0.3229*	0.3229	**0.2145**	**0.2675**	**0.2675**	**0.2222**	0.6667	0.6667	1
$I_{\text{chain}3}$	0	*0.5981*	0.5981	**0.0000**	**0.4142**	**0.4142**	**0.0000**	1	1	1
I_E	0	*0.1547*	0.1547	**0.0000**	**0.1381**	**0.1381**	**0.0000**	**0.0000**	0.3333	1
$I_{\text{chainCHSH}}$	0	*0.4142*	0.4142	**0.0000**	**0.2761**	**0.2761**	**0.0000**	0.6667	0.6667	1
M_{3322}	0	*0.0122*	0.0647	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	1

equivalent to the inequality M_{3322} of Ref. [53]. The M_{3322} inequality can be violated by two-qubit systems and this violation proves that there exist quantum correlations that cannot be simulated by any nonsignaling model respecting pairwise compatibility. Interestingly, an experimental violation of this inequality was reported in Ref. [54], but the observation of apparent signaling may require a reanalysis of its conclusions [55,56].

All Bell inequalities tested are explicitly written in Supplemental Material [26] and the code we used is available at [57].

Testing incompatibility structures in the EPR-steering scenario.—We finally consider the EPR-steering scenario, where no assumptions on Alice's measurements or the shared state are made but Bob can perform state tomography on his part of the system [58]. The experiment can be described by an assemblage $\sigma_{a|x} \coloneqq \mathrm{tr}_A(M_{a|x} \otimes \mathbb{1}\rho)$, which represents the unnormalized states held by Bob when Alice performs the measurements labeled by x and obtains the outcome a. We show that for any structure $\mathcal{C} = [C_1, C_2, \ldots, C_k]$, there exists a physical assemblage that allows us to rule out $\mathcal{C}$. This assemblage is given by local measurements $\{M_{a|x}\}$ applied on any pure entangled state with full Schmidt rank (e.g., the maximally entangled state). This extends the connection between measurement compatibility and EPR-steering established in Refs. [8,9,59]. See Supplemental Material for more details [26].

Conclusions and open questions.—In this Letter, we have shown that different structures of measurement compatibility give rise to constraints in the correlations that can be observed in Bell tests. These constraints can be interpreted as a partial locality, where the behaviors can be nonlocal but are seen to be local when restricted to some measurement choices. As a consequence, the violation of Bell inequalities by models satisfying incompatibility structures is reduced with respect to models in which measurements can be arbitrarily incompatible. This fact allows us to test different types of measurement incompatibility in a device-independent way.

Some open questions follow from our work. First, can any structure of genuine measurement incompatibility (for any number of measurements and outcomes) be realized by a quantum system? That would generalize the results of Ref. [60], where the authors have shown that any measurement structure can be realized in quantum mechanics. Also, can any structure of genuine measurement incompatibility be device-independently ruled out in QT (i.e., using quantum behaviors)? A second problem is that of mathematically characterizing the partially local sets for other scenarios and, in particular, finding tight inequalities that limit them.

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