Quasinormal Modes and the Hawking-Unruh Effect in Quantum Hall Systems: Lessons from Black Hole Phenomena

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In this work, we propose the quantum Hall system as a platform for exploring black hole phenomena. By exhibiting deep rooted commonalities between the lowest Landau level and spacetime symmetries, we show that features of both quantum Hall and gravitational systems can be elegantly captured by a simple quantum mechanical model: the inverted harmonic oscillator. Through this correspondence, we argue that radiation phenomena in gravitational situations, such as presented by W. G. Unruh and S. Hawking, bear a parallel with saddle-potential scattering of quantum Hall quasiparticles. We also find that scattering by the quantum Hall saddle potential can mimic the signature quasinormal modes in black holes, such as theoretically demonstrated through Gaussian scattering off a Schwarzschild black hole by C. V. Vishveshwara. We propose a realistic quantum Hall point contact setup for probing these temporally decaying modes in quasiparticle tunneling, offering a new mesoscopic parallel for black hole ringdown.

General relativity and quantum mechanics are the two cornerstones of modern physics, with the former describing physics at the astronomical scales and the latter primarily at micro- and mesoscopic scales. Outside of quantum gravity, these fields of studies are largely viewed as disjoint: each with a host of exotic physical phenomena. Recently, numerous works have drawn analogies between these disparate scales. Quantum systems, in fact, offer a multitude of ways to probe relativistic phenomena: from mimicking curved spacetimes [1] to investigating dynamic geometric backgrounds [2–6]. In this Letter, we show that the signature features of black holes and the scattering in quantum Hall systems can be remarkably unified by a mapping to single particle physics in the presence of an inverted harmonic oscillator (IHO) potential. The IHO model exhibits potential scattering and temporally decaying modes: features that have made the model invaluable in a broad variety of contexts since the birth of quantum mechanics [7–10]. From its infancy, phenomena such as particle decay [11] and metastability [12] have been analyzed using the IHO. In developments across the decades, the IHO has played key roles in the context of many disparate fields of physics and mathematics [13–24]. Thus, through its powerful simplicity, the IHO serves as an archetype for phenomena in numerous realms, much like the more familiar simple harmonic oscillator. Here, the IHO plays a crucial role in demonstrating how the confluence of black hole and quantum Hall physics can transfer insights between the astronomical and the mesoscopic realms.

In this Letter, we demonstrate that the physics of quantum Hall systems in strained or point contact geometries makes manifest two deep mathematical structures underlying black hole phenomena. Specifically, the quantum Hall situation describes lowest Landau level (LLL) physics in the presence of a saddle potential. (Quasi) particles moving in this potential are squeezed as a function of time; this operation of LLL squeezing is formally equivalent to time-evolution near the event horizon outside a black hole [25]. Hence, we show that the transmission probability near the point contact region manifest in conductance measurements naturally follows the thermal form associated with Hawking radiation. The second mathematical structure captured by such point contact physics is the scattering off the effective potential that lies outside a black hole event horizon. Quasi-normal modes, which are a hallmark temporally decaying feature of this scattering, are the fingerprint signatures of black holes in gravitational wave detection and have been hinted at by the Laser Interferometer Gravitational Wave Observatory (LIGO) discovery. Inspired by this black hole phenomenon, here, we highlight the presence of analogous temporally decaying modes in quantum Hall tunneling and propose a hitherto unexplored time-resolved measurement in a quintessential mesoscopic setting of quantum point contacts.

In what follows, we demonstrate how the semiclassical black hole scattering problem can be reduced to one-dimensional scattering off an IHO potential. We then show that the noncommutative structure of the lowest Landau level gives rise to the same scattering description in the presence of a saddle potential. Distinct from the black hole scattering context, we show that the Hamiltonian describing the IHO is isomorphic to the Rindler Hamiltonian associated with time evolution near the event horizon. This

symmetry-based link puts the previously observed connection between quantum Hall transmission probabilities and black hole thermality [26] on a stronger footing than a cursory analogy. Finally, we perform an in-depth analysis of quasinormal mode decay as an unexplored probe of quantum Hall dynamics.

Black hole scattering.—A black hole is formed by the gravitational collapse of a massive object. It is characterized by an event horizon that acts as a one-way membrane in spacetime: matter can enter the horizon but cannot leave. Several classical gravity and holographic phenomena, such as black hole collisions and quasinormal modes, are described through wave scattering in the black hole spacetime [27,28]. Here, we focus on such scattering for a massless scalar field Φ to provide a simple, illustrative derivation of IHO dynamics. The equation of motion for Φ in the background metric $g_{\mu\nu}$ is given by [27,29]

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(g^{\mu\nu}\sqrt{-g}\frac{\partial}{\partial x^\nu}\Phi\right) = 0. \tag{1}$$

For the Schwarzschild metric associated with an uncharged, nonrotating black hole for which the event horizon is at $r = 2GM/c^2$, we can decompose the field into its radial and angular parts. For each angular momentum component ℓ, the scalar field experiences an effective potential with its maximum outside the event horizon. The equation for the radial component $\Psi(r)$ reduces to the well-known form [27,30–33]

$$\frac{\partial^2\Psi}{\partial r_*^2} + V(\ell, r_*)\Psi = 0. \tag{2}$$

Here, r_* is the "tortoise coordinate" [29] in which the event horizon is at negative infinity, and $V(\ell, r_*)$ is the effective potential. Higher spin fields, such as for gravitational waves, can be treated analogously. Crucially, the potential has a maximum $V_0 = V(\ell, r_0)$ at a point $r_* = r_0$, and it can be approximated by an inverted harmonic potential nearby. Thus, in the semiclassical approximation, Eq. (2) takes the Weber form [33,34]:

$$-\frac{d^2\Psi}{d\tilde{r}_*^2} - \frac{1}{4}\tilde{r}_*^2\Psi = \left(\nu + \frac{1}{2}\right)\Psi. \tag{3}$$

Here, $\tilde{r}* = (2\overset{\sim}{V_0})^{1/4}(r_* - r_0)$ measures the dimensionless distance from the maximum. We identify $\nu + 1/2 = -V_0/(2\overset{\sim}{V_0})^{1/2}$ so as to cast Eq. (3) in the standard form. This derivation highlights the relevance of the IHO in black hole scattering [28,31,33,35]. In fact, the IHO captures essential dynamics even in the presence of gravitational backreaction [36,37] and the semiclassical limit of scattering off any smooth barrier [34].

Quantum Hall scattering.—We now show how the same scattering problem in Eq. (3) arises in a quantum Hall

system in the presence of a saddle potential. We consider a two-dimensional system of electrons confined to the x-y plane in the presence of a magnetic field $B\hat{\mathbf{z}} = \nabla \times \vec{A}$. Neglecting interactions, as is appropriate for an integer quantum Hall state, the Hamiltonian takes the form

$$H_0 = \frac{1}{2m}|\vec{\pi}|^2 + \lambda(x^2 - y^2), \tag{4}$$

where $\hat{\pi}_i = \hat{p}_i - e\hat{A}_i/c$ is the kinetic momentum, and the index $i = x$, y ranges over the spatial directions. The parameter λ captures the strength of the saddle potential. This potential models bulk quasiparticle tunneling through a point contact [26,38–40] and can serve as an anyon beam splitter [41].

We introduce the guiding center operators $\hat{R}_i = \hat{x}_i + (\ell_B^2/\hbar)\epsilon_{ij}\hat{\pi}_j$, where $\ell_B = \sqrt{\hbar c/(eB)}$ is the magnetic length. Upon projection to the lowest Landau level, the dynamics is determined entirely by the projected potential, which is a function of the R_i only [42]. The guiding center coordinates form a noncommutative plane, obeying $[\hat{R}_i, \hat{R}_j] = -i l_B^2 \epsilon_{ij}$. This commutation rule yields an effective "Planck's constant" $\hbar_{\mathrm{eff}} = l_B^2$ in this phase space that can be tuned through the applied magnetic field to access the semiclassical limit. Identifying $\hat{R}_x/(\sqrt{2}\ell_B)$ as a "momentum" $\tilde{P}$ and $\sqrt{2}\hat{R}_y/\ell_B$ as a "position" $\tilde{X}$, the projected Hamiltonian is that of the IHO:

$$H_{\mathrm{IHO}} = (2\hbar_{\mathrm{eff}}\lambda)\left(\tilde{P}^2 - \frac{1}{4}\tilde{X}^2\right) \tag{5}$$

Identifying $\tilde{X}$ with $\tilde{r}_*$ and the eigenenergy $E_\nu = 2\hbar_{\mathrm{eff}}\lambda(\nu + \frac{1}{2})$ for any real ν, the Schrödinger equation for this Hamiltonian takes the form of Eq. (3). Note that, even without LLL projection, dynamics in the presence of a saddle potential decouples into intra- and inter-Landau level components; the intra-Landau level dynamics is also governed by the applied potential [38].

The saddle potential can also be generated from area preserving shear deformations of two-dimensional electrons in a magnetic field. The generators $\hat{J}_{ij}$ of these transformations are [43,44]

$$\hat{J}_{ij} = -\frac{1}{2}\{\hat{x}_i, \hat{\pi}_j\} + \frac{1}{4}\{\hat{x}_k, \hat{\pi}_k\}\delta_{ij} + \frac{\hbar}{2\ell_B^2}\epsilon_{ij}\hat{x}_j\hat{x}_k. \tag{6}$$

The generators satisfy the $\mathfrak{sl}(2,\mathbb{R})$ commutation relations $i[J_{ij}, J_{kl}] = \delta_{il}J_{kj} - \delta_{jk}J_{il}$. These deformations are important in the study of Hall viscosity [43]. Of the three independent generators of the Lie algebra, the shear generator

$$V = \frac{2\lambda\ell_B^2}{\hbar}(J_{xy} + J_{yx}) = \frac{\lambda(R_x^2 - R_y^2)}{4} + \left(\frac{\lambda\ell_B^4}{\hbar^2}\right)(\pi_y^2 - \pi_x^2)$$

is of interest. Upon projecting it to the lowest Landau level, the kinetic momenta can be neglected due to vanishing matrix elements between any states in the same Landau level, leaving the H_{IHO} of Eq. (5). The Hamiltonian H_{IHO} is, in fact, a squeezing operator [45], which generates Bogoliubov transformations of the LLL eigenstates [26,41].

The Hawking-Unruh effect.—Given the common IHO starting point of Eq. (5), we now show that scattering off such a barrier captures both quantum Hall tunneling and radiation in black holes. To evaluate the scattering properties, we canonically transform to the basis $\hat{u}^{\pm} = (\hat{P} \pm \hat{X})/\sqrt{2}$ [21], where $\hat{u}^{\pm}$ are the incoming and outgoing states, as shown if Fig. 1. In this basis, H_{IHO} takes the form

$$H = \mp \, i2\hbar_{\text{eff}}\lambda\left(u^{\pm}\partial_{u^{\pm}} + \frac{1}{2}\right) \qquad (7)$$

which is the generator of rescalings (equivalently, dilatations) of $u^{\pm}$ [46]. The eigenstates are of the form $(1/\sqrt{2\pi})(\pm u^{\pm})^{iE-1/2}\Theta(\pm u^{\pm})$, where $\Theta(u^{\pm})$ is the Heaviside function and $E = \epsilon/2\hbar_{\text{eff}}\lambda$. The expansion in the eigenbasis of the dilatation operator is a Mellin transform [47,48], just as the Fourier transform is associated with the translation generator. The Mellin transform connects the incoming and outgoing states, thus yielding the scattering matrix (S matrix). The S matrix has the form [7,21,36]

$$\hat{S} = \frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{1}{2} - iE\right)\begin{pmatrix} e^{i\pi/4}e^{\pi E/2} & e^{-i\pi/4}e^{-\pi E/2} \\ e^{-i\pi/4}e^{-\pi E/2} & e^{i\pi/4}e^{\pi E/2} \end{pmatrix}, \quad (8)$$

where Γ is the Gamma function.

The transmission probability obtained from this S matrix yields the suggestive thermal form

$$|t|^2 = \frac{1}{1 + e^{\beta\epsilon}}, \quad \text{where } \beta = \frac{\pi}{\hbar_{\text{eff}}\lambda}. \qquad (9)$$

In the quantum Hall case, the transmission probability describes the tunneling of bulk quasiparticles between equipotential contours of the saddle potential. Such tunneling is ubiquitous in the quantum Hall system: for instance, in scattering between peaks or valleys in a disordered bulk potential landscape [49,50]. The tunneling form is also applicable for tunneling between edge channels across a pinched Hall bar, providing a derivation for the transmission probability via the bulk [38,51].

In the case of black holes, Eq. (9) can be interpreted as the thermal distribution associated with the Hawking-Unruh effect. Although there are many ways to understand the effect [25,52–55], we closely follow the treatment of Refs. [26,56–59]. To recapitulate, the thermal distribution stems from comparing vacua associated with inertial and accelerating observers. Specifically, we consider the vacuum state of a fermionic field [satisfying Eq. (1)] in a Minkowski (inertial) spacetime (t, x). A uniformly accelerating observer characterized by "Rindler coordinates" (τ, ξ), where $t = e^{\xi}\sinh(\tau)$ and $x = e^{\xi}\cosh(\tau)$, observes a different vacuum. Note that translations of τ generated by the "Rindler Hamiltonian" correspond to Lorentz boosts in (t, x). In Minkowski spacetime, we can mode expand the field in the basis of plane waves $e^{i\omega v}$ ($v = x - t$ is the "light-cone" coordinate). The expansion of the field in terms of the Rindler space is then in the eigenbasis $e^{i\Omega(\tau-\xi)} = v^{i\Omega}$. The Lorentz boost generator relates the modes in the two frames; a calculation shows that the fields are related by a Bogoliubov transformation [60,61]. Due to the Bogoliubov transformation, the Minkowski vacuum is perceived as a thermal distribution, such as depicted in Eq. (9), for the Rindler observer. In the presence of a black hole, such an analysis results in the Hawking radiation seen by an asymptotic observer [56,57].

It might be surprising that the two disparate platforms—one a relativistic spacetime and the other a quantum Hall system—exhibit such similarities. The connections are far from cursory and emerge from the underlying symmetry transformations: first, consider the $2 + 1$-dimensional Minkowski spacetime $(t, \vec{x})$ with the metric $ds^2 = dt^2 - d\vec{x}^2$. The Lorentz boost is one of the generators of the full spacetime symmetry algebra $\mathfrak{so}(2,1)$. In the quantum Hall system, the projected saddle potential (or the IHO/dilatation operator) is one of the generators of transformations that preserve the commutator $[R_x, R_y] = -il_B^2$. The generators respect a $\mathfrak{sp}(2,\mathbb{R})$ algebra. These two Lie algebras are isomorphic, and they coincide with that of area/flux preserving deformations given below [Eq. (6)]. Similarly, we can relate quantum states in the two situations as well. Note that the Rindler Hamiltonian and the IHO/dilatation operator in the LLL share similar eigenfunctions, upon identifying $\Omega = E - i/2$, where the extra $1/2$ is a result of the noncommutativity of the position and momentum in the quantum Hall case. In the relativistic derivation, this extra $1/2$ arises from the boost acting on the spin component of the fermionic fields [62]. As recently detailed [63], these modes determine the Bogoliubov coefficients through the Mellin transform and lead to Eq. (9) in both cases. This parallel between the quantum Hall transmission probability and the Hawking-Unruh effect was first explored for edge states in Ref. [26]; here, we have generalized the treatment to bulk quasiparticle scattering.

We emphasize that there is a distinction in the way the IHO relates to the two black hole phenomena considered in this work. The Hawking-Unruh effect explicitly involves second quantization and quantum effects across the event horizon, and the IHO appears here as the Rindler Hamiltonian corresponding to boosts. In contrast, black hole scattering concerns IHO potentials outside the event

horizon. Moreover, the associated QNMs that we now turn to are present in both quantum and classical situations.

Quasinormal modes.—QNM decay was originally predicted in the context of the stability of Schwarzschild black holes by Vishveshwara in Ref. [32] as damped outgoing oscillations in response to incoming Gaussian wave packets. This phenomenon has come to be a central point of study in black hole physics [28,64–70]. Here, our map of black hole scattering to the quantum IHO problem [33,35] in Eq. (3) and the resultant S matrix [Eq. (8)] enable us to explicitly characterize these damped oscillations, even if to the simplest approximation. The map also connects the black hole QNMs to previously unexplored dynamics of quantum Hall systems in a saddle potential.

The S matrix of Eq. (8) directly accesses the QNMs in its pole structure when analytically extended to the complex energy plane [71]. Specifically, the residues of the poles are resonant (quasistationary, or Gamow) states [72,73]. These states explicitly decay in time, and thus lie outside the standard Hilbert space [74,75]. Note that this decay does not require any nonunitary time evolution; rather, it is a consequence of the (purely outgoing) boundary conditions on the states. This can be compared with dissipation in the Landauer-Buttiker formalism for conductivity, where decay comes as a consequence of the boundary condition on states in the reservoir [76]. The S matrix for the IHO in Eq. (8) has poles at $E_n = -i(n + \frac{1}{2}); n = 0, 1, 2, \ldots$, coming from the Gamma function [see Fig. 2]. In order to probe these poles, as in both quantum scattering [71] and black hole perturbations [28,31], we employ a dynamic scattering process of impinging wave packets onto the potential rather than energy eigenfunctions.

A typical incident wave packet, as shown in Fig. 2, is composed of a superposition of energy states: $|\psi_{\text{in}}\rangle = \int d\epsilon f(\epsilon)|\epsilon\rangle$. For simplicity, we take $f(\epsilon)$ to be a Gaussian centered at an energy ϵ_* with width δ_ϵ. Importantly, the effect of the resonant QNM poles becomes manifest in the reflected wave at position x only for time $t > \tau \log |x/\ell|$, where $\tau = (2\hbar_{\text{eff}}\lambda)^{-1}$ and ℓ is the oscillator length. This can be understood from the classical trajectory of the model, $x \sim e^{t/\tau}$. For widths of $\delta_\epsilon \ll \hbar_{\text{eff}}\lambda$, the decay is governed by the residue of a single pole. For example, the reflected QNM decay due to the $n = 0$ pole has the spatiotemporal form

$$\Psi_{r0} \sim e^{-t/\tau} e^{\log |\sqrt{2}x/l|}. \tag{10}$$

This shows that QNM decay gives the asymptotic description of the change in probability as a scattered wave packet passes a (fixed) distant point.

The scattered packet also undergoes oscillations, as depicted in Fig. 2(c). For any fixed position, Fig. 2(d) depicts the characteristic temporal decay. These plots only take into account the pole at $n = 0$. Although we study resonances here as arising from the poles of the S matrix,

they can be equivalently cast as states in a rigged Hilbert space having complex eigenenergies [72–75,77] arising in an open system with purely outgoing boundary conditions. It is these boundary conditions that allow for seemingly nonunitary decay. Such QNM decay arises in any system having a potential landscape characterized by a local maxima, such as in Gamow's theory of radioactivity [11]. Thus, this analysis predicts the existence of black-hole-type QNMs in our quantum Hall setting.

QNMs in physical observables.—A direct measure of QNM decay would require a time-resolved nonequilibrium setting. In the black hole situation, the LIGO breakthrough recorded ringdown signals in cataclysmic black hole mergers, ushering in the era of multimessenger astronomy and highly suggestive of QNM behavior [78]. For a Schwarzschild black hole of one solar mass, the calculated decay time is 0.35 ms [79]; the decay time in the remarkable first measurement by LIGO from the binary black hole merger was 4 ms [80].

We now propose a setup in the quantum Hall situation for observing QNMs and derive analogous estimates. As shown in Fig. 1, a pinched point contact geometry creates a saddle potential in the bulk. The sources of bulk or edge state quasiholes undergo saddle-potential scattering and tunneling as described above. An indirect measure of QNM poles in the scattering matrix would be a Lorentzian form for the associated tunneling conductance, known as the Breit-Wigner distribution [81]. A direct measure would require a dilute beam of incoming quasiholes [82] that could enable time-resolving QNMs in the outgoing beam. Considering a point contact of width d, for applied voltage
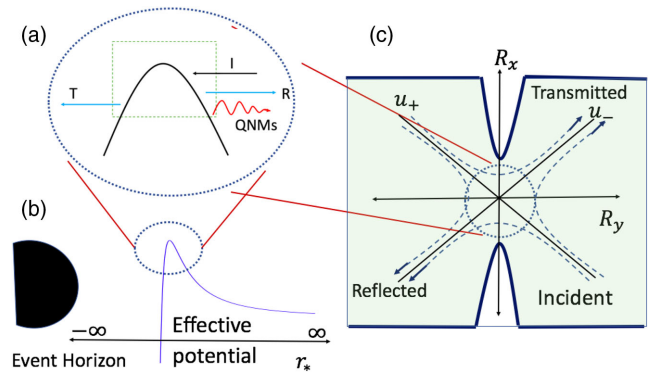

FIG. 1. (a) Schematic of the inverted harmonic oscillator potential and its scattering features of (i) regular transmission (T) and reflection (R) of an incident state (I); and (ii) temporally decaying quasinormal modes (QNMs), as applicable to two physically distinct phenomena [Figs. 1(b) and 1(c)]. (b) Dynamics of a field in a black hole spacetime resulting in (i) Hawking-Unruh radiation, and (ii) signature QNMs associated with black hole ringdown. (c) Scattering in the quantum Hall lowest Landau level via a saddle potential leads to (i) quasiparticle tunneling; and (ii) as we predict, QNMs that could be observed for dilute enough quasiparticle sources.
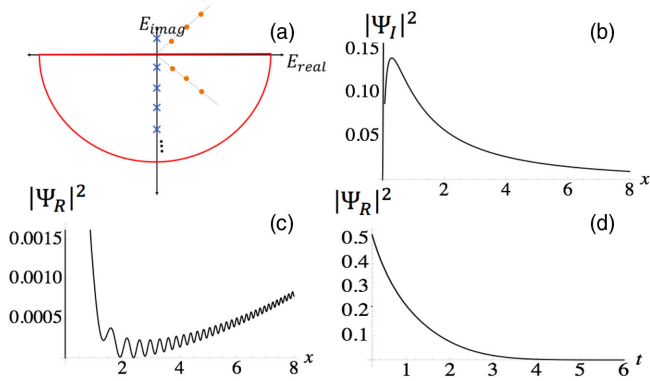
FIG. 2. Plots of wave-packet scattering off the IHO showing quasinormal mode behavior, as obtained form analytical calculations. (a) Pole structure of the S matrix of the IHO in the complex energy plane. Blue crosses indicate the resonant poles of the IHO, and purple boxes indicate generic resonance poles for some arbitrary potential barrier. Closing the contour in the lower half-plane is determined by the fact that we have picked outgoing boundary conditions. (b) A wave packet composed of scattering energy eigenstates, impinging on the barrier from the right. (c) The scattered wave packet shows the "quasinormal ringdown"; the form takes into account only a single pole for illustrative purposes. The scattered state escapes to infinity, as seen in its finite amplitude at large x but, as shown in Fig. 2(d), it exhibits an exponential time decay for a given point x after $t > \tau \log |x/\ell|$.

V and a background magnetic field B, we have that $\tau = d^2B/2V$ and $\lambda = eV/d^2$. Putting in typical numbers for a split-gate graphene junction [83], we find that $\tau \sim 400$ ps, whereas $\hbar_{\mathrm{eff}}\lambda \sim 125$ mK $\ll \hbar\omega_c$ and $\ell_B \sim 5$ nm. We anticipate then that the ringdown associated with quasihole wave packets should be observable at a distance of order 100 nm from the point contact at times on the order of nanoseconds. Finally, we note that realistic scattering potentials would have a finite range, unlike the unbounded IHO; the Pöschl-Teller potential yields a tractable candidate for such analyses [39,84]. We suggest that the actual recorded temporal decay in a given measurement would serve to recreate the underlying potential.

In conclusion, we have demonstrated that black hole and quantum Hall scattering dynamics naturally fall under an common umbrella dictated by properties of the inverted harmonic oscillator. These parallels not only draw connections between well-known effects but offer ground for unearthing new phenomena in both realms, such as QNMs in quantum Hall systems. We expect non-Poisson corrections to the current correlation function in quantum Hall noise experiments to provide an additional window into these temporally decaying modes. We hope that these parallels allow for future work to employ the quantum Hall system for even studying black holes and IHO physics in the context of recent connections to quantum chaos [13–15,85–88]. Our symmetry-based arguments also suggest several other experimentally accessible realizations of

relativistic kinematics in the quantum Hall system, such as Thomas precession. Thus, in drawing specific parallels between gravitational and quantum Hall physics, and transferring lessons from black hole scattering to new predictions in point contact geometries, we hope to have planted seeds of inspiration for the fertile exchange of ideas between the two realms.

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