Finite-Volume Effects in $(g-2)_\mu^{\mathrm{HVP,LO}}$

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An analytic expression is derived for the leading-order finite-volume effects arising in lattice QCD calculations of the hadronic-vacuum-polarization contribution to the muon's magnetic moment $a_\mu^{\mathrm{HVP,LO}} \equiv (g-2)_\mu^{\mathrm{HVP,LO}}/2$. For calculations in a finite spatial volume with periodicity L, $a_\mu^{\mathrm{HVP,LO}}(L)$ admits a transseries expansion with exponentially suppressed L scaling. Using a Hamiltonian approach, we show that the leading finite-volume correction scales as $\exp[-M_\pi L]$ with a prefactor given by the (infinite-volume) Compton amplitude of the pion, integrated with the muon-mass-dependent kernel. To give a complete quantitative expression, we decompose the Compton amplitude into the spacelike pion form factor $F_\pi(Q^2)$ and a multiparticle piece. We determine the latter through next-to leading order in chiral perturbation theory and find that it contributes negligibly and through a universal term that depends only on the pion decay constant, with all additional low-energy constants dropping out of the integral.

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Introduction.—The discrepancy between experimental [1,2] and theoretical [3–5] values and the ongoing measurements at Fermilab [6–9] and J-PARC [10,11] have motivated various lattice QCD (LQCD) collaborations to calculate the hadronic contributions to $(g-2)_\mu$, which currently dominate the theoretical uncertainty [12–28]. The relevant contributions divide into hadronic light-by-light, leading-order hadronic vacuum polarization (LO HVP) and electromagnetic as well as strong-isospin corrections to the HVP. As the dominant hadronic contribution, the LO HVP must be determined with subpercent uncertainties to reach a total theory uncertainty competitive with the expected experimental precision [29]. Depending on the central values of the theoretical and experimental updates, the improved precision on both sides will provide powerful constraints on, or else strong evidence for, new physics beyond the standard model.

As the only known, systematically improvable approach to nonperturbative QCD, numerical lattice QCD is a natural tool in the determination of the LO HVP where a systematic and precise value is of great importance. The most common approach is to estimate $a_\mu^{\mathrm{HVP,LO}} \equiv (g-2)_\mu^{\mathrm{HVP,LO}}/2$ via the integral [30]

$$a_\mu^{\mathrm{HVP,LO}}(T,L) = \frac{2\alpha^2}{m_\mu^2}\int_0^{T/2} dx_0 \hat{\mathcal{K}}(m_\mu x_0) G_{T,L}(x_0), \quad (1)$$

where $\alpha \approx 1/137$ is the fine-structure constant, m_μ the muon mass, and

$$G_{T,L}(x_0) \equiv -\frac{1}{3}\sum_{k=1}^{3}\int_{L^3} d^3\boldsymbol{x} \langle j_k(x_0,\boldsymbol{x}) j_k(0)\rangle_{T,L}, \quad (2)$$

$$\hat{\mathcal{K}}(t) \equiv t^2 - 2\pi t + (8\gamma_E - 2) + \frac{4}{t^2} + 8\log(t) - \frac{8K_1(2t)}{t} - 8\int_0^\infty dv \frac{e^{-t\sqrt{v^2+4}}}{(v^2+4)^{3/2}}. \quad (3)$$

Here $j_\mu(x) = \sum_f q_f \bar{\psi}_f(x)\gamma_\mu \psi_f(x)$ is the Euclidean-signature vector current and $K_1(z)$ a Bessel function. We have used notation to emphasize that the calculation is performed in a finite-volume $T \times L^3$ Euclidean spacetime with periodic geometry.

In Eq. (1) the finite temporal extent is accommodated by cutting the integral at $T/2$. We leave a detailed analysis of finite-T effects, arising both from the boundary conditions and the treatment of large x_0 in the integral, to a future work. In this work we consider only the finite-L effects, defining $a_\mu^{\mathrm{HVP,LO}}(L) \equiv \lim_{T\to\infty} a_\mu^{\mathrm{HVP,LO}}(T,L)$. We will show that this quantity has only exponentially suppressed finite-volume effects, and the suppression is controlled by the pion mass M_π.

Even when T is taken very large, the large-x_0 region of the integral in Eq. (1) cannot be calculated from the measured two-point function because of the well-known exponential degradation of the signal-to-noise ratio. In practice, one can calculate the two-point function $G_{T\to\infty,L}(x_0)$ for $x_0 < \tau_c$ from numerical simulations (possibly with a mild extrapolation to saturate the $T \to \infty$ limit), and then use additional inputs to reconstruct the $x_0 > \tau_c$ region. This yields a decomposition

$$a_\mu^{\mathrm{recon}}(L) = a_\mu(L|x_0 < \tau_c) + a_\mu^{\mathrm{recon}}(L|x_0 > \tau_c), \quad (4)$$

where the superscript recon stands for reconstructed. The first term is calculated by restricting the integration domain in Eq. (1) to $0 < x_0 < \tau_c$ and by using the measured two-point function. The second term is obtained from an analogous formula where the integral is taken over $\tau_c < x_0 < \infty$ and the reconstructed two-point function is used.

We will see that $a_\mu(L|x_0 < \tau_c)$ approaches the infinite-volume limit exponentially fast. On the other hand, $a_\mu^{\mathrm{recon}}(L|x_0 > \tau_c)$ may approach $L \to \infty$ more slowly, depending on the exact prescription used. As an extreme example, if one estimates $G_{\infty,L}(x_0)$ for $x_0 > \tau_c$ by summing over a fixed number of finite-volume states, the resulting contribution to the HVP will have power-law L dependence [31,32]. In practice, more sophisticated procedures are employed and the resulting scaling must be considered on a case-by-case basis. [As explained in Ref. [30], one can use the Lellouch-Lüscher formalism [31–33], or else some model [20], to extract the timelike pion form factor in infinite volume, and use this as an input in the spectral representation to calculate the contribution of states below the four-pion threshold to $a_\mu(L|x_0 > \tau_c)$, directly in infinite volume. In this case one trades the finite-volume effects for other systematics that depend on the particular chosen procedure.]

Our main result, the formula for the leading $\exp[-M_\pi L]$ finite-volume effect to $a_\mu^{\mathrm{HVP,LO}}(L)$, is presented in the next section, and is derived in the "Derivation" section by means of a Hamiltonian formalism in which quantization along a spatial direction is used to pick out the complete functional form nonperturbatively. In the "Implications" section we discuss the implications of our expression for ongoing calculations. We find that the dominant contribution enters through the spacelike pion form factor, and, since the latter is readily calculated on the lattice, this provides a viable method for correcting the leading L dependence. We estimate also the dominant contribution to the finite-volume effects of $a_\mu(L|x_0 < \tau_c)$, which can be useful information when devising a strategy along the lines of Eq. (4).

Our results differ from Refs. [34,35] in that these work to a fixed order in chiral perturbation theory (ChPT) whereas our result is the full nonperturbative expression, to leading order in the large L expansion. In this regard it is worth emphasizing that the strict chiral expansion is limited by the fact that, at next-to-next-to-next-to leading order, the momentum-space vector correlator receives a Q^6 contribution that leads to a divergence in the integral defining $a_\mu^{\mathrm{HVP,LO}}$.

Result.—We define

$$\Delta a_\mu(L) \equiv a_\mu^{\mathrm{HVP,LO}}(L) - \lim_{L\to\infty} a_\mu^{\mathrm{HVP,LO}}(L), \quad (5)$$

where, as in the Introduction, we ignore the effects of the finite temporal extent. These scale as $e^{-M_\pi T}$ and $e^{-M_\pi \sqrt{T^2+L^2}}$. Therefore, in the commonly used setup $T = 2L$, the finite-T corrections are subleading and should be dropped. The separation is plausible from the perspective of a generic effective field theory. Volume effects can be encoded via position-space propagators, summed over all periodic images. The propagator's form then leads to exponential decay falling according to the image distance multiplied with the pion mass. The detailed proof of this separation, based on the methods of Ref. [36], is given in a second longer publication.

In the next section we show that the leading finite-L corrections are given by

$$\Delta a_\mu(L) = -\frac{2\alpha^2}{m_\mu^2}\int\frac{dp_3}{2\pi}\frac{e^{-L\sqrt{M_\pi^2+p_3^2}}}{4\pi L}\int_0^\infty dx_0 \hat{\mathcal{K}}(m_\mu x_0)\int\frac{dk_3}{2\pi}\cos(x_0 k_3)\sum_{q=0,\pm1}\mathrm{Re}T_q(-k_3^2, -k_3 p_3) + O(e^{-\sqrt{2}M_\pi L}), \quad (6)$$

where T_q is the Compton amplitude,

$$T_q(k^2, k\cdot p) \equiv i\lim_{\boldsymbol{p}'\to\boldsymbol{p}}\int d^4x e^{ikx}\langle \boldsymbol{p}', q|T\mathcal{J}_\rho(x)\mathcal{J}^\rho(0)|\boldsymbol{p}, q\rangle_\infty, \quad (7)$$

in the forward limit. Here, $|\boldsymbol{p}, q\rangle$ is the relativistically normalized state of a single pion with momentum $\boldsymbol{p}$ and charge q, and $k^2 = k_0^2 - \boldsymbol{k}^2$ and $k\cdot p = k_0 p_0 - \boldsymbol{kp}$ are the Minkowski squared norm and scalar product. Following the discussion after Eq. (5), the subleading exponential, $e^{-\sqrt{2}M_\pi L}$, arises from an image displaced in two of the spatial directions.

$\mathcal{J}_\mu(x)$ is the Minkowski current. In the Schrödinger picture this is related to its Euclidean counterpart via

$$\mathcal{J}_0(\boldsymbol{x}) = j_0(\boldsymbol{x}), \qquad \mathcal{J}_k(\boldsymbol{x}) = -ij_k(\boldsymbol{x}), \quad (8)$$

and the corresponding Heisenberg operators are

$$j_\mu(x_0, \boldsymbol{x}) = e^{x_0 H} j_\mu(\boldsymbol{x}) e^{-x_0 H}, \tag{9}$$

$$\mathcal{J}_\mu(t, \boldsymbol{x}) = e^{itH} \mathcal{J}_\mu(\boldsymbol{x}) e^{-itH}. \tag{10}$$

Derivation.—Define $G_{L_\rho}(x_0)$ exactly as $G_{T,L}(x_0)$ in Eq. (2) but in a volume in which all four directions may differ, i.e., with $L_0 \times L_1 \times L_2 \times L_3$. Then introduce $\Delta G_3(x_0|L) \equiv [1 - \lim_{L_{0,1,2}\to\infty}] G_{L_\rho}(x_0)$ as the finite-volume residue due to compactification in the 3 direction only.

To determine $\Delta G_3(x_0|L)$, we study $G_{L_\rho}(x_0)$ with geometry $L_\rho = (L_\perp, L_\perp, L_\perp, L)$ and quantize along the 3 direction. Defining $\underline{x} = (x_1, x_2, x_0) = (x_\perp, x_0)$, the Hamiltonian representation of the Euclidean two-point function yields

$$G_{L_\rho}(x_0) = -\frac{1}{3}\int_0^{L_\perp} d^2x_\perp \int_0^L dx_3 \times \frac{\mathrm{tr}[e^{-(L-x_3)H} j_\mu(\underline{x}) e^{-x_3 H} j_\mu(\underline{0})]}{\mathrm{tr} e^{-LH}}, \tag{11}$$

where the Hamiltonian has a discrete finite-volume spectrum of states in $L_\perp^3$ and the trace is taken over this Hilbert space. Here we are using $L_\perp$ to ensure that intermediate expressions are well defined. This will be sent to infinity at the end of the calculation. For simplicity, in this formula we have assumed periodic boundary conditions for gluons and antiperiodic boundary conditions for fermions in the 3 direction. To account for the commonly used periodic boundary conditions for fermions one should introduce $(-1)^F$ in all traces, where F is the fermion number. This does not change the leading exponential contribution, since this is due to single-pion, hence bosonic, states.

Let $|n\rangle$ be a basis of simultaneous eigenstates of the Hamiltonian (eigenvalue E_n), the momentum (eigenvalue $\boldsymbol{p}_n$), the charge (eigenvalue q_n) operators. Inserting a complete set of such states in both the numerator and the denominator then gives

$$G_{L_\rho}(x_0) = -\frac{1}{3}\sum_{n,n'} \frac{e^{-LE_n}}{\sum_{n''} e^{-LE_{n''}}} \int_0^{L_\perp} d^2x_\perp \int_0^L dx_3 \times e^{-x_3(E_{n'}-E_n)} \langle n|j_\mu(\underline{x})|n'\rangle\langle n'|j_\mu(\underline{0})|n\rangle. \tag{12}$$

The role of the coordinates x_0 and x_3 in this analysis is potentially confusing. In our final results x_0 plays the role of the time coordinate. This is the coordinate of integration in Eq. (1), typically parametrizing the longest Euclidean direction. Here, to identify the leading L dependence, it is convenient to quantize along the 3 direction. One must only take care that, in any given expression, all energies and all states are consistently defined with respect to the same quantization direction.

Returning to Eq. (12), the integral over x_3 can be calculated explicitly. To avoid the need of separating $E_{n'} = E_n$ terms from the rest, we introduce the following identity, which holds for all values of E_n, $E_{n'}$:

$$e^{-LE_n}\int_0^L dx_3 e^{-x_3(E_{n'}-E_n)} = \lim_{\epsilon\to 0^+} \mathrm{Re} \frac{e^{-L(E_n+i\epsilon)} - e^{-L(E_{n'}-i\epsilon)}}{E_{n'} - E_n - 2i\epsilon}. \tag{13}$$

Substituting into Eq. (12) and exchanging $n' \leftrightarrow n$ in certain terms, we obtain

$$G_{L_\rho}(x_0) = -\frac{1}{3}\lim_{\epsilon\to 0^+}\sum_n \frac{e^{-LE_n}}{\sum_{n''} e^{-LE_{n''}}}\int_0^{L_\perp} d^2x_\perp \left\{\mathrm{Re}\langle n|j_\mu(\underline{x})\frac{e^{-iL\epsilon}}{H - E_n - 2i\epsilon} j_\mu(\underline{0})|n\rangle + (\epsilon \to -\epsilon)\right\}. \tag{14}$$

This expectation value can be expressed in terms of the (finite-volume) Minkowskian two-point function via

$$\mathrm{Re}\langle n|j_\mu(\underline{x})\frac{e^{-iL\epsilon}}{H - E_n - 2i\epsilon} j_\mu(\underline{0})|n\rangle + (\epsilon \to -\epsilon) = \mathrm{Re} i e^{-iL\epsilon}\int_{-\infty}^{\infty} dt e^{-2\epsilon|t|}\langle n|T\mathcal{J}_\mu(t,\underline{x})\mathcal{J}^\mu(0)|n\rangle, \tag{15}$$

which is valid for $\epsilon > 0$ and can be easily proven using Eq. (10) and integrating over t explicitly. We stress that this is a mathematical identity and the parameter t has no relation to any of the spacetime coordinates in the system.

The expansion about $L \to \infty$ is now straightforward as the exponentials are manifest and one can identify the relevant contribution. Neglecting terms of order $e^{-2M_\pi L}$, we reach

$$\Delta G_3(x_0|L) = -\frac{1}{3}\lim_{\epsilon\to 0^+}\sum_{M_\pi \le E_n < 2M_\pi} e^{-LE_n}\int_0^{L_\perp} d^2x_\perp \times \mathrm{Re} i e^{-iL\epsilon}\int_{-\infty}^{\infty} dt e^{-2\epsilon|t|}\langle n|T\mathcal{J}_\mu(t,\underline{x})\mathcal{J}^\mu(0)|n\rangle_c, \tag{16}$$

where the connected expectation value is defined as $\langle n|\mathcal{O}|n\rangle_c \equiv \langle n|\mathcal{O}|n\rangle - \langle 0|\mathcal{O}|0\rangle$ with $\mathcal{O}$ representing the time-ordered product. At this point we can take the $L_\perp \to \infty$ limit. This is done by replacing the sum over the states in the one-particle region with the phase-space integral

$$\sum_{M_\pi \le E_n < 2M_\pi} e^{-LE_n}|n\rangle\langle n| \underset{L_\perp\to\infty}{\longrightarrow} \sum_{q=0,\pm 1}\int_{E(\boldsymbol{p})<2M_\pi} \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{e^{-LE(\boldsymbol{p})}}{2E(\boldsymbol{p})}|\boldsymbol{p}, q\rangle\langle \boldsymbol{p}, q|, \tag{17}$$

and by replacing the connected expectation value with the forward limit

$$i\langle n|T\mathcal{J}_\mu(t,\underline{x})\mathcal{J}^\mu(0)|n\rangle_c \to \int \frac{d^4k}{(2\pi)^4} e^{-i(k_0t-k_\perp x_\perp - k_3x_0)} T_q(k^2, k\cdot p), \qquad (18)$$

where the definition Eq. (7) has been used. In the $L_\perp \to \infty$ limit, the integrals over t and $x_\perp$ are readily calculated, yielding delta functions in $k_0 = k_\perp = 0$, i.e.,

$$\Delta G_3(x_0|L) = -\frac{1}{3}\sum_{q=0,\pm1}\int_{E(\boldsymbol{p})<2M_\pi} \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{e^{-LE(\boldsymbol{p})}}{2E(\boldsymbol{p})} \times \mathrm{Re}\int \frac{dk_3}{2\pi} e^{ik_3x_0} T_q(-k_3^2, -k_3p_3). \qquad (19)$$

In the final expression note that any contribution to the integrand that is odd in k_3, $p_3 \to -k_3, -p_3$ must integrate to zero, which justifies the replacement $e^{ik_3x_0} \to \cos(k_3x_0)$. To complete the derivation we note that the restriction $E(\boldsymbol{p}) < 2M_\pi$ can be dropped, as this amounts to an error of the same order as terms that we are neglecting. Finally, the integral over $p_\perp$ can be explicitly calculated. We reach

$$\Delta G_3(x_0|L) = -\frac{1}{3}\sum_{q=0,\pm1}\int \frac{dp_3}{2\pi}\frac{e^{-L\sqrt{M_\pi^2+p_3^2}}}{4\pi L} \times \int \frac{dk_3}{2\pi}\cos(k_3x_0)\mathrm{Re}T_q(-k_3^2, -k_3p_3). \qquad (20)$$

Multiplying the result by 3 to account for the three directions with compactification L, we conclude Eq. (6). (This last step assumes a decomposition similar to that allowing us to neglect finite T and is demonstrated in detail in a subsequent publication.)

We close by commenting on different choices of boundary conditions. If fermions satisfy $e^{i\theta}$-periodic boundary conditions [37,38], i.e., $\psi_f(x + L_\rho\hat{\rho}) = e^{i\theta_\rho^f}\psi_f(x)$, Eq. (11) should be modified by inserting $(-1)^F e^{i\sum_f \theta_3^f N_f}$ in all traces, where N_f is the number operator for the flavor f. In this case, Eq. (6) is modified by replacing

$$3\sum_{q=0,\pm1}\mathrm{Re}T_q \to \sum_{k=1}^{3}\{2\cos(\theta_k^u-\theta_k^d)\mathrm{Re}T_{\pm1}+\mathrm{Re}T_0\}, \qquad (21)$$

where we have used $T_{+1} = T_{-1} \equiv T_{\pm1}$ as follows from charge-conjugation invariance.

Implications.—Having derived the leading-order functional form of $\Delta a_\mu(L)$, we close by considering the implications for ongoing numerical LQCD calculations. Here we mainly focus on periodic boundary conditions but comment again on the role of twisting below. For convenience, we define the charge-summed Compton amplitude, $T \equiv \sum_q T_q$.

We begin by rewriting Eq. (6) as (we drop terms of order $e^{-\sqrt{2}M_\pi L}$ throughout this section)

$$\Delta a_\mu(L) = -\frac{2\alpha^2 m_\mu}{\pi L}\int_0^\infty dk_3 \hat{g}(k_3/m_\mu)\mathcal{T}''(k_3^2|L), \qquad (22)$$

where we have introduced

$$\mathcal{T}(k_3^2|L) \equiv \int_{-\infty}^{\infty}\frac{dp_3}{2\pi}e^{-L\sqrt{M_\pi^2+p_3^2}}\mathrm{Re}T(-k_3^2, -p_3k_3), \qquad (23)$$

$$\hat{g}(\omega) \equiv \omega\int_{\omega^2}^\infty dy\int_y^\infty \frac{dx}{x^{3/2}}\frac{16}{\sqrt{x+4}(\sqrt{x+4}+\sqrt{x})^4}, \qquad (24)$$

and $\mathcal{T}''$ is the second derivative of $\mathcal{T}$ with respect to k_3^2.

We next decompose the Compton amplitude into its pole and analytical contributions,

$$T(k^2, k\cdot p) \equiv T^{\mathrm{reg}}(k^2, k\cdot p) + \left[\frac{2(4M_\pi^2 - k^2)F_\pi^2(-k^2)}{-k^2 - 2p\cdot k - i\epsilon} + (p \to -p)\right], \qquad (25)$$

where F_π is the spacelike pion form factor and the separation defines T^{reg}. This implies

$$\mathcal{T}(k_3^2|L) = 2(4M_\pi^2 + k_3^2)F_\pi^2(k_3^2)\zeta(k_3^2|L) + \mathcal{T}^{\mathrm{reg}}(k_3^2|L), \qquad (26)$$

where we have introduced

$$\zeta(k_3^2|L) \equiv 2\mathrm{Re}\int_{-\infty}^{\infty}\frac{dp_3}{2\pi}\frac{e^{-L\sqrt{M_\pi^2+p_3^2}}}{k_3^2 + 2p_3k_3 - i\epsilon}, \qquad (27)$$

which can be readily reduced to forms well suited to numerical evaluation. The second term in Eq. (26) is given by Eq. (23) with $T \to T^{\mathrm{reg}}$. [$T^{\mathrm{reg}}(-k_3^2, -p_3k_3)$ is an analytic function in the complex strip defined by $\mathrm{Im}k_3 < M_\pi/2$. In addition, for real k_3, both $T^{\mathrm{reg}}(-k_3^2, -p_3k_3)$ and $F_\pi(k_3^2)$ are real functions.]

In Table I we numerically estimate the contribution of the $F_\pi(Q^2)$-dependent term to $\Delta a_\mu(L)$, using two functional forms for the specelike form factor. We have confirmed that the $F_\pi(Q^2) = 1$ values match the prediction from ChPT. The monopole form, taken from Ref. [39], is known to

TABLE I. Contribution to $\Delta a_\mu(L)$ from the $F_\pi(Q^2)$ term for functional forms as indicated. Here we take $m_\mu/M_\pi = 106/137$ and $M/M_\pi = 727/137$. As a reference value, we take $a_\mu^{\mathrm{HVP,LO}} = 700 \times 10^{-10}$.

	$[\Delta a_\mu(L)/a_\mu^{\mathrm{HVP,LO}}] \times -10^2$		
$M_\pi L$	$F_\pi(Q^2) = 1$	$F_\pi(Q^2) = [1/(1+Q^2/M^2)]$	Treg
4.0	0.639	1.26	0.019
5.0	0.579	0.852	0.005
6.0	0.348	0.461	0.001
7.0	0.180	0.226	$\lvert\cdots\rvert < 10^{-3}$
8.0	0.0863	0.104	$\lvert\cdots\rvert < 10^{-3}$

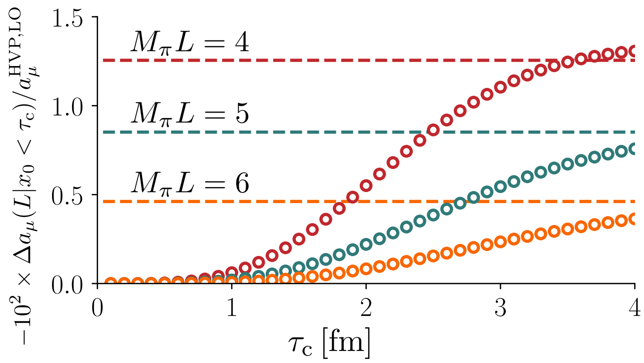

FIG. 1. Plot of the $F_\pi(Q^2)$ contribution to $\Delta a_\mu(L|x_0 < \tau_c)$ versus τ_c, taking the monopole ansatz as in Table I and $M_\pi = M_\pi^{\mathrm{phys}}$, for various values of $M_\pi L$. The horizontal lines give $\tau_c = \infty$, i.e., the full value for $\Delta a_\mu(L)$.

describe both experimental and lattice data very well up to $Q^2 = 2.45\ \mathrm{GeV}^2$.

We have also calculated the next-to leading order ChPT prediction for T^{reg} (summed over π_0 and $\pi^\pm$ external states),

$$T^{\mathrm{reg}}(-Q^2, k\cdot p) = c_0 + c_1 Q^2 + \frac{(7M_\pi^2 + 4Q^2)}{6\pi^2 f_\pi^2} z\coth^{-1} z\Big|_{z=\sqrt{1+4M_\pi^2/Q^2}}, \tag{28}$$

with the convention that $f_\pi \approx 132$ MeV. The coefficients c_0 and c_1 depend on various low-energy constants and on the pion mass. However, the contribution from these terms to $\Delta a_\mu(L)$ is identically zero, as can be seen explicitly from Eq. (22). Evaluating the remaining piece, we find that this contributes negligibly as shown in the third column of Table I. Note that, as can be seen from Eq. (21), using twisted boundary conditions with $\theta_k^u - \theta_k^d = \pi/2$ sets the $F_\pi(Q^2)$-dependent piece identically to zero, leaving only the contribution from the neutral pion in T^{reg}. This dramatically reduces the leading L dependence.

When separating the x_0 regions as in Eq. (4), it is useful to identify the cut value τ_c that minimizes the systematic errors given by the finite-volume effects of $a_\mu(L|x_0 < \tau_c)$, plus the uncertainties (finite volume or otherwise) entering through $a_\mu^{\mathrm{recon}}(L|x_0 > \tau_c)$. In Fig. 1 we plot the leading finite-L correction of $a_\mu(L|x_0 < \tau_c)$ versus τ_c for various $M_\pi L$.

The same data are presented in Table II, where we additionally vary the pion mass. At constant $M_\pi L$, increasing the pion mass leads to a decrease in $m_\mu L$ that translates into significantly enhanced volume effects. This behavior is predicted by an asymptotic expansion in $m_\mu L$, but the latter exhibits poor convergence so that the dependence is not obvious for these values. Nonetheless, the enhancement is clearly realized in these results, with a contribution of $\sim 2\%$ for $\Delta a_\mu(L)$ ($\tau_c \to \infty$) with $M_\pi L = 4$ and $M_\pi / M_\pi^{\mathrm{phys}} = 3$.

TABLE II. Tabulated values of the $F_\pi(Q^2)$ contribution to $\Delta a_\mu(L|x_0 < \tau_c)$ for various $M_\pi L$, M_π, and τ_c. We vary the monopole mass according to the result of Ref. [39]: $M^2 = 0.517(23)\ \mathrm{GeV}^2 + 0.647(30) M_\pi^2$ and hold the reference value fixed at $a_\mu^{\mathrm{HVP,LO}} = 700 \times 10^{-10}$.

$[\Delta a_\mu(L\|x_0 < \tau_c)/a_\mu^{\mathrm{HVP,LO}}] \times -10^2$					
$M_\pi = M_\pi^{\mathrm{phys}}$					
$M_\pi L$	$\tau_c = 1$ fm	1.5 fm	2 fm	2.5 fm	∞
4.0	0.0611	0.250	0.550	0.864	1.26
5.0	0.0198	0.0896	0.220	0.385	0.851
6.0	0.006 49	0.0313	0.0825	0.155	0.461
7.0	0.002 14	0.0108	0.0300	0.0593	0.226
8.0	0.000 72	0.003 74	0.0108	0.0221	0.104
$M_\pi = 2M_\pi^{\mathrm{phys}}$					
$M_\pi L$	$\tau_c = 1$ fm	1.5 fm	2 fm	2.5 fm	∞
4.0	0.231	0.682	1.08	1.28	1.38
5.0	0.0808	0.264	0.456	0.578	0.662
6.0	0.0281	0.0996	0.185	0.247	0.302
7.0	0.009 75	0.0369	0.0727	0.102	0.134
8.0	0.003 39	0.0135	0.0280	0.0411	0.0576
$M_\pi = 3M_\pi^{\mathrm{phys}}$					
$M_\pi L$	$\tau_c = 1$ fm	1.5 fm	2 fm	2.5 fm	∞
4.0	0.455	1.14	1.61	1.82	1.92
5.0	0.162	0.430	0.634	0.730	0.778
6.0	0.0574	0.162	0.249	0.293	0.316
7.0	0.0204	0.0609	0.0970	0.117	0.128
8.0	0.007 24	0.0227	0.0376	0.0462	0.0515

Conclusions.—We have presented a fully nonperturbative analysis of the leading finite-L effects in $a_\mu^{\mathrm{HVP,LO}}$. In particular, Eq. (6) relates the leading exponential, $\exp[-M_\pi L]$, to the Compton amplitude of an off-shell photon scattering against a pion in the forward limit. We also argue that the contribution coming from the one-pion exchange in the Compton amplitude (corresponding to the two-pion exchange in $a_\mu^{\mathrm{HVP,LO}}$) is the dominant contribution. We estimate the effect quantitatively using models for the electromagnetic spacelike pion form factor.

The results presented here provide an additional tool for systematically removing the finite-L effects in $a_\mu^{\mathrm{HVP,LO}}$. One option is to directly improve the result on each ensemble with a dedicated measurement of $F_\pi(Q^2)$. A limitation of this analysis is that the neglected $\exp[-\sqrt{2} M_\pi L]$ terms may not be small. As argued in Ref. [20], this is certainly true in the case of free pions with $M_\pi L \approx 4$ with leading-exponential domination setting in around $M_\pi L \approx 6$. In this vein we also stress that our full, nonperturbative result for the leading exponential can be used to assess and improve predictions, e.g., from ChPT, by correcting the leading exponential while keeping the fixed-order prediction for the higher exponentials in the series. This is well motivated since the structure of the pions becomes less important as the exponentials become more suppressed.

On a technical note, it will be interesting to pursue the Hamiltonian method (already used in Ref. [40]) for identifying finite-L effects in other contexts.

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