Anomaly Matching and Symmetry-Protected Critical Phases in $SU(N)$ Spin Systems in $1+1$ Dimensions

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We study $(1+1)$-dimensional $SU(N)$ spin systems in the presence of global $SU(N)$ rotation and lattice translation symmetries. Knowing the mixed anomaly of the two symmetries at low energy, we identify, by the anomaly matching argument, a topological index for the spin model—the total number of Young-tableau boxes of spins per unit cell modulo N—characterizing the "ingappability" of the system. A nontrivial index implies either a ground-state degeneracy in a gapped phase, which can be thought of as a field-theory version of the Lieb-Schultz-Mattis theorem, or a restriction of the possible universality classes in a critical phase, regarded as the symmetry-protected critical phases. As an example of the latter case, we show that only a class of $SU(N)$ Wess-Zumino-Witten theories can be realized in the low-energy limit of the given lattice model in the presence of the symmetries. Similar constraints also apply when a higher global symmetry emerges in the model with a lower symmetry. Our results agree with several examples known in previous studies of $SU(N)$ models, and predict a general constraint on the structure factor which is measurable in experiments.

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Introduction.—Identification of the phase of a many-body quantum system is an important but, in general, hard problem in condensed matter physics. Quite often, symmetries play an essential role in determining the phase. As a notable example, the Lieb-Schultz-Mattis (LSM) theorem and its generalizations [1–5] state that a lattice model cannot be a trivial symmetric insulator if the filling per unit cell is fractional and if the translation symmetry and particle number conservation are strictly imposed. This gives a strong constraint on possible gapped phases realized in a given microscopic model. On the other hand, classification of critical phases is also an important problem. In addition to the quantum critical points between gapped phases, many stable critical phases have been found numerically and experimentally [6–8]. However, the reason of such stability is not completely understood. Moreover, much less is known about universal constraints on the critical phases, while a related proposal has recently been addressed in $SU(2)$ spin chains [9].

Systems with global symmetries higher than the conventional $U(1)$ or $SU(2)$, in particular the $SU(N)$ symmetry, are of intense interest. For a long time, $SU(N)$ symmetric systems have been studied as a theoretical toy model to understand the "physical" $SU(2)$ spin systems. Recently, moreover, "spin" systems with an $SU(N)$ symmetry with $N>2$ are realized in ultracold atoms on optical lattices [10–17]. A realization in spin-orbital systems is also suggested [18]. Thus, the study related to the phase diagrams of $SU(N)$ spin systems is of realistic interest on its own. Furthermore, even when only the $SU(2)$ spin-rotation and translation symmetries are imposed, they can be enhanced to higher symmetries. For example, the spin-1 bilinear-biquadratic chain has an explicit $SU(3)$ symmetry at a special point called as the Uimin-Lai-Sutherland model [19–21]. Higher symmetries can also emerge in the thermodynamic limit, even if the microscopic model does not have such symmetries exactly. For example, an emergent $SU(3)$ symmetry is found in a critical spin-2 chain [22]. It would be important to find universal constraints on such symmetry enlargements.

In this Letter, we focus on fundamental constraints on the phase diagrams of $(1+1)$d $SU(N)$ spin systems with spin-rotation symmetry and lattice translation symmetry. Our approach is based on the idea of 't Hooft anomaly matching [23], which has been used to investigate infrared behaviors of strongly coupled systems (for example, see recent works [24–29]). Specifically, we identify a topological quantity for a lattice spin model that matches the "anomaly" of the relevant symmetries in the low-energy phases and is given by the total number of Young-tableau (YT) boxes of spins per unit cell modulo N. It predicts whether the system admits a unique gapped ground state and further restricts ground-state degeneracies (GSDs) in gapped phases. Our result agrees with the $SU(N)$ generalization of the LSM theorem [2]. In addition, it imposes constraints on possible critical phases, going beyond the scope of the LSM-type theorems. That is, we postulate a classification of symmetry-protected critical (SPC) phases regarding the

Wess-Zumino-Witten (WZW) universality classes of the $SU(N)$ spin models based on the anomaly matching, generalizing the proposal for the $SU(2)$ case [9]. Moreover, we discuss the consequence of our result on magnetic neutron scattering as an example.

Furthermore, we also obtain a constraint on a higher $SU(N')$ symmetry with $N' > N$ in a model with the $SU(N)$ symmetry only by matching their symmetry anomaly. As a special example, such restriction can explain that $SU(3)$ symmetry enlargement has not been (actually, cannot be) found in $SU(2)$ and translation-symmetric half-integer spin chains. This demonstrates the power of our anomaly based approach, and paves a new way to discuss symmetry enlargements in general.

Translationally invariant $SU(N)$ spin system in $1+1$ dimensions and the LSM index.—We consider a generic $(1+1)$d $SU(N)$ spin system described by a Hamiltonian $\mathscr{H}_{SU(N)}$ with both $SU(N)$ spin-rotation and translation symmetries. Here, the system is subject to periodic boundary conditions, and the translation defines a unit cell consisting of, for generality, multiple spins, forming a group $\mathbf{T}_{\mathrm{latt}} = \mathbb{Z}$ (infinite cyclic group) in the thermodynamic limit. A typical example of such a system is the $SU(N)$ Heisenberg antiferromagnetic (HAF) model

$$\mathscr{H}_{\mathrm{HAF}} = J \sum_{\langle i,j\rangle,\alpha,\beta} S^{\alpha}_{i,\beta} S^{\beta}_{j,\alpha}, \qquad J > 0. \tag{1}$$

where α and β are the spin indices that take values among 1 to N and the $SU(N)$ generators satisfy the following $su(N)$ Lie algebra commutation relations:

$$[S^{\alpha}_{i,\beta}, S^{\gamma}_{j,\delta}] = \delta_{i,j}(\delta^{\alpha}_{\delta} S^{\gamma}_{i,\beta} - \delta^{\gamma}_{\beta} S^{\alpha}_{i,\delta}). \tag{2}$$

The first question to be asked is whether $\mathscr{H}_{SU(N)}$ has a trivial symmetric gapped ground state [a translationally invariant $SU(N)$ singlet]. Here, we give an approach to answering this question, based on the idea of symmetry anomalies.

Let us consider, at first, the simplest HAF model with a single spin in the fundamental representation (rep.) per unit cell. The low-energy effective field theory (EFT) has been shown to lie in the level-1 $SU(N)$ WZW universality class [30]

$$\begin{aligned} I(g) &= \frac{1}{8\pi}\int_{M_2} dtdx \mathrm{Tr}(\partial_\mu g^{-1}\partial^\mu g) + \Gamma_{\mathrm{WZ}}, \\ \Gamma_{\mathrm{WZ}} &= \frac{1}{12\pi}\int_{B:\partial B = M_2} dtdxdy \mathrm{Tr}(dgg^{-1})^3, \end{aligned} \tag{3}$$

where $\{g^{\alpha}_{\beta}\}$ is an $SU(N)$ matrix-valued field, and "Tr" is the conventional matrix trace. To obtain the symmetry transformation behavior of $g^{\alpha}_{\beta}(x)$ field, we use the N-flavor spinor representation of $S^{\alpha}_{j,\beta}$ following [30]:

$$S^{\alpha}_{j,\beta} \equiv \Psi^{\dagger\alpha}\Psi_{\beta}(x = ja) - \delta^{\alpha}_{\beta}/N; \qquad \sum_{\alpha} \Psi^{\dagger\alpha}\Psi_{\alpha}(x) = 1, \tag{4}$$

where a is the lattice constant, and the canonical anticommutation of the Ψ field and particle number constraint, indeed, makes $S^{\alpha}_{j,\beta}$, defined above, satisfy the $su(N)$ Lie algebra in Eq. (2). Such a filling constraint also gives the low-energy mode expansion as

$$\Psi_{\alpha}(x) \approx \exp(-ik_F x)\psi_{L\alpha}(x) + \exp(ik_F x)\psi_{R\alpha}(x), \tag{5}$$

with $k_F = \pi/(aN)$ due to Eq. (4), where the left-moving (L) and right-moving (R) modes $\psi_{L,R\alpha}(x)$ are coarse grained continuous function: $\psi_{L,R\alpha}(x+a) \approx \psi_{L,R\alpha}(x)$ in the continuum limit $a \to 0$. Together with the mode expansion Eq. (5), the lattice translation symmetry: $\Psi_{\alpha}(x) \to \Psi_{\alpha}(x+a)$, acts on the low-energy field operator as, for all α's,

$$\psi_{\alpha}(x) \to \exp(ik_F a\sigma_3)\psi_{\alpha}(x) = \exp(i\pi\sigma_3/N)\psi_{\alpha}(x), \tag{6}$$

where we have packed the $\psi_{L,R}$ into the two component Dirac fermion operator $\psi_{\alpha} \equiv [\psi_{R\alpha}, \psi_{L\alpha}]$ acted by the Pauli matrix σ_3. Following the non-Abelian bosonization technique [30], the bosonic matrix field is $g^{\alpha}_{\beta}(x) \propto \psi^{\dagger\alpha}_{L}\psi_{R\beta}$ up to a $U(1)$ phase. Then, by Eq. (6), the lattice translation transformation is

$$g \to e^{2\pi i/N} g. \tag{7}$$

Equation (6) or (7) represents a discrete axial symmetry, implying that the non-on-site $\mathbf{T}_{\mathrm{latt}}$ appears as an internal symmetry $\mathbb{Z}_N$ in the low-energy effective field theory. For $SU(N)$-rotation symmetry, it is exactly the flavor rotation symmetry as $\psi \to u\psi$ or $u^{\dagger}gu$ with $u \in U(N)$. However, the actual global symmetry of such a high-spin rotation should be $PSU(N) \equiv U(N)/U(1)$, as the charge $U(1)_Q$ transformation $\psi(x) \to \exp[i\phi(x)]\psi(x)$ leaves the physical operator $S^{\alpha}_{j,\beta}$ invariant; such a $U(1)_Q$ is a gauge redundancy to be modded out from $U(N)$ flavor transformation. Thus, the full symmetry within our interest is $PSU(N) \times \mathbb{Z}_N$ in the WZW theory.

However, there might be a mixed 't Hooft anomaly between the $PSU(N)$ and the $\mathbb{Z}_N$ symmetries of the WZW theory. More specifically, when the WZW theory is coupled to a nontrivial background gauge field $A_{PSU(N)}$ with a unit $PSU(N)$ instanton, its partition function suffers a potential phase ambiguity under the $\mathbb{Z}_N$ transformation (7): $Z_{\mathrm{WZW}_1}(A_{PSU(N)}) \to \exp(2\pi i/N) Z_{\mathrm{WZW}_1}(A_{PSU(N)})$, as we will show later on a more general ground (12). The manifestation of such an anomaly in the EFT can be traced back to the non-on-site nature of the lattice translation in the microscopic lattice model [31–36]. Based on the

TABLE I. Examples of gapped and critical $SU(N)$ spin systems. For the first two gapped exactly solvable models, the actual GSDs are consistent with our constraint. For the following critical models, the numerically proposed infrared conformal field theories (IR CFTs) in the fifth column obey SPC classification specified by Eq. (13). VBS: Valence-bond-solid; TB: Takhtajan-Babujian; AJ: Andrei-Johannesson.

Model	YT	$\mathcal{I}_N$	GSD	IR CFT; m	Mixed anomaly
$SU(3)$ trimer model [48]	□	1 mod 3	$3 \in 3\mathbb{N}$	⋯	⋯
$SU(3)$ **10**-VBS model [48]	□□□	0 mod 3	$1 \in 1\mathbb{N}$	⋯	⋯
$SU(6)$ **70**-VBS model [49]	□□ □	3 mod 6	$2 \in 2\mathbb{N}$	⋯	⋯
$S-3/2$ TB model [50,51]	□□□	1 mod 2	⋯	$SU(2)_3$ WZW; 1	1 mod 2
$\mathscr{H}^{[3,2]}$ AJ model [52,53]	□□	2 mod 3	⋯	$SU(3)_2$ WZW; 1	2 mod 3
$SU(3)$ 1×2-YT HAF model [54,55]	□□	2 mod 3	⋯	$SU(3)_1$ WZW; 2	2 mod 3
$SU(9)$ 2×1-YT HAF model [56]	□ □	2 mod 9	⋯	$SU(9)_1$ WZW; 2	2 mod 9
$SU(3)$ 2-leg ladder [57]	□ ⊗ □	2 mod 3	⋯	$SU(3)_1$ WZW; 2	2 mod 3

concept of 't Hooft anomaly matching that an anomaly is robust against local symmetry-preserving interactions, the EFT of any $(1+1)$d $SU(N)$ spin system with a fundamental spin per unit cell shares the same anomaly factor $\exp(i2\pi\mathcal{I}_{N;1}/N)$ as that of HAF model, namely $\mathcal{I}_{N;1}=1$ mod N. Because of such universality of $\mathcal{I}_{N;1}$, we can use it to characterize the underlying lattice models as a lattice quantity to be called the LSM index, namely,

$$\text{LSM index}\,\mathcal{I}_N \leftrightarrow \text{mixed}PSU(N)-\mathbf{T}_{\mathrm{EFT}}\,\text{anomaly}, \tag{8}$$

where $\mathbf{T}_{\mathrm{EFT}}$ denotes a general EFT representation of lattice translation $\mathbf{T}_{\mathrm{latt}}$. $\mathbf{T}_{\mathrm{EFT}}$ equals $\mathbb{Z}_N$ for the case above in Eq. (7). Actually, the full 't Hooft anomaly of EFT also includes the one for $\mathbf{T}_{\mathrm{EFT}}$ alone [37], an "emergent" anomaly which only characterizes ingappability against infinitesimal perturbations around criticality [33,34] and is irrelevant to the universality properties we consider here.

Then, let us proceed to a more general case that there are b YT boxes in total per unit cell and start with b copies of fundamental chains discussed above. An $SU(N)$ spin in fundamental rep. is elementary, that is, any $SU(N)$ irreducible rep. with b boxes is contained in the tensor product of b fundamental rep.'s. Thus, we can introduce strong $PSU(N)\times\mathbf{T}_{\mathrm{latt}}$-preserving local interactions (of strength U) as a projection operator within each unit cell to leave only the b-box rep. in consideration at the relevant energy scale such as J in a HAF model (with $J\ll U$) [30,38,39]. Because of the robustness of the anomaly phase ambiguity against any local interaction and its multiplicity by anomaly matching, the EFT of this b-box case has the same anomaly as the b copies of fundamental chains: $\mathcal{I}_{N;b}=b\mathcal{I}_{N;1}=b$ mod N, independent of the detail of the YT rep. per unit cell. As a result, the LSM index associated with a generic $SU(N)$ system is identified by the number of YT boxes of spins per unit cell modulo N, namely [40]

$$\mathcal{I}_N=(b=\#\text{of YT boxes per unit cell})\quad\text{mod }N. \tag{9}$$

In Table I, we list (in the third column) the LSM indices for several $SU(N)$ models with given YT reps. per unit cell. A system with a nonzero LSM index ($\mathcal{I}_N\neq 0$ mod N) must exhibit nontrivial low-energy behaviors in connection with symmetry-respecting ingappability [41], that is, such a system is "ingappable" as long as both $PSU(N)$ and $\mathbf{T}_{\mathrm{latt}}$ symmetries are respected: the system can be in either a symmetric gapless phase or a phase with spontaneous symmetry breaking, e.g., a $\mathbf{T}_{\mathrm{latt}}$-broken gapped phase or a $PSU(N)$-broken gapless phase (the ferromagnetic phase). We will elaborate these scenarios in the following.

Ground-state degeneracy associated with a spontaneous broken translation symmetry.—First, let us assume that the system has a nonzero gap above the ground state(s). By Eq. (8), a nonzero LSM index $\mathcal{I}_N\neq 0$ mod N is related to the symmetry anomaly which is an obstruction to gapping the system to have a unique ground state, so it implies a nontrivial GSD, as in the case of the existing LSM-type theorems. Here, we derive the degeneracy based on the mixed anomaly. By considering a family of the LSM indices $p\mathcal{I}_N$ associated with lower translation symmetries $p\mathbf{T}_{\mathrm{latt}}\subseteq\mathbf{T}_{\mathrm{latt}}$ for $p\in\mathbb{N}$ (as $\mathbf{T}_{\mathrm{latt}}=\mathbb{Z}$) of a $SU(N)$ spin model, we obtain a restriction on the GSD of any gapped phase of this model as [39]

$$\mathrm{GSD}\in\frac{N}{\gcd(\mathcal{I}_N,N)}\mathbb{N}, \tag{10}$$

and the translation symmetry is spontaneously broken to at least $N/\gcd(\mathcal{I}_N,N)$ of unit cells, realizable by exactly solvable models [49], where "gcd" denotes "greatest common divisor." Indeed, this corresponds to the LSM theorem for the $SU(N)$ spin chain [2], with an explicit statement on GSD. In the first two rows of Table I, we list the exactly solvable $SU(3)$ trimer and **10**-VBS models, analogs of $SU(2)$ dimer and Affleck-Kennedy-Lieb-Tasaki models, as well as the $SU(6)$ **70**-VBS models with a quite nontrivial YT rep. with three boxes. Their GSDs (shown in the fourth column) are consistent with the constraint (10).

Constraint on critical WZW $SU(N)_k$ universality classes.—Next, we consider the other possibility, namely when the system is gapless. While the usual LSM-type theorems do not give any further restriction, in this case, the anomaly based approach leads to constraints on the possible universality class of the gapless critical phase. The most natural universality classes of a critical $SU(N)$ spin model is the level-k $SU(N)$ WZW theory [30]—as we have already seen in the HAF model. The action is given by $S_{\mathrm{WZW}_k}(g) = kI(g)$, where $I(g)$ is defined in Eq. (3) and k is an integer, and we denote it as "$SU(N)_k$ WZW." The infrared translation transformation $\mathbf{T}_{\mathrm{EFT}}$ takes the following general form [30]:

$$g \to e^{2\pi i m/N} g, \tag{11}$$

which forms a $\mathbb{Z}_n$ group with $n = N/\gcd(m, N)$, and the integer m is defined modulo N.

One way to calculate the mixed anomaly in the WZW theory is to make use of the equivalence between $SU(N)_k$ WZW theory to $U(kN)_1/U(k)$ constrained Dirac fermion (CDF) theory [58,59] and then compute the $\mathbb{Z}_n$ axial anomaly in the CDF theory coupled to a background $PSU(N)$ gauge field. More explicitly, we show that [39] there is a phase ambiguity of the WZW or CDF partition function under the $\mathbb{Z}_n$ transformation (11), which takes the form $\exp(2\pi i k m w/N)$, where w is a mod-N integer depending on the background $PSU(N)$ gauge field. From this fact, we deduce that the mixed $PSU(N)$-$\mathbb{Z}_n$ anomaly phase for any $SU(N)_k$ WZW theory with m defined by Eq. (11) is characterized by a mod-N integer

$$km \mod N, \tag{12}$$

which, after taking $k = m = 1$ and $w = 1$ (corresponding to the unit instanton of the $PSU(N)$ gauge field), reduces to the anomaly factor for EFT of fundamental chains: $\mathcal{I} = 1$ mod N.

Then, by the definition of the LSM index (8) and, also, the way we represent it, we conclude that an $SU(N)$ model with an index $\mathcal{I}_N$ at the lattice scale, when flowing to some $SU(N)_k$ WZW CFT in the infrared, must obey

$$\mathcal{I}_N = km \mod N. \tag{13}$$

Constraints on the (1 + 1)-d $SU(N)$ spin models and symmetry-protected critical phases.—Summarizing the above discussion, we obtain the following statement based on the matching condition, which includes the $SU(N)$ version of the LSM theorem: If a spin model with an exact $SU(N)$ spin-rotation and lattice translation symmetries has a nontrivial LSM index $\mathcal{I}_N$ (9), that is, the total umber of Young-tableau boxes per unit cell is not divisible by N, the system must either (i) have degenerate ground states below the gap, with the multiplicity (10), or (ii) have gapless excitations. If the low-energy EFT is given by an $SU(N)$ WZW theory, its level is constrained by the matching condition (13).

The latter half of the statement implies an SPC classification of the gapless critical systems with the global $SU(N)$ rotation and the lattice translation symmetries. That is, between the two fixed points corresponding to $SU(N)_k$ and $SU(N)_{k'}$ WZW theories with the representation of the translation symmetry (11) with the factors m and m', a renormalization-group (RG) flow is only possible if $km = k'm'$ mod N.

Our constraints are consistent with massless RG flows proposed in the literature [55,57,60,61]. In Table I, we list several critical phases of $SU(N)$ spin systems with their numerically proposed IR CFTs in the fifth column. The mixed anomalies of these CFTs in the sixth column calculated by Eq. (12) exactly match their LSM indices in the third column, consistent with our "anomaly matching" in Eq. (13) and each CFT belongs to a certain SPC class of the underlying spin system. More specifically, the quantity m of each IR CFT is calculated by the peak $\kappa = 2\pi m/N$ of the structure factor defined as the Fourier transform of spin-spin correlations or structure factors $C(r) \equiv \sum_{\alpha,\beta} \langle S^{\alpha}_{r,\beta} S^{\beta}_{0,\alpha} \rangle$ [56]. In addition, perturbative half-integer-spin TB models are permitted to flow to $SU(2)_1$ WZW theory with $m = 1$ since the CFT mixed anomaly matches their LSM indices, reproducing the $SU(2)$ SPC classification [9].

Experimental consequences.—Structure factors are among fundamental quantities of interest in spin systems, and are measured for example by neutron scattering at non-zero scattering momenta [62–65]. The static structure factor, which is the total magnetic scattering differential cross section, corresponds to the Fourier transform of the spin-spin correlation function $C(r)$. At an RG fixed point, the correlation function $C(r)$ is expected to behave as

$$C(r) \sim \frac{\mathrm{const}}{r^2} + \frac{\cos k_o r}{r^\eta}. \tag{14}$$

Thus, both k_o and η can be extracted from the static structure factor. On the other hand, Eq. (11) implies $k_o = 2\pi m/N$, and for the $SU(N)_k$ universality class, $\eta \equiv 2(N^2 - 1)/[N(N + k)]$ is known [66]. Combining Eqs. (9), (13), (14), we arrive at a universal constraint between the behavior of $C(r)$ (infrared property) and the number of YT boxes (ultraviolet property determined by the microscopic model). That is, for an $SU(N)$ spin chain with b boxes per unit cell, the asymptotic behavior of Eq. (14) must obey

$$\left(\frac{2(N^2-1)}{N\eta} - N\right)\frac{Nk_o}{2\pi} = b \mod N. \tag{15}$$

Critical phases with higher symmetries.—It is possible that an $SU(N)$ model can have a critical phase described by an $SU(N')_{k'}$ WZW CFT with $N' > N$. In this case, the critical theory is constrained, regarding the level k' and the $\mathbb{Z}$ symmetry represented by $g \to e^{2\pi i m'/N'} g$, by the $SU(N)$ LSM index $\mathcal{I}_N$ of the spin model through the following condition [39]

$$\mathcal{I}_N \frac{N'}{\gcd(N', k'm')} = 0 \mod N. \tag{16}$$

As a special example, an $SU(2)$ spin chain with a half-integer spin per unit cell does not admit any critical phase described by the $SU(N')$ WZW CFT for any odd integer N', which explains $SU(3)$ symmetries are only found in integer-spin models [19–22]. Furthermore, if the $SU(N)$ spin model has an explicit symmetry enhancement [from $SU(N)$ to $SU(N')$] leading to the occurrence of a higher-symmetry critical phase, one can use a finer LSM index $\mathcal{I}_{N'}$ associated to the enlarged $SU(N')$ symmetry than the original index $\mathcal{I}_N$ to make further constraints on the possible critical theories. For example, the $SU(2)$ spin-1 Uimin-Lai-Sutherland model [19–21], which exhibits an explicit $SU(3)$ symmetry and can be expressed as an $SU(3)$ HAF model with fundamental rep. per site (unit cell), has a nontrivial $SU(3)$ LSM index $\mathcal{I}_3 = 1 \bmod 3$ [while its $SU(2)$ LSM index $\mathcal{I}_2$ is trivial], consistent with the existence of the $SU(3)_1$ WZW critical theory (with $m' = 1$) of this model.

Conclusions.—We proposed a topological "LSM index" to diagnose the ingappability of a generic $SU(N)$ spin system in $1+1$d with the spin-rotation and translation symmetries, based on the mixed 't Hooft anomaly. It leads to a constraint on the GSD if the system is gapped (LSM theorem), or on the possible critical theory if the system is gapless. Another implication is that $SU(N)$ WZW universality classes, which are SPC phases of the spin systems, fall into a $\mathbb{Z}_N$ classification. Furthermore, the formalism can be applied to cases where a higher $SU(N')$ symmetry emerges in $SU(N)$-symmetric systems, to derive constraints on the possible phases with the emergent symmetry. We have verified that our results are consistent with several previous results listed in Table I. Our approach is systematic and is not restricted to $SU(N)$. We believe that it will be useful to further explore SPC phases, in particular, those with emergent symmetries.

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