Bose-Einstein Condensation on the Surface of a Sphere

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Motivated by the recent achievement of space-based Bose-Einstein condensates (BEC) with ultracold alkali-metal atoms under microgravity and by the proposal of bubble traps which confine atoms on a thin shell, we investigate the BEC thermodynamics on the surface of a sphere. We determine analytically the critical temperature and the condensate fraction of a noninteracting Bose gas. Then we consider the inclusion of a zero-range interatomic potential, extending the noninteracting results at zero and finite temperature. Both in the noninteracting and interacting cases the crucial role of the finite radius of the sphere is emphasized, showing that in the limit of infinite radius one recovers the familiar two-dimensional results. We also investigate the Berezinski-Kosterlitz-Thouless transition driven by vortical configurations on the surface of the sphere, analyzing the interplay of condensation and superfluidity in this finite-size system.

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Introduction.—From the theoretical prediction in a series of articles of Bose [1] and Einstein [2,3], to its experimental achievement in 1995 [4,5], the paradigm of Bose-Einstein condensation (BEC) guided and marked the development of a large part of modern physics. Despite the fact that a huge variety of phenomena emerges from the combination of different trapped configurations and a tunable interaction strength [6], some basic analytical problems of fascinating beauty are currently unexplored. Among them is the condensation of a bosonic gas of ultracold atoms confined on the surface of a sphere. Our theoretical study of this system is triggered by the experimental possibility to confine the atoms on a spherically symmetric bubble trap [7], and by zero-temperature computational works [8–10] on the same topic. This configuration is produced by a radio frequency dressing of the atoms, which allows us to engineer a large variety of radial configurations [9,11–14]. However, a spherical atom distribution cannot be observed in conventional experiments since the atoms fall in the bottom of the trap due to gravitational effects [15]. Until now, many experiments with BEC have been carried on in microgravity settings [16–18], and bubble trap experiments in microgravity are planned for an orbiting cold atom laboratory inside the International Space Station [19–22]. It is thus pressing to obtain analytical results for these systems, whose better understanding would offer an efficient benchmark for precise atom interferometry, improved description of compact stars [23], and fundamental physics testing [24]. Here we calculate the critical temperature for a BEC of noninteracting bosons confined on the surface of a sphere and we derive an expression for their condensate fraction as a function of the temperature. Then we consider the addition of a zero-range two-body interaction. Within a functional integration approach we extend the noninteracting results in a Gaussian (one-loop) approximation. Despite a different topology with respect to a planar condensate, it is expected that a thin spherical shell undergoes the Berezinski-Kosterilitz-Thouless (BKT) transition [25,26]. We investigate the relationship between BEC and BKT, whose understanding is of general interest for any superfluid system on a curved surface described by an angle-valued field [27].

Noninteracting Bose gas.—The energy of a particle of mass m moving on the surface of a sphere of radius R is quantized according to the formula

$$\epsilon_l = \frac{\hbar^2}{2mR^2} l(l+1), \tag{1}$$

where $\hbar$ is the reduced Planck constant and $l = 0, 1, 2, \ldots$ is the integer quantum number of the angular momentum. This energy level has the degeneracy $2l + 1$ due to the magnetic quantum number $m_l = -l, -l + 1, \ldots, l - 1, l$ of the third component of the angular momentum. In quantum statistical mechanics the total number N of noninteracting bosons moving on the surface of a sphere and at equilibrium with a thermal bath of absolute temperature T is given by

$$N = \sum_{l=0}^{+\infty} \frac{2l+1}{e^{(\epsilon_l - \mu)/(k_B T)} - 1}, \tag{2}$$

where k_B is the Boltzmann constant and μ is the chemical potential. In the Bose-condensed phase, we can set $\mu = 0$ and

$$N = N_0 + \sum_{l=1}^{+\infty} \frac{2l+1}{e^{\epsilon_l/(k_B T)} - 1}, \tag{3}$$

where N_0 is the number of bosons in the lowest single-particle energy state, i.e., the number of bosons in the Bose-Einstein condensate. From this equation one gets a critical temperature T_{BEC} above which $N_0 = 0$. Strictly speaking, in our system with a finite radius R and finite particle number N, μ cannot be zero and there is never a full depletion of the condensate above the critical temperature. This residual population of the condensate is, however, rapidly vanishing for a finite but macroscopic system. Within the semiclassical approximation, where $\sum_{l=1}^{+\infty} \to \int_1^{+\infty} dl$, Eq. (3) becomes

$$n = n_0 + \frac{m k_B T}{2\pi\hbar^2} \left(\frac{\hbar^2}{m R^2 k_B T} - \ln\left(e^{\hbar^2/(m R^2 k_B T)} - 1\right) \right), \tag{4}$$

where $n = N/(4\pi R^2)$ is the 2D number density and $n_0 = N_0/(4\pi R^2)$ is the 2D condensate density. We emphasize that in the low-temperature limit of $T \to 0$ the second term of Eq. (4) vanishes and the system density n is coincident with the condensate density n_0. At the critical temperature T_{BEC}, the condensate density must be zero: from Eq. (4) one finds

$$k_B T_{\mathrm{BEC}} = \frac{\frac{2\pi\hbar^2}{m} n}{\frac{\hbar^2}{m R^2 k_B T_{\mathrm{BEC}}} - \ln\left(e^{\hbar^2/(m R^2 k_B T_{\mathrm{BEC}})} - 1\right)}, \tag{5}$$

which is an implicit analytical formula for the critical temperature T_{BEC} as a function of the 2D number density n and the radius R of the sphere. As expected [28], in the limit $R \to +\infty$ one gets $T_{\mathrm{BEC}} \to 0$. However, for any finite value of R the critical temperature T_{BEC} is larger than zero. This can also be seen in the top panel of Fig. 1, where we report the critical temperature T_{BEC} for noninteracting bosons as a function of the parameter nR^2. The semiclassical approximation (solid line) works very well because the strong convergence to zero of the Bose distribution for high values of l cuts off the pathological behavior of the density of states for $l \gg 1$. Combining Eqs. (4) and (5) one immediately obtains the condensate fraction of the system for $0 \le T \le T_{\mathrm{BEC}}$; namely,

$$\frac{n_0}{n} = 1 - \frac{1 - k_B T \frac{m R^2}{\hbar^2} \ln\left(e^{\hbar^2/(m R^2 k_B T)} - 1\right)}{1 - k_B T_{BEC} \frac{m R^2}{\hbar^2} \ln\left(e^{\hbar^2/(m R^2 k_B T_{BEC})} - 1\right)}. \tag{6}$$

The numerical solution of Eq. (6) is reported in the bottom panel of Fig. 1, in which we represent the condensate fraction n_0/n of noninteracting bosons in terms of the rescaled temperature T/T_{BEC} for different values of the nR^2 parameter, with $n = N/(4\pi R^2)$. Experimentally, one can tune nR^2 simply changing the total number N, but also changing the radius R at fixed density n.

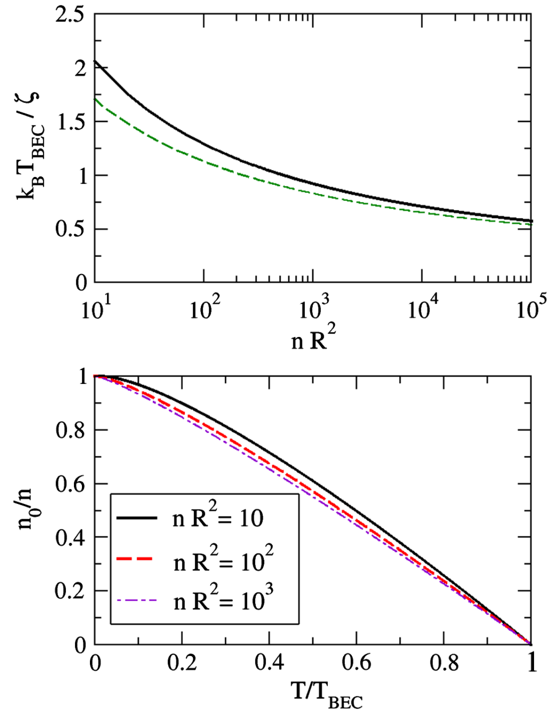

FIG. 1. Top: Critical temperature for Bose-Einstein condensation in terms of the product nR^2, where R is the radius of the sphere and $k_B T_{\mathrm{BEC}}$ is rescaled with the energy $\zeta = \hbar^2 n/m$. Notice how our result in the semiclassical approximation (solid line) converges for $nR^2 \gg 1$ to the numerical evaluation of the summation of Eq. (3) (dashed line). As expected, for fixed n the critical temperature tends to zero in the thermodynamic two-dimensional limit $nR^2 \to \infty$. Bottom: Condensate fraction n_0/n for noninteracting bosons on the sphere surface in terms of the rescaled temperature T/T_{BEC}, obtained from the numerical solution of Eq. (6) with nR^2 fixed.

Interacting Bose gas.—We now consider a system of interacting bosons on the surface of a sphere. The main thermodynamic function describing a system of particles in the grand canonical ensemble is the grand potential $\Omega = -\beta^{-1} \ln(\mathcal{Z})$, where $\beta^{-1} = k_B T$ and $\mathcal{Z}$ is the grand canonical partition function. Within the formalism of functional integration, we calculate $\mathcal{Z}$ as the functional integral,

$$\mathcal{Z} = \int \mathcal{D}[\bar{\psi}, \psi] e^{-S[\bar{\psi},\psi]/\hbar}, \tag{7}$$

where

$$S[\bar{\psi}, \psi] = \int_0^{\beta\hbar} d\tau \int_0^{2\pi} d\varphi \int_0^{\pi} \sin(\theta) d\theta R^2 \mathcal{L}(\bar{\psi}, \psi) \tag{8}$$

is the Euclidean action, and

$$\mathcal{L} = \bar{\psi}(\theta,\varphi,\tau)\left(\hbar\partial_\tau + \frac{L^2}{2mR^2} - \mu\right)\psi(\theta,\varphi,\tau) + \frac{g}{2}|\psi(\theta,\varphi,\tau)|^4 \tag{9}$$

is the Euclidean Lagrangian, i.e., the Lagrangian with imaginary time τ. Notice that the kinetic energy is written in terms of the angular momentum L, which is proportional to the angular components of the Laplacian operator in spherical coordinates; namely,

$$L^2 = -\hbar^2\left[\frac{1}{\sin(\theta)}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin^2(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right)\right]. \tag{10}$$

Our Lagrangian models interacting bosons on the surface of a sphere. Experimentally, this corresponds to the thin-shell limit of a bubble trap potential [12], in which the atoms are confined by the radial shifted harmonic potential $V_{\rm shell} = m\omega_{\rm sh}^2(r-R)^2/2$. Following Refs. [8,11], we suggest that one can tune the experimental parameters to confine the atoms on a shell with a radial harmonic length $l_{\rm sh} = [\hbar/(m\omega_{\rm sh})]^{1/2}$ of the order $l_{\rm sh} \approx 0.1$ μm, which is much smaller than the radius of the sphere $R \approx 10$ μm. In this case the radial excitations are inhibited and it is safe to impose our constraint in the Lagrangian of Eq. (9). Therein, the real two-body interatomic potential $V(r)$ has been substituted with the effective two-dimensional zero-range interaction coupling g. Therefore, for a given interatomic potential, one can calculate the exact value of g [29]. One can also use the scattering theory to link g to the two-dimensional s-wave scattering length a_s. In 2D the latter quantity is, however, energy dependent and a physical cutoff, usually identified with the chemical potential of the system, is needed [30]. Besides this, the relation between g and a_s should also include the corrections due to the curvature of the scattering surface [31]. To keep the compatibility with different experimental setups and interparticle interactions, in the following we will simply employ g, which could also be used as a phenomenological fitting parameter for thicker spherical shells.

Let us now explicitly perform the functional integration of the Lagrangian $\mathcal{L}$. The spontaneous breaking in the condensate phase of the U(1) symmetry of the complex order parameter ψ is introduced with the Bogoliubov shift,

$$\psi(\theta,\varphi,\tau) = \psi_0 + \eta(\theta,\varphi,\tau), \tag{11}$$

where the real field configuration ψ_0 describes the condensate component with angular momentum $l = 0$ and $m_l = 0$. By substituting this field parametrization and keeping only second order terms in the field η, we rewrite the Lagrangian as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_g, \tag{12}$$

with $\mathcal{L}_0 = -\mu\psi_0^2 + g\psi_0^4/2$, and

$$\mathcal{L}_g = \bar{\eta}(\theta,\varphi,\tau)\left(\hbar\partial_\tau + \frac{L^2}{2mR^2} - \mu\right) \times \eta\left(\theta,\varphi,\tau) + \frac{g}{2}[\bar{\eta}^2(\theta,\varphi,\tau) + \eta^2(\theta,\varphi,\tau)\right]. \tag{13}$$

The mean-field Lagrangian $\mathcal{L}_0$ gives the mean-field grand potential,

$$\Omega_0 = 4\pi R^2(-\mu\psi_0^2 + g\psi_0^4/2), \tag{14}$$

while the functional integral of the Gaussian Lagrangian $\mathcal{L}_g$ can be calculated explicitly with the following decomposition of the complex fluctuation field $\eta(\theta,\varphi,\tau)$,

$$\eta(\theta,\varphi,\tau) = \sum_{\omega_n}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l}\frac{e^{-i\omega_n\tau}}{R}\mathcal{Y}_{m_l}^l(\theta,\varphi)\eta(l,m_l,\omega_n), \tag{15}$$

and similarly for $\bar{\eta}(\theta,\varphi,\tau)$, where ω_n are the Matsubara frequencies, and we introduce the orthonormal basis of the spherical harmonics $\mathcal{Y}_{m_l}^l$ [32]. Substituting this decomposition into the Gaussian Lagrangian (13) and using the orthonormality properties of $\mathcal{Y}_{m_l}^l$ and of the complex exponentials, we rewrite the Gaussian action S_g as

$$S_g[\bar{\eta},\eta] = \frac{\hbar}{2}\sum_{\omega_n}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l}\begin{pmatrix}\bar{\eta}(l,m_l,\omega_n)\\ \eta(l,-m_l,-\omega_n)\end{pmatrix}^T \times \mathbf{M}\begin{pmatrix}\eta(l,m_l,\omega_n)\\ \bar{\eta}(l,-m_l,-\omega_n)\end{pmatrix}, \tag{16}$$

where the elements $\mathbf{M}_{ij}$ of the matrix $\mathbf{M}$ are defined as

$$\mathbf{M}_{ii} = (-1)^i i\hbar\omega_n + \epsilon_l - \mu + 2g\psi_0^2, \qquad i = 1,2,$$
$$\mathbf{M}_{12} = \mathbf{M}_{21} = (-1)^m g\psi_0^2. \tag{17}$$

The Gaussian action S_g can be integrated in the $\omega_n \geq 0$ field sector and the corresponding contribution to the Gaussian grand potential Ω_g reads

$$\Omega_g(\mu,\psi_0^2) = \frac{1}{2\beta}\sum_{\omega_n}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l}\ln\{\beta^2[\hbar^2\omega_n^2 + E_l^2(\mu,\psi_0^2)]\}, \tag{18}$$

where $E_l(\mu,\psi_0^2)$ is the excitation spectrum of the interacting system:

$$E_l(\mu,\psi_0^2) = \sqrt{(\epsilon_l - \mu + 2g\psi_0^2)^2 - g^2\psi_0^4}. \tag{19}$$

One can easily sum over the Matsubara bosonic frequencies ω_n [33], and remembering the mean-field grand potential Ω_0 of Eq. (14), we obtain the total grand potential Ω as

$$\Omega(\mu,\psi_0^2) = 4\pi R^2(-\mu\psi_0^2 + g\psi_0^4/2) + \frac{\alpha}{2}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l} E_l(\mu,\psi_0^2) + \frac{\alpha}{\beta}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l} \ln(1-e^{-\beta E_l(\mu,\psi_0^2)}) + o(\alpha^2), \tag{20}$$

where we include the parameter $\alpha = 1$, whose power counts the perturbative order of the grand potential expansion [34,35]. We fix the value of the order parameter ψ_0 with the variational saddle-point condition $\partial\Omega/\partial\psi_0 = 0$, which determines a relation between ψ_0, the chemical potential μ, and the contact interaction strength g. Since the condensate density n_0 is defined as $n_0 = \psi_0^2$, the saddle-point condition implies that

$$n_0(\mu) = \frac{\mu}{g} - \frac{\alpha}{4\pi R^2}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l} \frac{\frac{\hbar^2 l(l+1)}{2mR^2} + \mu}{E_l^B}\left(\frac{1}{2} + \frac{1}{e^{\beta E_l^B} - 1}\right) + o(\alpha^2), \tag{21}$$

which at the lowest perturbative order gives $n_0 = \mu/g + o(\alpha)$, and where the excitation spectrum $E_l^B = E_l[\mu, n_0(\mu)]$ takes the Bogoliubov-like form [36]

$$E_l^B = \sqrt{\epsilon_l(\epsilon_l + 2\mu)}. \tag{22}$$

With the mean-field condition, the grand potential of Eq. (20) can be rewritten as

$$\Omega[\mu, n_0(\mu)] = -4\pi R^2\frac{\mu^2}{2g} + \frac{\alpha}{2}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l} E_l^B + \frac{\alpha}{\beta}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l} \ln(1-e^{-\beta E_l^B}) + o(\alpha^2), \tag{23}$$

and we can introduce the system density n as

$$n(\mu) = -\frac{1}{4\pi R^2}\frac{\partial\Omega[\mu, n_0(\mu)]}{\partial\mu}. \tag{24}$$

Since we are interested in the relation between the density n and the condensate density n_0, we substitute in the last equation the value of $\mu = \mu(n_0)$ given by Eq. (21), obtaining

$$n(n_0) = n_0 + f_g^{(0)}(n_0) + f_g^{(T)}(n_0), \tag{25}$$

where

$$f_g^{(0)}(n_0) = \frac{\alpha}{4\pi R^2}\frac{1}{2}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l} \frac{\epsilon_l + g n_0}{E_l[\mu(n_0), n_0]} \tag{26}$$

is the zero-temperature Gaussian density, and

$$f_g^{(T)}(n_0) = \frac{\alpha}{4\pi R^2}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l} \frac{\epsilon_l + g n_0}{E_l[\mu(n_0), n_0]}\frac{1}{e^{\beta E_l[\mu(n_0), n_0]} - 1} \tag{27}$$

is the finite-temperature Gaussian density. We emphasize that, at a Gaussian level, this procedure is equivalent to the one adopted in our previous article [37]. The number density of Eq. (25) constitutes a reliable result in a low-temperature and weakly interacting regime, in which the quantum and thermal depletion are small, namely, $n \approx n_0$. Within the variational perturbation theory (VPT) [38,39], one can employ the weakly interacting perturbative expansion of Eq. (25) to derive a self-consistent approximation for the number density, valid also for larger values of the depletion. In our case, the VPT procedure outlined in Ref. [35] is equivalent at the order $o(\alpha^2)$ to the substitution of the condensate density n_0 in Eqs. (26), (27) with the total number density n, obtaining $f_g^{(0)}(n)$ and $f_g^{(T)}(n)$. We stress that this VPT method for a 3D homogeneous Bose gas leads to a critical temperature that scales with the square root of the gas parameter, while Monte Carlo simulations suggest a linear scaling [40]. Setting $\alpha = 1$, we now calculate explicitly $f_g^{(0)}(n)$, which is ultraviolet divergent and needs a regularization procedure. We rewrite the sum as an integral over l in which the degeneration over m_l gives a $2l+1$ factor. Using the variable $t = \hbar^2 l(l+1)/(4mngR^2)$, we integrate $f_g^{(0)}(n)$ subtracting the pathological asymptotic behavior of the integrand function at $+\infty$, thus obtaining

$$f_g^{(0)}(n) = \frac{mgn}{4\pi\hbar^2} + \frac{1}{4\pi R^2}\left[1 - \sqrt{1 + \frac{2gmnR^2}{\hbar^2}}\right], \tag{28}$$

which vanishes for noninteracting bosons for which the quantum depletion does not occur. We emphasize that $f_g^{(0)}(n)$ generalizes the quantum depletion result by Schick for a weakly interacting Bose gas in $2D$ [41], by including a correction due to the finite size of the sphere radius. In particular, Schick's result is reproduced in the $R \to \infty$ limit in which the interaction coupling g can be identified with $g = 2\pi\hbar^2/[m|\ln(na_s^2)|]$, where a_s is the two-dimensional s-wave scattering length [30]. Similarly, we calculate the thermal density $f_g^{(T)}(n)$, obtaining

$$f_g^{(T)}(n) = \frac{1}{2\pi R^2}\sqrt{1+\frac{2gmnR^2}{\hbar^2}} - \frac{mk_BT}{2\pi\hbar^2} \times \ln\left(e^{(\hbar^2/mR^2k_BT)\sqrt{1+(2gmnR^2/\hbar^2)}} - 1\right). \quad (29)$$

Putting together the density contributions Eqs. (28) and (29) with n_0, we get the VPT-improved self-consistent condensate fraction of an interacting Bose gas on the surface of a sphere as

$$\frac{n_0}{n} = 1 - \frac{mg}{4\pi\hbar^2} - \frac{1}{4\pi R^2 n}\left(1+\sqrt{1+\frac{2gmnR^2}{\hbar^2}}\right) + \frac{mk_BT}{2\pi\hbar^2 n}\ln\left(e^{(\hbar^2/mR^2k_BT)\sqrt{1+(2gmnR^2/\hbar^2)}} - 1\right). \quad (30)$$

With $n_0 = 0$ in Eq. (30), we calculate an implicit relation for the condensation critical temperature of interacting bosons [with $\beta_{\mathrm{BEC}} = (k_BT_{\mathrm{BEC}})^{-1}$]

$$k_BT_{\mathrm{BEC}} = \frac{\frac{2\pi\hbar^2 n}{m} - \frac{gn}{2}}{\frac{\hbar^2\beta_{\mathrm{BEC}}}{2mR^2}\left(1+\sqrt{1+\frac{2gmnR^2}{\hbar^2}}\right) - \ln\left(e^{(\hbar^2\beta_{\mathrm{BEC}}/mR^2)\sqrt{1+(2gmnR^2/\hbar^2)}} - 1\right)}. \quad (31)$$

Note that the critical temperature for the noninteracting system of Eq. (5) is reproduced if $g = 0$ is set. In Fig. 2 we report the critical temperature $k_BT_{\mathrm{BEC}}/\zeta$ (dashed line), rescaled with the energy $\zeta = \hbar^2 n/m$, in terms of $gm/\hbar^2$.

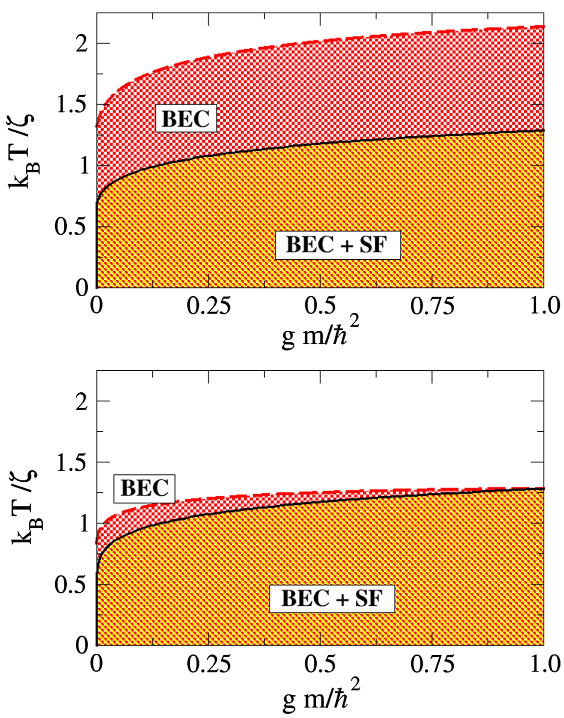

FIG. 2. Phase diagram of the system for two values of nR^2: 10^2 in the upper panel, 10^4 in the lower panel. The dashed lines represent the critical temperature $k_BT_{\mathrm{BEC}}/\zeta$, rescaled with the typical energy $\zeta = \hbar^2 n/m$, plotted in terms of the adimensionalized zero-range interaction strength $gm/\hbar^2$. The solid curve represents the critical temperature $k_BT_{\mathrm{BKT}}/\zeta$ of the Berezinski-Kosterlitz-Thouless transition. Note that, depending on the values of nR^2 and of $gm/\hbar^2$ and within the approximations adopted (see text), the system can show coexistence of condensation and superfluidity (BEC + SF), or condensation in the absence of superfluidity (BEC).

The shaded area under each of the dashed curves with nR^2 fixed is where BEC occurs: if the density n is kept fixed and the sphere radius R is increased, this area diminishes. Indeed, the expected result of $T_{\mathrm{BEC}} = 0$ is reobtained in the 2D flat-system limit $R \to \infty$.

A spherical surface is topologically inequivalent to the 2D flat plane. In particular, the presence of a point at infinity allows only for couples of topological defects to exist, being vortex-antivortex dipoles, or free vortices [42]. Despite this fact, the Kosterlitz-Nelson criterion [43] for the jump of the superfluid density $n_s(T)$ was recovered extending the Berezinski-Kosterlitz-Thouless theory on the sphere [26]; i.e., $k_BT_{\mathrm{BKT}}/[\hbar^2 n_s(T_{\mathrm{BKT}})/m] = \pi/2$. Here, in analogy to the Landau formula for the 2D plane, we calculate $n_s(T)$ as

$$n_s = n - \frac{1}{k_BT}\int_1^{+\infty}\frac{dl(2l+1)}{4\pi R^2}\frac{\hbar^2(l^2+l)}{2mR^2}\frac{e^{E_l^B/(k_BT)}}{(e^{E_l^B/(k_BT)}-1)^2}, \quad (32)$$

and applying the Kosterlitz-Nelson criterion, we evaluate numerically the critical temperature T_{BKT}, represented as the solid curve of Fig. 2. We find that T_{BKT} has a weak dependence on nR^2 and goes to zero in an exponentially small region where $gm/\hbar^2 \to 0$: this result is in agreement with the classical field simulations of Ref. [44] for bosons in a 2D uniform configuration. Indeed, the spherical surface is locally isomorphic to the Euclidean plane [26], where superfluidity is absent in the noninteracting limit. At the same time, we stress that for any tiny but physically meaningful interaction strength, the critical BKT temperature T_{BKT} is practically finite, while T_{BEC} coincides with the noninteracting T_{BEC}. In this regime of vanishing interaction, the unification of BEC and BKT transitions observed with bosons in a 2D harmonic trap [45] is obtained only when $nR^2 \gtrsim 10^4$.

In the phase diagram of Fig. 2, within the approximations involved in the calculation of $n_s(T)$ and at the zero order of VPT, we find a region where condensation and superfluidity coexist, and a region where condensation is not accompanied by superfluidity. Note that the latter condition is more pronounced for $nR^2 \lesssim 10^2$ and was experimentally observed in a quasi-2D finite-size Bose gas [46]: at low density, the curvature of the sphere may play the same role of their 2D weak external potential. Finally, in the regime of $gm/\hbar^2 \gg 1$, where our perturbative scheme is not expected to hold, we point out that $T_{\rm BKT}$ becomes essentially constant while $T_{\rm BEC} \to 0$ due to a large depletion of the condensate. The liability of this result should be established with more refined methods.

Conclusions.—The condensate fraction for noninteracting and interacting bosons on the surface of a sphere, i.e., Eqs. (5), (6), (30) and (31), can be experimentally observed in microgravity conditions with bubble traps in the thin-shell limit. These results are concrete predictions to test quantum statistical mechanics in regimes where finite-size and curvature effects play a relevant role. Further experimental and theoretical investigations should also focus on the expected interplay of superfluidity and Bose-Einstein condensation. We expect that a typical 2D configuration, with $N \approx 10^5$ ^{87}Rb atoms confined on a shell with radius $R = 10$ μm and thickness $l_{\rm sh} = 0.1$ μm, has a critical temperature of $T_{\rm BEC} = 670$ nK for $gm/\hbar^2 = 2^{3/2}\pi^{1/2}a_s/l_{\rm sh} = 0.26$, where $a_s = 5$ nm is the bare s-wave scattering length. With the Feshbach resonance one can tune a_s and investigate also regimes with higher values of $gm/\hbar^2$.

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