How to Quantify a Dynamical Quantum Resource

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We show that the generalization of the relative entropy of a resource from states to channels is not unique, and there are at least six such generalizations. Then, we show that two of these generalizations are asymptotically continuous, satisfy a version of the asymptotic equipartition property, and their regularizations appear in the power exponent of channel versions of the quantum Stein's lemma. To obtain our results, we use a new type of "smoothing" that can be applied to functions of channels (with no state analog). We call it "liberal smoothing" as it allows for more spread in the optimization. Along the way, we show that the diamond norm can be expressed as a max relative entropy distance to the set of quantum channels, and prove a variety of properties of all six generalizations of the relative entropy of a resource.

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Introduction.—In recent years, it has been recognized that many properties of physical systems, such as quantum entanglement, asymmetry, coherence, athermality, contextuality, and many others, can be viewed as resources circumventing certain constraints imposed on physical systems (see [1] and references therein). Each resource can be classified as being classical or quantum, static (e.g., entangled state) or dynamic (e.g., quantum channel), noisy or noiseless, leading to numerous interesting quantum information processing tasks [2] (e.g., quantum teleportation [3]). While there are many ways to quantify the resourcefulness of such properties, all quantifiers of a resource must satisfy certain conditions such as monotonicity under the set of free operations. Typically, there are numerous measures that satisfy these conditions, but what can single out a given measure is an operational interpretation, giving it meaning beyond its sheer ability to quantify, somewhat vaguely, the resource.

The relative entropy of a resource, which was originally defined in [4] for entanglement theory, is an example of a measure that has such an operational interpretation in many quantum resource theories (QRTs). First, it was shown in [5,6] to be a unique measure in reversible QRTs and, then, was shown to be the unique asymptotic rate of interconversion among static resources under resource nongenerating operations [7]. Moreover, it was shown, very recently [8,9], that resource erasure as a universal operational task leads to the (regularized) relative entropy of a resource as the optimal rate (this idea was first laid out in [10]). In addition, this measure satisfies the asymptotic equipartition property (AEP) [11], appears as an optimal rate in the generalized quantum Stein's lemma [11], and is asymptotically continuous [12,13], a property also linked to it being a nonlockable measure [14]. Because of all of these properties, the relative entropy of a resource plays a major role in many QRTs [1].

In this Letter, we study six generalizations of the quantum relative entropy of a resource from static resources (i.e., states) to dynamic ones (i.e., channels). Four of these measures were introduced very recently in [15,16]. We show that for two of them, the relative entropy of the dynamical resource is asymptotically continuous, satisfies a version of the AEP, and a version of their regularization appears as optimal rates in a version of the quantum Stein's lemma for channels. In addition, we show that all these measures are, indeed, generalizations to dynamical resources in the sense that they reduce to the relative entropy of a static resource for replacement (i.e., constant) channels.

Resource theories of quantum processes.—A QRT consists of a function $\mathfrak{F}$ taking any pair of physical systems A and B to a subset of completely positive and trace preserving (CPTP) maps $\mathfrak{F}(A \to B) \subset \mathrm{CPTP}(A \to B)$, where $\mathrm{CPTP}(A \to B)$ is the set of all CPTP maps (i.e., quantum channels) from $\mathcal{B}(A)$ (bounded operators on Hilbert space of system A) to $\mathcal{B}(B)$ [15–21]. The mapping $\mathfrak{F}$ is a quantum resource theory if the following two conditions hold: (1) For any physical system A the set $\mathfrak{F}(A \to A)$ contains the identity map id_A. (2) For any three systems A, B, C, if $\mathcal{M} \in \mathfrak{F}(A \to B)$ and $\mathcal{N} \in \mathfrak{F}(B \to C)$, then $\mathcal{N} \circ \mathcal{M} \in \mathfrak{F}(A \to C)$. Denoting by 1 the trivial Hilbert space, we identify $\mathfrak{F}(1 \to A)$ with the set of free density matrices in $\mathcal{B}(A)$. That is, a density matrix $\rho \in \mathfrak{F}(1 \to A)$ can be viewed as the CPTP map $\rho(z) = z\rho$ for all $z \in \mathbb{C}$. For simplicity, we will write $\mathfrak{F}(1 \to A) \equiv \mathfrak{F}(A)$. Typically, QRTs are physical in the sense that they arise from some physical constraints and, therefore, admit a tensor product structure. That is, the set of free operations $\mathfrak{F}$ satisfies the following additional conditions: (3) The free operations are "completely free": For any three physical systems A, B, and C, if $\mathcal{M} \in \mathfrak{F}(A \to B)$, then $\mathrm{id}_C \otimes \mathcal{M} \in \mathfrak{F}(CA \to CB)$. (4) Discarding a system (i.e., the trace) is a free operation: For any system A, the set $\mathfrak{F}(A \to 1)$ is not empty. The above additional conditions are very natural and satisfied by almost all QRTs studied in literature. They imply the following properties [1]: (a) If $\mathcal{M}_1$ and $\mathcal{M}_2$ are free channels, then $\mathcal{M}_1 \otimes \mathcal{M}_2$ is also free. (b) Appending free states is a free operation: For any given free state $\sigma \in \mathfrak{F}(B)$, the CPTP map $\mathcal{M}_\sigma(\rho) \coloneqq \rho \otimes \sigma$ is a free map; i.e., it belongs to $\mathfrak{F}(A \to AB)$. (c) The replacement map $\mathcal{M}_\sigma(\rho) \coloneqq \sigma$, for any density matrix $\rho \in \mathcal{B}(A)$ and a fixed free state $\sigma \in \mathfrak{F}(B)$, is a free channel; i.e., $\mathcal{M}_\sigma \in \mathfrak{F}(A \to B)$. It is also physical to assume that $\mathfrak{F}(A \to B)$ is a closed set, since otherwise, there exists a sequence of free channels whose limit is a resource channel. Finally, we will assume that for any integer n, free channel $\mathcal{N} \in \mathfrak{F}(A_1 \cdots A_n \to B_1 \cdots B_n)$, and two permutation channels $\mathcal{P}_A^\pi$ and $\mathcal{P}_B^{\pi^{-1}}$ corresponding to a permutation π on n elements, we have

$$\mathcal{P}_B^{\pi^{-1}} \circ \mathcal{N}_{A_1 \cdots A_n \to B_1 \cdots B_n} \circ \mathcal{P}_A^\pi \in \mathfrak{F}(A_1 \cdots A_n \to B_1 \cdots B_n).$$

Note that almost all QRTs discussed in literature satisfy this last condition including entanglement theory, coherence, athermality, etc. In the rest of this Letter, we will assume that $\mathfrak{F}$ satisfies all the above conditions.

The most general physical operation that can be performed on a dynamical resource $\mathcal{N} \in \mathrm{CPTP}(A \to B)$ can be characterized with a superchannel [17,18], Θ, defined for all $\mathcal{N} \in \mathrm{CPTP}(A \to B)$ as a transformation of the form

$$\Theta[\mathcal{N}_{A \to B}] = \mathcal{E}^{\mathrm{post}}_{BE \to B'} \circ \mathcal{N}_{A \to B} \circ \mathcal{E}^{\mathrm{pre}}_{A' \to AE}, \tag{1}$$

where $\mathcal{E}^{\mathrm{post}} \in \mathrm{CPTP}(BE \to B')$ and $\mathcal{E}^{\mathrm{pre}} \in \mathrm{CPTP}(A' \to AE)$ are quantum channels. We say that the superchannel Θ is free if, in addition, $\mathcal{E}^{\mathrm{post}} \in \mathfrak{F}(BE \to B')$ and $\mathcal{E}^{\mathrm{pre}} \in \mathfrak{F}(A' \to AE)$ (i.e., $\mathcal{E}^{\mathrm{post}}$ and $\mathcal{E}^{\mathrm{pre}}$ are free). Therefore, any measure of a resource $E\colon \mathrm{CPTP} \to \mathbb{R}$ must satisfy

$$E(\Theta[\mathcal{N}_{A \to B}]) \le E(\mathcal{N}_{A \to B}), \tag{2}$$

for all $\mathcal{N} \in \mathrm{CPTP}(A \to B)$ and all free superchannels Θ. In addition, we require that $E(\mathcal{N}) = 0$ if $\mathcal{N} \in \mathfrak{F}(A \to B)$. This condition implies that E is non-negative. To see it, take $\mathcal{E}^{\mathrm{post}}_{BE \to B'}$ in (1) to be the replacement map whose output is some free state in $\mathfrak{F}(B')$, and observe that, for this case, $0 = E(\Theta[\mathcal{N}]) \le E(\mathcal{N})$ for all $\mathcal{N} \in \mathrm{CPTP}(A \to B)$.

The relative entropy of a resource.—Here, we will consider two generalizations of the relative entropy of a resource from the state domain to the channel domain, and leave four further generalizations to the Supplemental Material (SM) [19]. The first relative entropy of a dynamical resource $\mathcal{N} \in \mathrm{CPTP}(A \to B)$ is defined as

$$D_{\mathfrak{F}}(\mathcal{N}) \coloneqq \inf_{\mathcal{M} \in \mathfrak{F}(A \to B)} D(\mathcal{N} \| \mathcal{M}), \tag{3}$$

with the channel divergence [17,22,23]

$$D(\mathcal{N} \| \mathcal{M}) \coloneqq \max_{\varphi \in \mathcal{D}(RA)} D[\mathcal{N}_{A \to B}(\varphi_{RA}) \| \mathcal{M}_{A \to B}(\varphi_{RA})], \tag{4}$$

and $D(\rho \| \sigma) = \mathrm{Tr}[\rho \log \rho - \rho \log \sigma]$ is the relative entropy. The optimization is over all states φ_{RA}, where, without loss of generality, we can take $R \cong A$, and φ_{RA} is pure [22,23]. If the optimization over $\mathcal{D}(RA)$ is replaced with optimization over the set of all density matrices $\mathfrak{F}(RA)$, then one gets the second generalization [15]

$$E_{\mathfrak{F}}(\mathcal{N}) \coloneqq \min_{\mathcal{M} \in \mathfrak{F}(A \to B)} \sup_{\rho \in \mathfrak{F}(RA)} D[\mathcal{N}_{A \to B}(\rho_{RA}) \| \mathcal{M}_{A \to B}(\rho_{RA})], \tag{5}$$

where the supremum is over all free states $\rho \in \mathfrak{F}(RA)$ and all dimensions $|R|$, and the minimum is over all free channels in $\mathfrak{F}(A \to B)$. Both $D_{\mathfrak{F}}$ and $E_{\mathfrak{F}}$, as well as other generalizations, were introduced very recently in [15,16], and in the SM [19], we list all of them along with a few new ones and discuss some of their properties. For clarity, we leave the technical details of all proofs to the SM [19].

Theorem 1.—The above relative entropies have the following properties: (1) Monotonicity: $D_{\mathfrak{F}}$ and $E_{\mathfrak{F}}$ behave monotonically under free superchannels. Specifically, let $\mathcal{E}^{\mathrm{post}} \in \mathrm{CPTP}(BE \to B')$ and $\mathcal{E}^{\mathrm{pre}} \in \mathrm{CPTP}(A' \to AE)$ be completely resource nongenerating (RNG) channels, and let Θ have the form: (1). Then, for all $\mathcal{N} \in \mathrm{CPTP}(A \to B)$

$$D_{\mathfrak{F}}(\Theta[\mathcal{N}]) \le D_{\mathfrak{F}}(\mathcal{N}); \qquad E_{\mathfrak{F}}(\Theta[\mathcal{N}]) \le E_{\mathfrak{F}}(\mathcal{N}). \tag{6}$$

(2) Reduction: Let $\mathcal{N} \in \mathrm{CPTP}(A \to B)$ be a constant channel $\mathcal{N}(X_A) = \mathrm{Tr}[X_A]\omega_B$ for all $X_A \in \mathcal{B}(A)$ and a fixed density matrix $\omega_B \in \mathcal{D}(B)$. Then,

$$D_{\mathfrak{F}}(\mathcal{N}) = E_{\mathfrak{F}}(\mathcal{N}) = D_{\mathfrak{F}}(\omega_B) \coloneqq \min_{\sigma \in \mathfrak{F}(B)} D(\omega_B \| \sigma_B). \tag{7}$$

(3) Faithfulness: $D_{\mathfrak{F}}(\mathcal{N}_{A \to B}) = 0$ if and only if $\mathcal{N} \in \mathfrak{F}(A \to B)$. If $E_{\mathfrak{F}}(\mathcal{N}) = 0$ for some $\mathcal{N} \in \mathrm{CPTP}(A \to B)$, then $\mathcal{N}$ must be completely RNG. Moreover, if for $|R| = |A|$, the set $\mathfrak{F}(RA)$ contains a pure state with full Schmidt rank, then

$$E_{\mathfrak{F}}(\mathcal{N}_{A \to B}) = 0 \Leftrightarrow \mathcal{N} \in \mathfrak{F}(A \to B). \tag{8}$$

In contrast to the monotonicity property above, the function $D_{\mathfrak{F}}$ behaves monotonically under any RNG superchannel. This follows directly from the following:

$$
\begin{aligned}
D_{\mathfrak{F}}(\Theta[\mathcal{N}]) &= \min_{\Omega\in\mathfrak{F}(A'\to B')} D(\Theta[\mathcal{N}_{A\to B}]\|\Omega_{A'\to B'}) \\
&\leq \min_{\mathcal{M}\in\mathfrak{F}(A\to B)} D(\Theta[\mathcal{N}_{A\to B}]\|\Theta[\mathcal{M}_{A\to B}]) \\
&\leq \min_{\mathcal{M}\in\mathfrak{F}(A\to B)} D(\mathcal{N}_{A\to B}\|\mathcal{M}_{A\to B}) = D_{\mathfrak{F}}(\mathcal{N}),
\end{aligned}
$$

where the first inequality follows from the fact that Θ is RNG, and the second from the data processing inequality of the channel divergence [17]. Note also that, from their definitions, we always have

$$E_{\mathfrak{F}}(\mathcal{N}) \leq D_{\mathfrak{F}}(\mathcal{N}) \quad \forall\ \mathcal{N}\in \mathrm{CPTP}(A\to B). \tag{9}$$

One may wonder if exchanging the min-max order in (3) and (5) would yield other relative entropy based measures that are in general different than $D_{\mathfrak{F}}$ and $E_{\mathfrak{F}}$. However, in the following theorem, we show that this is not the case.

Theorem 2.—Let $d:\mathcal{D}(A)\times\mathcal{D}(A)\to\mathbb{R}$ be any function satisfying non-negativity, contractivity (monotonicity) under CPTP maps, and joint concavity under orthogonally flagged mixtures: This means that for any two families $\{\rho_x\}$ and $\{\sigma_x\}$ of states, and any probability distribution $\{p_x\}$,

$$
\begin{aligned}
&d\left(\sum_x p_x\rho_x\otimes|x\rangle\langle x|, \sum_x p_x\sigma_x\otimes|x\rangle\langle x|\right) \\
&\quad\geq \sum_x p_x d(\rho_x,\sigma_x),
\end{aligned}
\tag{10}
$$

where $|x\rangle$ are orthonormal basis states of an auxiliary system. Moreover, suppose d is convex in the second argument, and suppose $\mathfrak{F}(A\to B)$ is convex. Then,

$$
\begin{aligned}
&\inf_{\mathcal{M}\in\mathfrak{F}(A\to B)}\ \sup_{\rho\in\mathfrak{F}(RA)} d(\mathcal{N}_{A\to B}(\rho_{RA}),\mathcal{M}_{A\to B}(\rho_{RA})) \\
&\quad= \sup_{\rho\in\mathfrak{F}(RA)}\ \inf_{\mathcal{M}\in\mathfrak{F}(A\to B)} d(\mathcal{N}_{A\to B}(\rho_{RA}),\mathcal{M}_{A\to B}(\rho_{RA})).
\end{aligned}
$$

Note that the relative entropy D (as well as the trace distance and all the Renyi divergences) satisfies (10) with equality, and therefore, $E_{\mathfrak{F}}$ and $D_{\mathfrak{F}}$ will not change by swapping the min-max order.

Asymptotic continuity.—Since we only consider QRTs that admit the tensor product structure here, the replacement channels $\mathcal{M}_\sigma(X)=\mathrm{Tr}[X]\sigma$ are free [i.e., in $\mathfrak{F}(A\to B)$] for any free $\sigma\in\mathfrak{F}(B)$. In the SM [19], we show that this implies that $E_{\mathfrak{F}}$ is bounded as long as the set of free states contains a full rank state. For example, if $\mathfrak{F}(B)$ contains the maximally mixed (uniform) state $I_B/|B|$ (were $|B|$ is the dimension of system B), then

$$E_{\mathfrak{F}}(\mathcal{N}) \leq D_{\mathfrak{F}}(\mathcal{N}) \leq \log(|B|^2|A|). \tag{11}$$

The fact that $E_{\mathfrak{F}}$ and $D_{\mathfrak{F}}$ are bounded enable us to prove that they are also asymptotically continuous.

Definition 3.—A function $E:\mathrm{CPTP}\to\mathbb{R}_+$ is said to be asymptotically continuous if for any $\mathcal{M},\mathcal{N}\in\mathrm{CPTP}(A\to B)$,

$$|E(\mathcal{M})-E(\mathcal{N})| \leq \log(|AB|)f(\|\mathcal{M}-\mathcal{N}\|_\diamond), \tag{12}$$

where $f:\mathbb{R}\to\mathbb{R}$ is some function independent on the dimensions and satisfies $\lim_{\epsilon\to 0^+}f(\epsilon)=0$.

Theorem 4.—Suppose that for any system A, $\mathfrak{F}(A)$ contains a full rank state. Then, $D_{\mathfrak{F}}$ is asymptotically continuous. Moreover, if in addition, for any system A, the extreme points of $\mathfrak{F}(A)$ are pure states (e.g., entanglement theory, coherence, etc.), then $E_{\mathfrak{F}}$ is also asymptotically continuous.

Remark: The proof of the theorem above is based on a key observation that the diamond norm can be expressed in terms of the max relative entropy distance of $\mathcal{N}-\mathcal{M}$ to the set of all quantum channels $Q(A\to B)$ (see SM [19] for more details). For $E_{\mathfrak{F}}$, the condition that the extreme points of the set of free states are pure states, ensures that the supremum in (5) can be replaced with a maximum since, in this case, $|R|$ can be shown to be bounded by $|A|$. If the extreme points of the set of free states are not pure states, but $|R|$ is polynomially bounded in $|AB|$, then in this case, $E_{\mathfrak{F}}$ is also asymptotically continuous. This happens, for example, in the QRT of thermodynamics. Finally, we point out that asymptotic continuity for certain amortized measures of entanglement was recently proved in [24].

Asymptotic equipartition property.—The logarithmic robustness of a dynamical resource $\mathcal{N}\in\mathrm{CPTP}(A\to B)$ is defined as [16]

$$
\begin{aligned}
LR_{\mathfrak{F}}(\mathcal{N}_{A\to B}) &\coloneqq \min_{\mathcal{M}\in\mathfrak{F}(A\to B)} D_{\max}(\mathcal{N}_{A\to B}\|\mathcal{M}_{A\to B}) \\
&\coloneqq \log_2\min\{t:\ t\mathcal{M}\geq\mathcal{N};\mathcal{M}\in\mathfrak{F}(A\to B)\},
\end{aligned}
\tag{13}
$$

where the ordering $t\mathcal{M}\geq\mathcal{N}$ means that $t\mathcal{M}-\mathcal{N}$ is completely positive (CP). Here, we also define

$$
\begin{aligned}
&\underline{L}R_{\mathfrak{F}}(\mathcal{N}_{A\to B}) \\
&\coloneqq \min_{\mathcal{M}\in\mathfrak{F}(A\to B)}\ \sup_{\varphi\in\mathfrak{F}(RA)} D_{\max}[\mathcal{N}_{A\to B}(\varphi_{RA})\|\mathcal{M}_{A\to B}(\varphi_{RA})].
\end{aligned}
\tag{14}
$$

Like $D_{\mathfrak{F}}$ and $E_{\mathfrak{F}}$, the functions $LR_{\mathfrak{F}}$ and $\underline{L}R_{\mathfrak{F}}$ are resource monotones (see SM [19]). Note that, by Theorem 2 the order sup-min can be exchanged, and furthermore,

$$\underline{L}R_{\mathfrak{F}} \leq LR_{\mathfrak{F}}, \tag{15}$$

with equality if $\mathfrak{F}(RA)$ contains a pure state of full Schmidt rank. For example, in entanglement theory, system A is replaced with AB and R with R_AR_B so that $\mathfrak{F}(R_AR_BAB)$ contains the state $\phi^+_{(R_AR_B)(AB)}=\phi^+_{R_AA}\otimes\phi^+_{R_BB}$, where ϕ^+

stands for the maximally entangled state between the respective spaces. Hence, $\phi^+_{(R_AR_B)(AB)}$ has full Schmidt rank between R_AR_B and AB [even though it is a product state between Alice (R_AA) and Bob (R_BB)]. Therefore, in entanglement theory, $\underline{LR}_{\mathfrak{F}} = LR_{\mathfrak{F}}$.

The smoothed version of the logarithmic robustness can be defined as [16]

$$\widetilde{LR}^{\epsilon}_{\mathfrak{F}}(\mathcal{N}) \coloneqq \min_{\mathcal{N}' \in B_\epsilon(\mathcal{N})} LR_{\mathfrak{F}}(\mathcal{N}'), \tag{16}$$

where

$$B^\epsilon(\mathcal{N}) \coloneqq \{\mathcal{N}' \in \mathrm{CPTP}(A \to B) : \|\mathcal{N}' - \mathcal{N}\|_\diamond \le \epsilon\}. \tag{17}$$

The above diamond-smoothed log robustness is a straightforward generalization from states to channels and has an operational interpretation in the setting of resource erasure [16], generalizing the single-shot part of [9]. However, our goal here is to define a method for smoothing that is the least restrictive possible. This will be necessary for a proof of an AEP for the logarithmic robustness of channels.

For this reason, we consider another (more "liberal") way to define smoothing for channels for which there is no analog in the state domain. For any state $\varphi \in \mathcal{D}(RA)$ and a channel $\mathcal{N} \in \mathrm{CPTP}(A \to B)$ define $B^{\varphi}_{\epsilon}(\mathcal{N})$ to be the set of all *CP* maps (not necessarily trace preserving) $\mathcal{N}' \in \mathrm{CP}(A \to B)$ satisfying

$$\|\mathcal{N}'_{A\to B}(\varphi_{RA}) - \mathcal{N}_{A\to B}(\varphi_{RA})\|_1 \le \epsilon. \tag{18}$$

Clearly, $B_\epsilon(\mathcal{N}) \subset \bigcap_{\varphi \in \mathcal{D}(RA)} B^{\varphi}_{\epsilon}(\mathcal{N})$. We define the smoothing of $LR_{\mathfrak{F}}$ as

$$LR^{\epsilon}_{\mathfrak{F}}(\mathcal{N}) \coloneqq \max_{\varphi \in \mathcal{D}(RA)} \min_{\mathcal{N}' \in B^{\varphi}_{\epsilon}(\mathcal{N})} LR_{\mathfrak{F}}(\mathcal{N}'). \tag{19}$$

Similarly, we denote by $\underline{LR}^{\epsilon}_{\mathfrak{F}}$ the above smoothing of $\underline{LR}_{\mathfrak{F}}$. Note that the above types of smoothing respect the condition that, for $\epsilon = 0$, the smoothed quantities reduce to the nonsmoothed ones. Furthermore, from its definition, it follows that (see SM [19] for more details)

$$LR^{\epsilon}_{\mathfrak{F}}(\mathcal{N}) \le \widetilde{LR}^{\epsilon}_{\mathfrak{F}}(\mathcal{N}), \tag{20}$$

justifying the name "liberal smoothing."

In the SM [19], we show that $LR^{\epsilon}_{\mathfrak{F}}(\mathcal{N})$ is a resource monotone, and the regularized versions

$$LR^{\infty}_{\mathfrak{F}}(\mathcal{N}) \coloneqq \varliminf_{n\to\infty} \frac{LR^{\epsilon}_{\mathfrak{F}}(\mathcal{N}^{\otimes n})}{n}; \quad D^{\infty}_{\mathfrak{F}}(\mathcal{N}) \coloneqq \varliminf_{n\to\infty} \frac{D^{\epsilon}_{\mathfrak{F}}(\mathcal{N}^{\otimes n})}{n},$$

satisfy $D^{\infty}_{\mathfrak{F}}(\mathcal{N}) \le LR^{\infty}_{\mathfrak{F}}(\mathcal{N})$. We believe that, in general, this inequality can be strict. However, as we show now, if we also revise the type of regularization, then it is possible to get an equality.

The type of regularization that we consider here is as follows. For each $n \in \mathbb{N}$, and a channel $\mathcal{N} \in \mathrm{CPTP}(A \to B)$, we define the quantities

$$D^{(n)}_{\mathfrak{F}}(\mathcal{N}) \coloneqq \frac{1}{n} \max_{\varphi \in \mathcal{D}(RA)} \min_{\mathcal{M} \in \mathfrak{F}(A^n \to B^n)} D[\mathcal{N}^{\otimes n}_{A\to B}(\varphi^{\otimes n}_{RA}) \| \mathcal{M}_{A^n \to B^n}(\varphi^{\otimes n}_{RA})], \tag{21}$$

and $E^{(n)}_{\mathfrak{F}}$ is defined exactly as above with $\mathfrak{F}(RA)$ replacing $\mathcal{D}(RA)$.

In the SM [19], we show that the limit $n \to \infty$ of $D^{(n)}_{\mathfrak{F}}$ and $E^{(n)}_{\mathfrak{F}}$ exists. Therefore, we define the "regularized" version of $D_{\mathfrak{F}}$ and $E_{\mathfrak{F}}$ to be

$$D^{(\infty)}_{\mathfrak{F}}(\mathcal{N}) = \lim_{n\to\infty} D^{(n)}_{\mathfrak{F}}(\mathcal{N}); \qquad E^{(\infty)}_{\mathfrak{F}}(\mathcal{N}) = \lim_{n\to\infty} E^{(n)}_{\mathfrak{F}}(\mathcal{N}).$$

We can also use this regularization method for the liberal smoothed logarithmic robustness quantities $LR^{\epsilon}_{\mathfrak{F}}$ and $\underline{LR}^{\epsilon}_{\mathfrak{F}}$. We define

$$LR^{\epsilon,n}_{\mathfrak{F}}(\mathcal{N}) \coloneqq \frac{1}{n} \max_{\varphi \in \mathcal{D}(RA)} \min_{\mathcal{N}' \in B^{\varphi^{\otimes n}}_{\epsilon}(\mathcal{N}^{\otimes n})} LR_{\mathfrak{F}}(\mathcal{N}') \tag{22}$$

$$\text{and} \quad LR^{(\infty)}_{\mathfrak{F}} \coloneqq \lim_{\epsilon\to 0} \varliminf_{n\to\infty} LR^{\epsilon,n}_{\mathfrak{F}}(\mathcal{N}). \tag{23}$$

The quantities $\underline{LR}^{\epsilon,n}_{\mathfrak{F}}$ and $\underline{LR}^{(\infty)}_{\mathfrak{F}}$ are defined analogously with $\mathfrak{F}(RA)$ replacing $\mathcal{D}(RA)$.

Theorem 5.—For all $\mathcal{N} \in \mathrm{CPTP}(A \to B)$

$$D^{(\infty)}_{\mathfrak{F}}(\mathcal{N}) = \lim_{\epsilon\to 0} \varlimsup_{n\to\infty} \frac{1}{n} LR^{\epsilon,n}_{\mathfrak{F}}(\mathcal{N}^{\otimes n}) = LR^{(\infty)}_{\mathfrak{F}}(\mathcal{N}).$$

Moreover, if for any system A the extreme points of $\mathfrak{F}(A)$ are pure states, then

$$E^{(\infty)}_{\mathfrak{F}}(\mathcal{N}) = \lim_{\epsilon\to 0} \varlimsup_{n\to\infty} \frac{1}{n} \underline{LR}^{\epsilon,n}_{\mathfrak{F}}(\mathcal{N}^{\otimes n}) = \underline{LR}^{(\infty)}_{\mathfrak{F}}(\mathcal{N}).$$

Quantum channel Stein's lemma (See related work [17,22,23,25,26]).—Consider the task of discriminating between n copies of a fixed channel $\mathcal{N} \in \mathrm{CPTP}(A \to B)$ and one of the free channels in $\mathfrak{F}(A^n \to B^n)$. There are two types of errors in such a task: (1) The observer guesses that the channel belongs to $\mathfrak{F}(A^n \to B^n)$ while the channel is $\mathcal{N}^{\otimes n}_{A\to B}$. This occurs with probability

$$\alpha^{(n)}(\mathcal{N}, P_n, \varphi_{RA}) \coloneqq \mathrm{Tr}[\mathcal{N}^{\otimes n}_{A\to B}(\varphi^{\otimes n}_{RA})(I - P_n)].$$

Here, we consider the "parallel" case, in which the observer only provides n copies of a free state $\varphi \in \mathfrak{F}(RA)$, and $0 \le P_n \le I_{A^nB^n}$. (2) The observer guesses that the channel is $\mathcal{N}_{A\to B}^{\otimes n}$ while the channel is some $\mathcal{M}_n \in \mathfrak{F}(A^n \to B^n)$. This occurs with probability

$$\beta^{(n)}(P_n, \mathcal{M}_n, \varphi_{RA}) \coloneqq \mathrm{Tr}[\mathcal{M}_n(\varphi_{RA}^{\otimes n})P_n],$$

and the worst case for a given $\varphi \in \mathfrak{F}(RA)$ is

$$\beta_{\mathfrak{F}}^{(n)}(P_n, \varphi_{RA}) \coloneqq \max_{\mathcal{M}_n \in \mathfrak{F}(A^n\to B^n)} \mathrm{Tr}[\mathcal{M}_n(\varphi_{RA}^{\otimes n})P_n].$$

We further define

$$\beta_{\mathfrak{F},\epsilon}^{(n)}(\mathcal{N}, \varphi_{RA}) \coloneqq \min \beta_{\mathfrak{F}}^{(n)}(P_n, \varphi_{RA}), \tag{24}$$

where the minimum is over all P_n satisfying $\alpha^{(n)}(\mathcal{N}, P_n, \varphi_{RA}) \le \epsilon$ and $0 \le P_n \le I_{R^nB^n}$.

Theorem 6.—Let $\mathfrak{F}$ be a convex resource theory satisfying all the conditions discussed in the introduction, and suppose further that the set of free states contains a full rank state. Then, for all $\epsilon \in (0, 1)$,

$$\tilde{E}_{\mathfrak{F}}^{(\infty)}(\mathcal{N}) = \max_{\varphi\in\mathfrak{F}(RA)} \lim_{n\to\infty} -\frac{\log \beta_{\mathfrak{F},\epsilon}^{(n)}(\mathcal{N}, \varphi_{RA})}{n}, \tag{25}$$

where

$$\tilde{E}_{\mathfrak{F}}^{(\infty)}(\mathcal{N}) \coloneqq \max_{\varphi\in\mathfrak{F}(RA)} \lim_{n\to\infty} \min_{\mathcal{M}\in\mathfrak{F}(A^n\to B^n)} \frac{D[\mathcal{N}^{\otimes n}(\varphi_{RA}^{\otimes n})\|\mathcal{M}(\varphi_{RA}^{\otimes n})]}{n}.$$

Note that the only difference between $\tilde{E}_{\mathfrak{F}}^{(\infty)}(\mathcal{N})$ and $E_{\mathfrak{F}}^{(\infty)}(\mathcal{N})$ is the order between the limit and the maximum. Therefore, we must have $\tilde{E}_{\mathfrak{F}}^{(\infty)}(\mathcal{N}) \le E_{\mathfrak{F}}^{(\infty)}(\mathcal{N})$, and it is left open to determine if this inequality can be strict. If the latter holds, that would mean that $\tilde{E}_{\mathfrak{F}}^{(\infty)}(\mathcal{N})$ is yet another (distinct) generalization of the relative entropy of a resource.

Conclusions.—We have seen that $D_{\mathfrak{F}}$ and $E_{\mathfrak{F}}$ are asymptotically continuous, satisfy the AEP, and are related to a channel version of the quantum Stein's lemma. To establish these results, we had to adopt two unconventional strategies, liberal smoothing and product-state channel regularization. In this way, lots of the properties in the state domain carry over to the channel domain. In the SM [19], we also introduce four additional generalizations of the relative entropy of a resource. This variety of generalizations indicates that, in the channel domain, things are much more complicated. We believe that the results and techniques presented here will provide an initial step towards the development of QRT with dynamical resources.

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