Nonergodic Extended States in the Sachdev-Ye-Kitaev Model

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We analytically study spectral correlations and many body wave functions of a Sachdev-Ye-Kitaev model deformed by a random Hamiltonian diagonal in Fock space. Our main result is the identification of a wide range of intermediate coupling strengths where the spectral statistics is of Wigner-Dyson type, while wave functions are nonuniformly distributed over Fock space. The structure of the theory suggests that such manifestations of nonergodic extendedness may be a prevalent phenomenon in many body chaotic quantum systems.

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Introduction.—In recent years, classifications of many body quantum systems as either "ergodic" or "many body localized" (MBL) have become mainstream. This reflects the discovery of a growing number of systems supporting MBL phases [1–12] and naturally extends the distinction between single particle ergodic and Anderson localized systems to many body quantum disorder. However, recently, we are seeing mounting evidence [13–21] that the above dichotomy may be too coarse to capture the complexity of chaotic many body systems. Specifically, recent work has put the focus on the study of statistical properties of many body wave functions. It has been reasoned that, sandwiched between the extremes ergodic and many body localized, there might exist intermediate phases of nonergodic extended (NEE) states, i.e., quantum states different from localized in that they have unbounded support, and different from ergodic in that their amplitudes are not uniformly distributed. One reason why this option comes into focus only now is that standard tools in diagnosing chaos—spectral statistics applied to systems of small size of $\mathcal{O}(10^1)$ physical sites—are too coarse to resolve the spatial structure of quantum states in Fock space. Indeed, the above indications are indirect in that they are based on numerical and analytic work on disordered graphs with high coordination numbers, artificial systems believed to share key characteristics with genuine random Fock spaces. The complexity of the matter shows in that, even for this synthetic system, there is a controversy between work suggesting an NEE phase [13–16] and other refuting it [17].

In this Letter, we present a first principles analytic description of NEE states in a deformed version of the Sachdev-Ye-Kitaev (SYK) model [22,23]. The standard SYK model is a system of $2N \gg 1$ Majorana fermions, $[\chi_i, \chi_j]_+ = 2\delta_{ij}$, governed by the interaction Hamiltonian

$$\hat{H}_0 = \frac{1}{4!} \sum_{i,j,k,l=1}^{2N} J_{ijkl} \hat{\chi}_i \hat{\chi}_j \hat{\chi}_k \hat{\chi}_l, \tag{1}$$

where the coupling constants are drawn from a Gaussian distribution, $\langle |J_{ijkl}|^2 \rangle = 6J^2/(2N)^3$, and the constant J defines the effective bandwidth of the system as $\gamma = (J/2)(2N)^{1/2}$ [24]. The model (1) is known to be in an ergodic phase with eigenfunctions uniformly distributed in Fock space [24,25]. To make the situation more interesting, we generalize the Hamiltonian to $\hat{H} = \hat{H}_0 + \hat{H}_V$, where

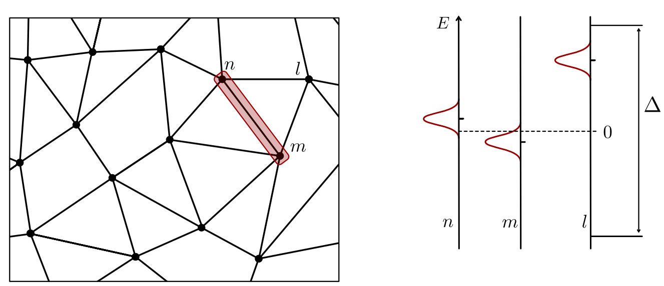

FIG. 1. Left: Cartoon of Fock space sites $n, m, l, \ldots$ (indicated by dots) connected by hopping operator $\mathcal{P}$ (solid lines). For $\Delta \gg 1$ exceeding the bandwidth of the unperturbed model, one may approach the problem perturbatively, i.e., taking the isolated eigenstates of levels v_n, v_m, v_l as a starting point. The hybridization leads to level broadening κ of resonant neighbors (indicated by hatched link) which have both energies v_n, $v_m \lesssim 1$ within the SYK band. Right side: Typical energy distributions of Fock-space neighbors connected by $\mathcal{P}$. The hybridization does, in general, not generate overlap between neighboring sites. For $\Delta < N^2$ wave functions are thus extended ($\rightarrow$ Wigner-Dyson statistics) yet confined to only a fraction $\sim 1/\Delta^2$ of the total Fock space.

$$\hat{H}_V = \gamma \sum_{n}^{D} v_n |n\rangle\langle n|, \tag{2}$$

is a sum over projectors onto the occupation number eigenstates $|n\rangle = |n_1, n_2, \ldots, n_N\rangle$, $n_i = 0, 1$, of a system of complex fermions $c_i = \frac{1}{2}(\chi_{2i-1} + i\chi_{2i})$, $i = 1, \ldots, N$ defined via the Majorana operators. The coefficients v_n can be chosen to represent any operator diagonal in the occupation number basis $\{|n\rangle\}$ pertaining to a fixed one-body basis. For example, any one-body operator [26,27] $\hat{H}_0 = \frac{1}{2}\sum_{i,j} J_{ij}\hat{\chi}_i\hat{\chi}_j$ can be diagonalized in the fermion representation and described in this way. However, for our discussion below it will be sufficient to consider realizations of maximal entropy with coefficients v_n drawn from a box distribution of width Δ symmetric around zero. In this way Δ sets the effective strength of the coupling in units of the SYK bandwidth, and in the limit of asymptotically large Δ enforces Fock space localization in states n with energies v_n. The Hamiltonian $\hat{H}_0$ perturbs this "Poisson limit" via transitions $|n\rangle \to |m\rangle$ between states nearby in Fock space. (The two-body $\hat{H}_0$ changes the occupation of a state $|n\rangle$ by at most four, and it preserves the number parity, where we focus on even parity states throughout.) It does so via only an algebraically small number $\sim N^4 \sim \ln(D)$ of independent matrix elements, and thus defines an operator with strong statistical correlation. However, we will see that $\hat{H}_0$ is very efficient in introducing many body chaos, as evidenced by the onset of Wigner-Dyson (WD) spectral statistics, including for values $\Delta \gg 1$, where the diagonal still dominates. Our main objective is to explore the profile of the many body wave functions in this setting.

Qualitative picture.—Before turning to the quantitative analysis of the problem, let us outline an intuitive picture of nonergodic wave function statistics. Let us work in dimensionless units, where the SYK bandwidth $1 \sim JN^{1/2}$ is set to unity, or $J \sim N^{-1/2}$. Consider a situation where the strength of the diagonals $\Delta \sim N^{\alpha}$, $\alpha > 0$ parametrically exceeds the bandwidth. In this case, we have a situation where the "hopping" in Fock space induced by the SYK Hamiltonian does not effectively hybridize the majority of the $\sim N^4$ states, $m, l, \ldots$, neighboring a given n, cf. Fig. 1. With the characteristic hopping amplitude $t \sim JN^{-3/2} \sim N^{-2}$, a self-consistent golden rule argument may be applied to estimate the residual smearing κ of n as $\kappa \sim |t|^2 [N^4(\kappa/\Delta)](1/\kappa) \sim (1/\Delta) \sim N^{-\alpha}$, where the term in parentheses is the number of neighbors that are in resonance, and $\sim\kappa^{-1}$ is the broadened energy denominator. The effective hybridization of two nearest neighbors requires overlap of their smeared levels, a condition satisfied only by a fraction $(\kappa/\Delta) \sim \Delta^{-2}$ of neighbors. From this argument we infer that typical wave functions occupy only a number $D/\Delta^2 \sim D/N^{2\alpha}$ of the available D sites in Fock space. We also note that for $N^4/\Delta^2 = N^{4-2\alpha} \sim 1$ the number of resonant neighboring levels becomes of $\mathcal{O}(1)$. This is when we expect the wave functions to fragment and a transition to the Poisson regime to take place.

Matrix integral representation.—To obtain a more quantitative picture, we start from a first quantized representation, where the Hamiltonian $\hat{H}$ is considered as a sparse matrix acting in a huge Fock space. This perspective is complementary to that of the more conventional many body $G\Sigma$ formalism [22] probing the physics of collective fluctuations close to the ground state. Formulated in this language, the problem becomes one of random matrix diagonalization and methods such as the powerful supersymmetry technique, originally designed to solve single particle hopping problems, become applicable. Specifically, the occupation number basis $\{|n\rangle\}$ plays a role analogous to the position basis of a fictitious quantum state and $\hat{H}_0$ and $\hat{H}_V$ act as hopping and "on-site potential" Hamiltonians, respectively. Within the first quantized approach, information on the statistics of the many body wave functions $|\psi\rangle$ at the band center, $\epsilon_\psi = 0$ (generalization to generic energies is straightforward but omitted for simplicity), is contained in the matrix elements of the resolvent, $G^{\pm}_{nn'} = \langle n|\pm i\delta - \hat{H})^{-1}|n'\rangle$. Specifically, the qth moment is defined as $I_q \equiv (1/\nu_0) \sum_n \langle |\langle\psi|n\rangle|^{2q}\delta(\epsilon_\psi)\rangle$, where $\langle\ldots\rangle$ denotes averaging over the randomness in the model, and $\nu_0 = \langle\sum_\psi \delta(\epsilon_\psi)\rangle$ is the density of states in the band center. Using the eigenfunction decomposition $G^{+}_{nn} = \sum_\psi |\langle\psi|n\rangle|^2(i\delta - \epsilon_\psi)^{-1}$, this can be expressed as $I_q = -(1/\pi\nu_0)\lim_{\delta\to 0}(2i\delta)^{q-1} \sum_n \mathrm{Im}G^{+}_{nn}G^{+(q-1)}_{nn}$ [28], where the last equality relies on the absence of degeneracies $E_\psi \neq E_{\psi'}$, for $\psi \neq \psi'$ in a disordered system. (For completeness, we apply the same setup to compute the eigen*value* statistics and diagnose Wigner-Dyson or Poisson statistics. See Supplemental Material [29] for details.) Our principal workhorse in computing the realization average of these expressions is an exact integral representation $\langle I_q\rangle = \partial_\beta\partial_\alpha^{q-1}\int dY e^{-S(Y,\alpha,\beta)}$. Here, the integration variables $Y = \{Y^{ss',\sigma\sigma'}_{nn'}\}$ are $2\times 2\times D$-dimensional matrices which on top of the Fock space index n contain an index $s, s' = \pm$ labeling advanced and retarded states, and a two-component index $\sigma, \sigma' = \mathrm{b}, \mathrm{f}$ distinguishing between commuting (Y^{bb}, Y^{ff}) and Grassmann valued (Y^{bf}, Y^{fb}) matrix blocks [31]. This "supermatrix structure" [29] is required to cancel unwanted fermion determinants appearing in the computation of purely commuting or anticommuting matrix integrals. (We cannot use replicas to achieve determinant cancellation because the analysis will involve one nonperturbative integration, not defined in the replica formalism.)

Referring for a derivation of the above integral, and the discussion of the source parameters α, β required to generate the wave function moments to the Supplemental Material [29], the action $S(Y) \equiv S(Y, 0, 0)$ of the field integral is given by

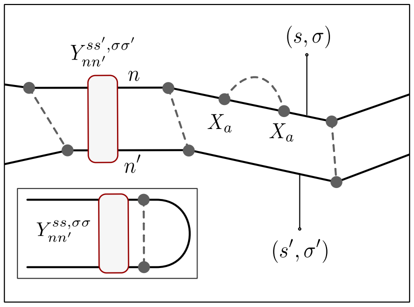

FIG. 2. The scattering of wave function amplitudes in Fock space. Variables $Y_{nn'}^{ss',\sigma\sigma'}$ describe the correlated propagation of resolvents (solid lines) labeled by a conserved index (s,σ). Scattering processes (indicated by dots) can be distinguished into those dressing propagators by "self energies" (dashed lines connecting same resolvent) and vertex contributions (dashed lines connecting different resolvents). Hatched regions summarize repeated, ladder-diagrams of vertex contributions and define the slow modes in the system. Inset: Self consistency equation for self-energy Eq. (4).

$$S(Y) = -\frac{1}{2}\mathrm{STr}(Y\mathcal{P}^{-1}Y) + \mathrm{STr}\ln(i\delta\sigma_3 - \hat{H}_V + i\gamma Y), \qquad (3)$$

where $\mathrm{STr}(X) \equiv \sum_{n,s,\sigma}(-)^\sigma X_{nn}^{ss,\sigma\sigma}$ is the canonical trace operation for supermatrices [32]. To understand the structure of the action, notice that the Green functions describe the propagation of wave functions subject to random scattering in Fock space. Contributions surviving the configuration average are correlated as indicated in Fig. 2. The first term in the action describes how the pair amplitudes $Y_{n,n'}^{ss',\sigma\sigma'}$ represent the propagation of two such states, specified by a doublet of indices (n,s,σ) and (n',s',σ'). It is defined by an operator $\mathcal{P}$, which acts as $\mathcal{P}Y \equiv (1/\mathcal{N})\sum_a X_a Y X_a^\dagger$, where $\mathcal{N} = \binom{2N}{4}$, i.e., the multiplication of the two states represented by Y by the Majorana product operators contained in the Hamiltonian, where $X_a \equiv \chi_i\chi_j\chi_k\chi_l$, and the shorthand $a = (i,j,k,l)$ is used. The second term couples the Y matrices to the fermion propagator effectively describing the propagation in between SYK-scattering events, where $(\sigma_3)^{ss'} = (-)^s\delta^{ss'}$ does the bookkeeping on causality.

Stationary phase approach.—Our strategy is to evaluate the matrix integral by stationary phase methods backed by excitation gaps present in the limit $\delta \to 0$. The structure of the action suggests looking for solutions of the stationary phase equations $\delta_{\bar{Y}}S(\bar{Y}) = 0$ diagonal in Fock space $Y_{nn'} = Y_n\delta_{nn'}$. Physically, this restriction means that for a fixed realization of the diagonals $v_n \neq 0$ phase coherence of the pair propagation requires $n = n'$ in the representation of Fig. 2. The stationary phase equation then assumes the form

$$Y_n = i\sum_m \Pi_{nm}\frac{1}{i\frac{\delta}{\gamma}\sigma_3 - v_m + iY_m}, \qquad (4)$$

where the projection of the pair-scattering operator $P_d\mathcal{P}P_d \equiv \Pi$ on the space of diagonal matrix configurations acts on diagonal configurations as $(\Pi X)_n = \sum_m \Pi_{nm}X_m = (1/\mathcal{N})\sum_{a,m}|(\hat{X}_a)_{mn}|^2 X_m$. The solution of the equation now essentially depends on the structure of this operator. We first note that the operators $\hat{X}_a$ change at most four of the N binary occupation numbers contained in n, implying that Π is a local hopping operator in the space of n states. The permutation symmetry inherent to the sum over all configurations $a = (i,j,k,l)$ further implies that the hopping strengths $\Pi_{nm} = \Pi_{|n-m|}$ depend only on the occupation number difference between Fock space states, where a straightforward counting procedure yields $\Pi_0 = N(N-1)/2\mathcal{N}$, $\Pi_2 = 4(N-2)/\mathcal{N}$, and $\Pi_4 = 16/\mathcal{N}$, and all other matrix elements vanish. Armored with this result, we interpret the right-hand side (r.h.s.) of the mean field equation Eq. (4) as a sum over a large number of terms, which are effectively random due to the presence of the coefficients v_m. In this way, $Y_n(v)$ becomes a random variable depending on the realizations $v = \{v_m\}$.

The structure of the mean field equation, and the transition rates Π_{nm} identifies the components Y_n^{ss} as the self energies dressing the retarded $(s = +)$ and advanced $(s = -)$ Fock space propagators (also cf. inset of Fig. 2.). The solutions Y_n are obtained as sums over large numbers of random contributions, which for small Δ implies a self averaging property $Y_n \simeq \langle Y_n\rangle_v \equiv Y_0$, where the r.h.s. denotes the average over the independent distribution over v_m. Ignoring the imaginary part of Y_n (which does no more than inducing a weak shift $v_n \to v_n + \mathrm{Im}Y_n \simeq v_n$ of the random energies), and averaging v over a box distribution $\langle\ldots\rangle_v = \prod_m\int_{-\Delta/2}^{\Delta/2}(dv_m/\Delta)(\ldots)$, we obtain $Y_0 = \kappa\sigma_3$, where the self energy κ obeys the equation $\kappa = (2/\Delta)\arctan(\Delta/2\kappa)$. The solution smoothly interpolates between $\kappa \simeq 1$ for the weakly perturbed model $\Delta \ll 1$ and $\kappa \simeq \pi/\Delta$ for $\Delta \gg 1$. In accordance with the qualitative discussion above, this decay reflects that for $\Delta \gg 1$ the majority of sites neighboring a fixed n are off resonant and decouple from the self energy. We also note that the averaged density of states $\nu_0 = -\mathrm{Im}\langle\mathrm{tr}(G^+)\rangle = D\kappa/\pi\gamma$ shows the same behavior. Before proceeding, let us ask when the above approximations break down and the stationary solutions become strongly fluctuating in the sense $\mathrm{var}(Y_n) > Y_0^2$. Assuming that $Y_m \simeq Y_0$ on the r.h.s. of Eq. (4), a straightforward calculation leads to $\mathrm{var}(Y_n) \simeq (10\pi/N^4\kappa^2)\mathcal{F}(\Delta/2\kappa)$, where $\mathcal{F}(x)$ is a function monotonically increasing from $\mathcal{F}(0) = 0$ to $\mathcal{F}(x) = \mathcal{O}(1)$ at $x \sim 1$ before decaying as $\mathcal{F}(x) \sim 1/x$ at $x \gg 1$ [33]. A balance $\mathrm{var}(Y_n) \sim Y_0^2$ is reached when $\kappa^2 \sim \Delta^{-2} \sim (1/N^4\kappa^2)(\kappa/\Delta) \sim N^{-4}$, where $\kappa \sim \Delta^{-1}$ was used. This shows that only for disorder strength $\Delta > \Delta_P \sim N^2$ parametrically larger than the bandwidth, the homogeneity of the stationary phase configuration in Fock space gets compromised. This observation is one of the most important results of this Letter. As we will demonstrate in the following, it provides the basis for the analytical extraction of wave functions and spectra.

Wave function statistics.—In the limit $\delta \to 0$, $Y_0 = \kappa\sigma_3$ is but one element of a manifold of stationary solutions, $Y_0 = \kappa T\sigma_3 T^{-1} \equiv \kappa Q$, where $T = \{T^{ss',\sigma\sigma'}\}$ is a 4×4 rotation matrix in advanced-retarded and super space. The absence of Fock-space indices implies $[\mathcal{P}, Q] = 0$, which in combination with $Q^2 = \mathbb{1}$ means that the first term in Eq. (3) is independent of T. We conclude that the stationary phase action of the matrix integral is given by

$$S[Q] = \mathrm{STr}\ \ln(i\delta\sigma_3 - \hat{H}_V + i\gamma\kappa Q). \qquad (5)$$

This action is known to describe [34] the Rosenzweig-Porter (RP) model [35]: a D-dimensional Gaussian random matrix ensemble perturbed by a fixed diagonal, $\hat{H}_V$. We thus conclude that for diagonals with $\Delta < \Delta_P$ the deformed SYK model and this much simpler model are in the same universality class. The first step of the computation of the wave function statistics [34] based on Eq. (5) is the integration over the matrix T. This integration is not innocent, because the 2×2 block T^{bb} defines a noncompact integration manifold [32]. The convergence of the corresponding integral is safeguarded only by the infinitesimal symmetry breaking parameter $i\delta$, and integration over T [29] indeed produces a singular factor δ^{-q+1} canceling the δ dependence in the definition of the wave function moments, and leading to the result

$$I_q = \frac{q!}{\nu_0^q}\sum_n \langle \nu(n)^q\rangle_v, \qquad \nu(n) \equiv \frac{\nu_0}{D(v_n^2+\kappa^2)}. \qquad (6)$$

Intuitively, the r.h.s. contains the qth moments of local Green's function matrix elements, with energy denominators broadened by the self energy κ. It is straightforward to average this expression over the box distribution of the individual v_n and obtain

$$I_q = -(-2)^q q D^{1-q}\partial_{y_0^2}^{q-1}(1/y_0\Delta)\arctan(\Delta/2y_0). \qquad (7)$$

For $\Delta \ll 1$ smaller than the SYK bandwidth, this asymptotes to the random matrix result $I(q) = q!(D/2)^{1-q}$, demonstrating a uniform state distribution. In the opposite case, $\Delta \gg 1$, $y_0 = \pi/\Delta$ and the moments $I_q = (2\pi^2)^{1-q}q(2q-3)!!\Delta^{2(q-1)}D^{1-q}$, show power law scaling in Δ. Finally, for $\Delta \sim N^\alpha$ the wave functions become nonergodic $I_q \sim [D/N^{2\alpha}]^{1-q}$, and now only occupy a $\sim 1/N^{2\alpha}$ fraction of Hilbert space, in line with the qualitative discussion above. In Fig. 3, these predictions are compared to wave function moments obtained by exact diagonalization for $N = 13$ as a function of the deformation parameter (main panel), or as a function of system size $N = 7, \ldots, 13$ at fixed deformation (lower left panel).

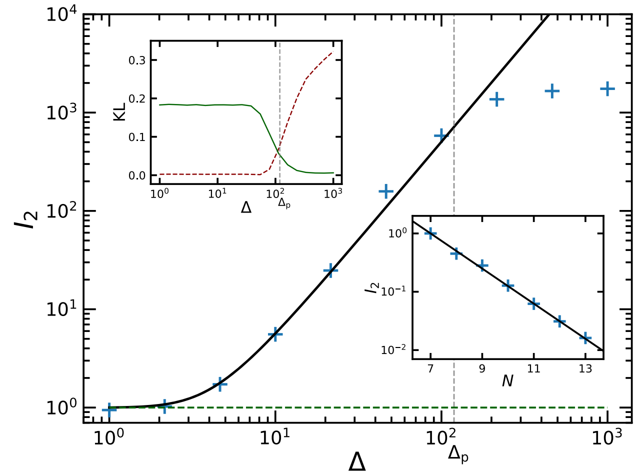

FIG. 3. Inverse participation ratio as a function of Δ, normalized by $I_2(\Delta = 0)$, from exact diagonalization $N = 13$; the analytical prediction Eq. (7) is indicated by the solid line. Left inset: Relative entropy (Kullback-Leibler) between numerical and Wigner Dyson (dashed), respectively, Poisson (solid) distributions. Right inset: Inverse participation ratio as a function of N, normalized by $I_2(N = 7)$, from exact diagonalization at $\Delta = 10$; solid line is the analytical prediction from Eq. (7) [36].

The figure demonstrates excellent, and parameter free agreement with the analytic result.

The figure also confirms the statement that throughout the entire window $\Delta < \Delta_P$, or $0 \le \alpha < 2$, the spectral statistics remains Wigner-Dyson like. This is probed by comparing the relative, or Kullback-Leibler entropies [37] $\mathrm{KL}(p|q) \equiv \sum_k p_k \ln(p_k/q_k)$ between the numerically obtained moments q_k and the Wigner-Dyson, or Poisson distribution p_k, respectively. The upper inset of Fig. 3 shows that the change between the two statistics takes place at the deformation strength analytically predicted as $\Delta \sim \Delta^P \simeq 120$, beyond which both saturation of the wave function moments [36], and the level statistics indicate Poissonian behavior.

Conceptually, the robustness of spectral correlations follows from the equivalence (SYK $\overset{\Delta<\Delta_P}{\sim}$ RP), the latter being a model demonstrating the strong resilience of a single random matrix against perturbations on its diagonal. The domain of the above equivalence is limited by both the deformation strength of SYK $\Delta \lesssim N^2$, and the width of the probed spectrum $\epsilon \lesssim \delta N^2$, where δ is the many body level spacing [38]. Outside this window, for $\Delta \gtrsim N^2$, the theory predicts a fragmentation of the Fock space homogeneous mean field (equivalent to the fluctuations of a single random matrix ensemble) into inhomogeneous stationary configurations, $\kappa \to \kappa_n$. On the background of this inhomogeneous configuration one may construct a lattice field theory that indeed predicts a Fock space localization transition at $\Delta \sim \Delta_P$ [39]. Finally, models of the perturbation different from the identically distributed v_n, lead to similar results. Specifically, a random one-body term, $\hat{H}_1 \equiv \sum_{j=1}^{N} \eta_{2j-1}\eta_{2j}v_j$ is equivalent to $\hat{H}_V$ with statistically correlated $v_n(\{v_j\})$. Referring to Ref. [39] for details, this leads to similar scaling over a slightly higher tolerance window, $\Delta_P \lesssim N^{9/4}$.

Summary and discussion.—The model considered in this Letter defines the perhaps simplest many body system showing a competition between Fock space localization and ergodicity. We are seeing unambiguous evidence that the passage between the two limits is not governed by a single many body localization transition but contains a parametrically extended intermediate phase characterized by a coexistence of Wigner-Dyson spectral statistics and non-trivial extension of wave functions over Fock space. Methodologically, this phenomenon emerged as the result of a competition: the hopping in Fock space generated by the SYK two-body interaction stabilized a uniform mean field against the "localizing" tendency of the Fock-space diagonal operator $\hat{H}_v$. We have identified an intermediate regime, where the corresponding low energy theory is governed by a homogeneous fluctuation mode T_0, acting on top of a background containing inhomogeneous energy denominators. This mechanism appears to be of a rather general nature and makes one suspect that nonergodic wave function statistics in coexistence with random matrix theory spectral correlations could be a more frequent phenomenon than previously thought.

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