Fluctuation Theorem Uncertainty Relation

Yoshihiko Hasegawa[*] and Tan Van Vu[†]

Department of Information and Communication Engineering, Graduate School of Information Science and Technology, The University of Tokyo, Tokyo 113-8656, Japan

(Received 3 March 2019; published 12 September 2019)

The fluctuation theorem is the fundamental equality in nonequilibrium thermodynamics that is used to derive many important thermodynamic relations, such as the second law of thermodynamics and the Jarzynski equality. Recently, the thermodynamic uncertainty relation was discovered, which states that the fluctuation of observables is lower bounded by the entropy production. In the present Letter, we derive a thermodynamic uncertainty relation from the fluctuation theorem. We refer to the obtained relation as the fluctuation theorem uncertainty relation, and it is valid for arbitrary dynamics, stochastic as well as deterministic, and for arbitrary antisymmetric observables for which a fluctuation theorem holds. We apply the fluctuation theorem uncertainty relation to an overdamped Langevin dynamics for an antisymmetric observable. We demonstrate that the antisymmetric observable satisfies the fluctuation theorem uncertainty relation but does not satisfy the relation reported for current-type observables in continuous-time Markov chains. Moreover, we show that the fluctuation theorem uncertainty relation can handle systems controlled by time-symmetric external protocols, in which the lower bound is given by the work exerted on the systems.

DOI: 10.1103/PhysRevLett.123.110602

Introduction.—During the last two decades, stochastic thermodynamics [1–3] accelerated the understanding of nonequilibrium systems through the discovery of several thermodynamic relations. Among them, the fluctuation theorem (see [4–12] and reviews [13,14]) is the central relation in nonequilibrium systems because this theorem leads to important thermodynamic relations, such as the second law of thermodynamics, the Green-Kubo relation [15], and the Jarzynski equality [16], to name a few. Recently, a remarkable relation between fluctuation and the entropy production was found, which is known as the thermodynamic uncertainty relation (TUR) [17–39]. The TUR states that the fluctuation of observables, such as the current, is lower bounded by the reciprocal of the entropy production. The proof of the TUR has been carried out using the large deviation principle [19–21,23–25,29,30,32], the fluctuation-response inequality [33,34], the Cramér-Rao inequality [35–38], and the linear response around equilibrium [17,39]. Although, as stated above, the fluctuation theorem can be used to derive many other thermodynamic relations, the relations between the TUR and the fluctuation theorem remain unclear. The universality of the fluctuation theorem leads us to posit that the TUR can be derived through the fluctuation theorem.

In the present Letter, we answer this question by obtaining the TUR for observables, which are antisymmetric under time reversal, from the fluctuation theorem. We refer to the obtained relation as the fluctuation theorem uncertainty relation (FTUR). Considering a detailed fluctuation theorem with respect to the entropy production and the observable, we derive the FTUR [see Eq. (10)]. As long as the fluctuation theorem holds, the FTUR is valid for arbitrary systems regardless of underlying dynamics and observables, and for arbitrary observation times. Notably, the FTUR holds for deterministic dynamical ensembles, which cannot be handled by the above-mentioned previous approaches. This is in contrast to existing TURs, which assume particular stochastic dynamics (mostly Markovian), and their proofs were given for each dynamics. The obtained results indicate that the TUR is a direct consequence of the fluctuation symmetry of the total entropy production. We apply the FTUR to the signum function of the current in an overdamped Langevin dynamics. We show that the signum function of the current does not satisfy the previously reported TUR [cf. Eq. (11)], which holds for a current-type observable in continuous-time Markov chains. Furthermore, the FTUR holds for systems controlled by time-symmetric external protocols. In particular, when the systems are initially in equilibrium, the FTUR holds, with the total entropy production replaced by the work exerted on the systems. As an example of the FTUR with external protocols, we consider an overdamped dragged Brownian particle.

Model.—We consider a system that is continuous in space and time and assume that its time evolution is governed by a Markov process. Although our description is based on continuous time and continuous space, generalizations to discrete time or discrete space are straightforward. We set the Boltzmann constant to unity. Let $x(t)$ be the position of the system at time t [$x(t)$ can be multidimensional], Γ a trajectory from $t = 0$ to $t = T$ ($T > 0$), $\Gamma \equiv [x(t)]_{t=0}^{t=T}$, and $\Gamma^\dagger$ its reversed trajectory, i.e., $\Gamma^\dagger \equiv [x(T-t)]_{t=0}^{t=T}$. The system (i.e., the transition rate) can depend on an external protocol $\lambda(t)$. In the ensemble level, the state of the system is depicted by $P(x,t)$, which is the probability density that the system is in x at time t. As is often considered in stochastic thermodynamics, we consider forward and reverse processes. We define $\mathcal{P}(\Gamma|x(0))$, the probability of observing a trajectory Γ in the forward process starting from $x(0)$ at $t = 0$, and $\mathcal{P}^\dagger(\Gamma^\dagger|x(T))$, the probability of observing a trajectory $\Gamma^\dagger$ in the reverse process starting from $x(T)$ at $t = T$. According to the local detailed balance assumption, the total entropy production $\sigma(\Gamma)$ satisfies [40] $\sigma(\Gamma) = \ln[\mathcal{P}(\Gamma)/\mathcal{P}^\dagger(\Gamma^\dagger)]$, where $\mathcal{P}(\Gamma) \equiv P(x(0),0)\mathcal{P}(\Gamma|x(0))$ and $\mathcal{P}^\dagger(\Gamma^\dagger) \equiv P(x(T),T)\mathcal{P}^\dagger(\Gamma^\dagger|x(T))$. Throughout the present Letter, we consider cases in which σ satisfies the (strong) detailed fluctuation theorem $P(\sigma)/P(-\sigma) = e^\sigma$. This condition is met when the system satisfies the following two conditions: (i) the initial and final probability distributions agree, $P(x,0) = P(x,T)$; and (ii) the external protocol is time symmetric, $\lambda(t) = \lambda(T-t)$ [14]. These conditions are typically satisfied by systems in a steady state or in a periodic steady state with the periodic protocol satisfying $\lambda(t) = \lambda(T-t)$. When (i) and (ii) are satisfied, $\mathcal{P}(\Gamma) = \mathcal{P}^\dagger(\Gamma)$. Moreover, satisfying (i) and (ii) implies that $\sigma(\Gamma)$ is antisymmetric under time reversal:

$$\sigma(\Gamma^\dagger) = -\sigma(\Gamma). \tag{1}$$

Let $\phi(\Gamma)$ be an observable that is a function of Γ. Similar to the total entropy production, we assume that $\phi(\Gamma)$ is antisymmetric under time reversal, i.e.,

$$\phi(\Gamma^\dagger) = -\phi(\Gamma). \tag{2}$$

As long as Eq. (2) holds, $\phi(\Gamma)$ can be an arbitrary function of Γ. The condition of Eq. (2) is typically satisfied by the current, but there exist many other quantities that can satisfy the condition.

Let $P(\sigma,\phi)$ be the probability that we observe the total entropy production σ and the observable ϕ in the forward process. From Eqs. (1) and (2), we can show that σ and ϕ obey the following strong detailed fluctuation theorem [41]:

$$\begin{aligned} P(\sigma,\phi) &= \int \mathcal{D}\Gamma\,\delta(\sigma - \sigma(\Gamma))\delta(\phi - \phi(\Gamma))\mathcal{P}(\Gamma) \\ &= e^\sigma \int \mathcal{D}\Gamma^\dagger \delta(\sigma + \sigma(\Gamma^\dagger))\delta(\phi + \phi(\Gamma^\dagger))\mathcal{P}(\Gamma^\dagger) \\ &= e^\sigma P(-\sigma,-\phi), \end{aligned} \tag{3}$$

where $\int \mathcal{D}\Gamma$ is the path integral.

We now derive the FTUR solely from Eq. (3). Reference [42] examined the statistical properties of entropy production from the fluctuation theorem. Inspired by Ref. [42], we introduce a probability density function $Q(\sigma,\phi)$ as follows:

$$Q(\sigma,\phi) \equiv (1 + e^{-\sigma})P(\sigma,\phi). \tag{4}$$

Here, $Q(\sigma,\phi)$ is normalized such that $\int_0^\infty d\sigma \times \int_{-\infty}^\infty d\phi Q(\sigma,\phi) = 1$, which directly follows from Eq. (3) and $\int_{-\infty}^\infty d\sigma = \int_{-\infty}^0 d\sigma + \int_0^\infty d\sigma$. Then, $\langle\phi\rangle$ can be represented as the expectation with respect to $Q(\sigma,\phi)$:

$$\begin{aligned} \langle\phi\rangle &\equiv \int_{-\infty}^\infty d\sigma \int_{-\infty}^\infty d\phi P(\sigma,\phi)\phi \\ &= \int_0^\infty d\sigma \int_{-\infty}^\infty d\phi P(\sigma,\phi)\phi(1 - e^{-\sigma}) \\ &= \left\langle \phi \tanh\left(\frac{\sigma}{2}\right)\right\rangle_Q, \end{aligned} \tag{5}$$

where $\langle\alpha(\sigma,\phi)\rangle_Q \equiv \int_0^\infty d\sigma \int_{-\infty}^\infty d\phi Q(\sigma,\phi)\alpha(\sigma,\phi)$ for the arbitrary function $\alpha(\sigma,\phi)$. Equation (5) holds for any observable $\phi(\Gamma)$ that is antisymmetric under time reversal [Eq. (2)]. Similarly, $\langle\sigma\rangle$ and $\langle\phi^2\rangle$ are

$$\langle\sigma\rangle \equiv \int_{-\infty}^\infty d\sigma \int_{-\infty}^\infty d\phi P(\sigma,\phi)\sigma = \left\langle \sigma \tanh\left(\frac{\sigma}{2}\right)\right\rangle_Q, \tag{6}$$

$$\langle\phi^2\rangle \equiv \int_{-\infty}^\infty d\sigma \int_{-\infty}^\infty d\phi P(\sigma,\phi)\phi^2 = \langle\phi^2\rangle_Q. \tag{7}$$

Applying the Cauchy-Schwarz inequality to Eq. (5), we obtain

$$\langle\phi\rangle^2 = \left\langle \phi \tanh\left(\frac{\sigma}{2}\right)\right\rangle_Q^2 \le \langle\phi^2\rangle_Q \left\langle \tanh\left(\frac{\sigma}{2}\right)^2\right\rangle_Q. \tag{8}$$

Next, we want to show the following series of inequalities:

$$\left\langle \tanh\left(\frac{\sigma}{2}\right)^2\right\rangle_Q \le \left\langle \tanh\left[\frac{\sigma}{2}\tanh\left(\frac{\sigma}{2}\right)\right]\right\rangle_Q \le \tanh\left(\frac{\langle\sigma\rangle}{2}\right). \tag{9}$$

In order to show the first inequality part in Eq. (9), we define $\Delta(\sigma) \equiv (\sigma/2)\tanh(\sigma/2) - \mathrm{atanh}[\tanh(\sigma/2)^2]$. We find that $\Delta(0) = 0$ and $\Delta'(\sigma) = (\sigma - \tanh(\sigma))/(2 + 2\cosh(\sigma)) \ge 0$ for $\sigma \ge 0$, which shows $\Delta(\sigma) \ge 0$ for $\sigma \ge 0$ [note that the integration of $\langle\cdots\rangle_Q$ with respect to σ is in $[0,\infty)$, and thus we only have to consider the $\sigma \ge 0$ domain]. Since $\tanh(\sigma)$ is a strictly increasing function, we prove the first inequality in Eq. (9). The second inequality part in Eq. (9) can be proved as follows. Since $\tanh(\sigma)$ is a concave function for $\sigma \ge 0$, by using the Jensen inequality, we find $\langle\tanh[(\sigma/2)\tanh(\sigma/2)]\rangle_Q \le \tanh[\frac{1}{2}\langle\sigma\tanh(\sigma/2)\rangle_Q]$, which proves the second inequality part in Eq. (9) by using

Eq. (6). Combining Eqs. (7)–(9), we obtain $\langle\phi^2\rangle/\langle\phi\rangle^2 \geq \tanh(\langle\sigma\rangle/2)^{-1}$, which yields

$$\frac{\mathrm{Var}[\phi]}{\langle\phi\rangle^2} \geq \frac{2}{e^{\langle\sigma\rangle}-1}. \tag{10}$$

Here, $\mathrm{Var}[\phi] \equiv \langle\phi^2\rangle - \langle\phi\rangle^2$ is the variance of ϕ. We refer to Eq. (10) as the FTUR, which is the main result of the present Letter.

We make some remarks on Eq. (10). Equation (10) is valid for arbitrary dynamics as long as the fluctuation theorem of Eq. (3) holds. Therefore, Eq. (10) can be applied to continuous-time, as well as discrete-time, Markov chains. Indeed, the expression of Eq. (10) is equivalent to the bound obtained for discrete-time Markov chains [24]. The bound of Eq. (10) is always smaller than that of the well-known TUR

$$\frac{\mathrm{Var}[\phi]}{\langle\phi\rangle^2} \geq \frac{2}{\langle\sigma\rangle}, \tag{11}$$

which is valid for continuous-time Markov chains. Discrete-time Markov chains do not satisfy Eq. (11) [24,43]. The bound of Eq. (11) has been proven for current-type observables and for the first-passage time (the former case was proven for a finite-time case). Still, as will be demonstrated, Eq. (11) is not satisfied even in continuous-time Markov chains when we consider an observable other than the current.

The quantity $\phi(\Gamma)$ can be arbitrary as long as Eq. (2) holds. This condition is typically satisfied by the current but can be satisfied by other quantities as well. Let $\jmath(\Gamma)$ be the current, which can of course satisfy $\jmath(\Gamma^\dagger) = -\jmath(\Gamma)$. Then any observable $h(\jmath(\Gamma))$, where $h(x)$ is an arbitrary odd function, satisfies $h(\jmath(\Gamma^\dagger)) = -h(\jmath(\Gamma))$, and thus the FTUR holds for $h(\jmath(\Gamma))$ (this case is considered in the example section). Moreover, σ can be an observable other than the total entropy production. Although, for clarity, we have assumed that σ is the total entropy production in Eq. (3), any set of observables σ and ϕ that satisfy the fluctuation theorem of Eq. (3) admit the FTUR of Eq. (10). In particular, we can obtain the FTUR for which the thermodynamic cost is the work exerted on the system. Suppose that the initial distributions for both the forward and reverse processes are equilibrium distributions. Then, w, the work exerted on the system, satisfies the Crooks work relation [9,44] $P(w)/P^\dagger(-w) = e^{(w-\Delta F)/\mathcal{T}}$, where $\mathcal{T}$ is the temperature, $P^\dagger(-w)$ is the probability of observing $-w$ in the reverse process, and ΔF is the free energy difference between equilibrium distributions corresponding to $\lambda(T)$ and $\lambda(0)$. Furthermore, when a symmetric external protocol $\lambda(t) = \lambda(T-t)$ is applied, the forward and reverse processes are indistinguishable, and the free energy difference vanishes, $\Delta F = 0$, resulting in $P(w)/P(-w) = e^{w/\mathcal{T}}$. Therefore, under these conditions, any observables $\phi(\Gamma)$ satisfying Eq. (2) obey the following FTUR:

$$\frac{\mathrm{Var}[\phi]}{\langle\phi\rangle^2} \geq \frac{2}{e^{\langle w\rangle/\mathcal{T}}-1}. \tag{12}$$

Thus far, we have been concerned with stochastic systems. Historically, the fluctuation theorem was first demonstrated on deterministic dynamical ensembles [4]. We can show that the FTUR also holds in such deterministic systems. Consider an N-particle system, where $\boldsymbol{q}_i(t)$ and $\boldsymbol{p}_i(t)$ denote the coordinates and the momenta of the ith particle at time t. Let $\Gamma(t) \equiv [\boldsymbol{q}(t), \boldsymbol{p}(t)] \equiv [\boldsymbol{q}_1(t), \ldots, \boldsymbol{q}_N(t), \boldsymbol{p}_1(t), \ldots, \boldsymbol{p}_N(t)]$ be a point in a phase space at time t, and let $\rho(\Gamma, t)$ be the distribution function of the phase space at time t. The time evolution of $\Gamma(t)$ is governed by the deterministic differential equation of $\dot{\Gamma}$ (the overdot denotes the time derivative), which is assumed to be time reversible so that the conjugate dynamics exists. We assume that the initial ensemble ($t=0$) obeys a given distribution (e.g., equilibrium distribution), and for $t>0$, a constant field is applied to the system. We define a dissipation from $t=0$ to $t=T$ as $\Sigma \equiv \ln(\rho(\Gamma(0),0)/\rho(\Gamma(T),0)) - \int_0^T \Upsilon(\Gamma(s))ds$ [10], where $\Upsilon(\Gamma) \equiv (\partial/\partial\Gamma)\cdot\dot{\Gamma}$ is the phase space compression factor [45] (the dot "$\cdot$" denotes the inner product). It is known that Σ satisfies the fluctuation theorem, $P(\Sigma)/P(-\Sigma) = e^{\Sigma}$, under mild conditions on the initial ensemble and dynamics [10,45]. Analogous to Eq. (2), we consider an arbitrary observable Φ defined from $t=0$ to $t=T$, which is assumed to be antisymmetric under time reversal. Extending the derivation of Ref. [10], we can show that the fluctuation theorem $P(\Sigma,\Phi)/P(-\Sigma,-\Phi) = e^{\Sigma}$ holds in the deterministic dynamical ensembles, which indicates the satisfaction of the FTUR (see [46] for details of the derivation and numerical verification).

We next discuss the equality condition of Eq. (10). When the equality is attained in both Eqs. (8) and (9), the equality of the FTUR is satisfied. According to the equality condition of the Cauchy-Schwarz inequality, the equality of Eq. (8) is satisfied only when $\phi \propto \tanh(\sigma/2)$. The first inequality part in Eq. (9) becomes equality only at $\sigma = 0$. From the equality condition of the Jensen inequality, the second inequality part in Eq. (9) saturates only when $\tanh(\sigma)$ is a linear function, which is asymptotically achieved for $\sigma \to 0$. Combining all of these conditions, we find that the equality of Eq. (10) is asymptotically satisfied if and only if $\phi \propto \sigma$ and $\sigma \to 0$. When $\sigma \to 0$, the system reduces to equilibrium. It has been reported that the total entropy production satisfies the equality of the TUR near equilibrium [19,26,35,39], which agrees with our equality condition.

Example 1.—We apply the FTUR to an overdamped particle on a ring [Fig. 1(a)], which has been extensively investigated in the literature [49,50]. Without loss of generality, we assume that the circumference of the ring is 1. We consider

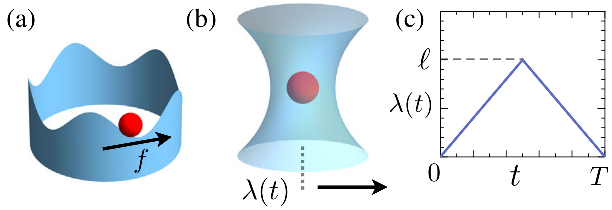

FIG. 1. Models considered in examples 1 and 2. (a) Particle on a ring topology with the drift term $A(x)$ in example 1. (b) Dragged Brownian particle, where the potential is manipulated by a protocol $\lambda(t)$ in example 2. (c) Protocol $\lambda(t)$, defined by Eq. (15), as a function of t, which is applied to the dragged Brownian particle shown in diagram (b). Here, ℓ denotes the height of the protocol.

$$\dot{x} = A(x) + \sqrt{2D}\xi(t), \tag{13}$$

where $A(x)$ is a periodic drift function ($A(x) = A(x+1)$), $D > 0$ is the noise intensity, and $\xi(t)$ is the white Gaussian noise with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = \delta(t-t')$. Let $P(x,t)$ be the probability density of x at time t. The Fokker-Planck equation (13) is $\partial_t P(x,t) = -\partial_x J(x,t)$, where $J(x,t) \equiv A(x)P(x,t) - D\partial_x P(x,t)$ is the probability current. We use the following generalized current: $\jmath(\Gamma) \equiv \int_0^T \Lambda(x) \circ \dot{x} dt$, where $\circ$ is the Stratonovich product, and $\Lambda(x)$ is a projection function. We consider observables defined by $\phi_{\mathrm{sgn}}(\Gamma) \equiv \mathrm{sign}(\jmath(\Gamma))$, where $\mathrm{sign}(x)$ is the signum function. Note that ϕ_{sgn} simply returns the sign of $\jmath$. Since $\mathrm{sign}(x)$ is an odd function, ϕ_{sgn} obeys the FTUR of Eq. (10).

We explicitly calculate $\mathrm{Var}[\phi_{\mathrm{sgn}}]$ and $\langle \phi_{\mathrm{sgn}} \rangle$. We consider $A(x) = f$, where f is a constant force applied to the particle. Since we consider a ring of circumference of 1, $P(x,t) \to 1$ for $t \to \infty$. Therefore, the steady-state current is $J^{\mathrm{ss}} = f$. We use $\Lambda(x) = 1$ in $\jmath(\Gamma)$, with which the current simply gives the position at time $t = T$ on the infinite line, $\jmath(\Gamma) = x(T) - x(0)$. Furthermore, on the infinite line, $P(x,t)$ is a Gaussian distribution with the mean ft and the variance $2Dt$ when $x(0) = 0$ [51]. Let $\mathbb{P}(\phi_{\mathrm{sgn}}, t)$ be the probability of $\phi_{\mathrm{sgn}} \in \{-1, 1\}$ at time t, which is expressed by $\mathbb{P}(1,t) = \frac{1}{2}\{1 + \mathrm{erf}[(f/2)\sqrt{t/D}]\}$ and $\mathbb{P}(-1,t) = \frac{1}{2}\{1 - \mathrm{erf}[(f/2)\sqrt{t/D}]\}$. The explicit expression of $\mathbb{P}(\phi_{\mathrm{sgn}}, t)$ confers $\mathrm{Var}[\phi_{\mathrm{sgn}}]/\langle \phi_{\mathrm{sgn}} \rangle^2 = -1 + \mathrm{erf}(f\sqrt{t/D}/2)^{-2}$. Since the entropy production from $t = 0$ to $t = T$ is given by $\langle \sigma \rangle = T \int_0^1 dx D^{-1} A(x) J^{\mathrm{ss}} = Tf^2/D$, we obtain [46]

$$\frac{\mathrm{Var}[\phi_{\mathrm{sgn}}]}{\langle \phi_{\mathrm{sgn}} \rangle^2} = -1 + \mathrm{erf}\left(\frac{\sqrt{\langle \sigma \rangle}}{2}\right)^{-2}. \tag{14}$$

The right-hand side of Eq. (14) is larger than the lower bound of Eq. (10), $-1 + \mathrm{erf}(\sqrt{\langle \sigma \rangle}/2)^{-2} \geq 2/[e^{\langle \sigma \rangle} - 1]$. This relation is obvious when evaluating both sides numerically, but we provide a proof in [46].

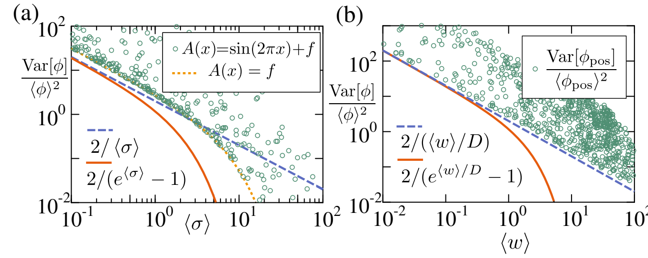

FIG. 2. $\mathrm{Var}[\phi]/\langle \phi \rangle^2$ and the lower bounds of the TUR and FTUR as a function $\langle \sigma \rangle$ (or $\langle w \rangle$). The lower bounds of Eqs. (10) and (11) are depicted by solid and dashed lines, respectively. (a) Results of the particle on a ring in example 1. The dotted line and circles denote $\mathrm{Var}[\phi_{\mathrm{sgn}}]/\langle \phi_{\mathrm{sgn}} \rangle^2$ for $A(x) = f$ and $A(x) = \sin(2\pi x) + f$, respectively, where f, T, and D are randomly selected from $f \in [0.01, 3.0]$, $T \in [0.1, 3.0]$, and $D \in [0.01, 1.0]$. (b) Results of the dragged Brownian particle in example 2. Circles denote $\mathrm{Var}[\phi_{\mathrm{pos}}]/\langle \phi_{\mathrm{pos}} \rangle^2$ for randomly selected β, ℓ, and T with $D = 1$. Note that β, ℓ, and T are selected as $\beta \in [0.01, 10.0]$, $\ell \in [0.01, 10.0]$, and $T \in [0.01, 10.0]$.

We plot $\mathrm{Var}[\phi_{\mathrm{sgn}}]/\langle \phi_{\mathrm{sgn}} \rangle^2$ [Eq. (14)] in Fig. 2(a) for $A(x) = f$ and $A(x) = \sin(2\pi x) + f$. For $A(x) = f$, Eq. (14) is depicted by a dotted line, and the lower bounds of Eqs. (10) and (11) are shown by solid and dashed lines, respectively. Although Eq. (14) is larger than the bound of Eq. (10), it does not satisfy Eq. (11), which indicates that the continuous TUR [Eq. (11)] does not generally hold for quantities that are antisymmetric under time reversal. We check the inequality for $A(x) = \sin(2\pi x) + f$, which has a non-Gaussian distribution, with computer simulation. We randomly select f, T, and D, and calculate $\mathrm{Var}[\phi_{\mathrm{sgn}}]/\langle \phi_{\mathrm{sgn}} \rangle^2$ and $\langle \sigma \rangle$ for the selected parameter values as the average of 10^6 trajectories [the range of the parameters is shown in the caption of Fig. 2(a)], and the realizations are shown by circles in Fig. 2(a). We can see that $\mathrm{Var}[\phi_{\mathrm{sgn}}]/\langle \phi_{\mathrm{sgn}} \rangle^2$ for $A(x) = \sin(2\pi x) + f$ is larger than the result of $A(x) = f$, indicating that the case of $A(x) = f$ appears to be the lower bound case of this particular example. We again confirm that the conventional TUR is not satisfied for larger $\langle \sigma \rangle$.

Example 2.—Next, we consider an overdamped dragged Brownian particle [Fig. 1(b)] [52,53] to test the FTUR of Eq. (12). The dragged Brownian particle is important in stochastic thermodynamics from both theoretical and experimental viewpoints. We consider the following Langevin equation: $\dot{x} = -\partial_x U(x, \lambda(t)) + \sqrt{2D}\xi(t)$, where $U(x, \lambda) \equiv \beta(x - \lambda)^2/2$ is a potential function ($\beta > 0$ is a model parameter), $\lambda(t)$ is an external protocol, and $\xi(t)$ and D are the same as in Eq. (13). We consider a time-symmetric protocol defined by

$$\lambda(t) = \begin{cases} \frac{2\ell}{T} t & 0 \leq t < \frac{T}{2} \\ -\frac{2\ell}{T} t + 2\ell & \frac{T}{2} \leq t \leq T, \end{cases} \tag{15}$$

where ℓ is the height of the signal [Fig. 1(c)]. Note that $\lambda(t)$ of Eq. (15) satisfies time symmetry, $\lambda(T-t)=\lambda(t)$. Suppose that the system is in equilibrium at $t=0$, $P(x,0)=P^{\mathrm{eq}}(x,\lambda(0))$, where $P^{\mathrm{eq}}(x,\lambda)\equiv \mathcal{N}\exp(-U(x,\lambda)/D)$ is the equilibrium distribution corresponding to λ ($\mathcal{N}$ is a normalization constant). We consider an observable $\phi_{\mathrm{pos}}(\Gamma)\equiv \jmath(\Gamma)$ with $\Lambda(x)=1$. Here, $\phi_{\mathrm{pos}}(\Gamma)$ simply gives the position of the particle at time $t=T$, $\phi_{\mathrm{pos}}(\Gamma)=x(T)-x(0)$. Based on these assumptions, $\phi_{\mathrm{pos}}(\Gamma)$ satisfies the FTUR given by Eq. (12), with $\mathcal{T}$ replaced by D. In the dragged Brownian particle, the work w exerted on the particle is given by [2] $w(\Gamma)=\int_0^T dt\partial_\lambda U(x,\lambda)\dot{\lambda}$. Since the probability density $P(x,t)$ is a Gaussian distribution for all t and $\lambda(t)$ is a piecewise linear function [Eq. (15)], $\mathrm{Var}[\phi_{\mathrm{pos}}]/\langle\phi_{\mathrm{pos}}\rangle^2$ and $\langle w\rangle$ can be calculated analytically [46]:

$$\frac{\mathrm{Var}[\phi_{\mathrm{pos}}]}{\langle\phi_{\mathrm{pos}}\rangle^2}=\frac{D\beta T^2(1-e^{-\beta T})}{2\ell^2(e^{-\beta T}-2e^{-\beta T/2}+1)^2}, \tag{16}$$

$$\langle w\rangle=\frac{4\ell^2(-e^{-\beta T}+\beta T+4e^{-\beta T/2}-3)}{\beta T^2}. \tag{17}$$

We randomly select β, T, and ℓ with $D=1$, and calculate $\mathrm{Var}(\phi_{\mathrm{pos}})/\langle\phi_{\mathrm{pos}}\rangle^2$ and the average work $\langle w\rangle$ for the selected parameter values [the range of the parameters is shown in the caption of Fig. 2(b)]. We repeat this calculation many times and plot $\mathrm{Var}(\phi_{\mathrm{pos}})/\langle\phi_{\mathrm{pos}}\rangle^2$ as a function of $\langle w\rangle$ in Fig. 2(b). Here, $2/[e^{\langle w\rangle/D}-1]$ and $2/(\langle w\rangle/D)$ are depicted by solid and dashed lines, respectively. We can confirm that all the realizations (circles) are above the bound of Eq. (12), indicating that Eq. (12) holds for the dragged Brownian particle. Still, we can see that all the realizations are even above $2/(\langle w\rangle/D)$ (dashed line). This tighter bound is an analogue of Eq. (11) for the system subject to the external protocol. Indeed, we can prove that $\mathrm{Var}[\phi_{\mathrm{pos}}]/\langle\phi_{\mathrm{pos}}\rangle^2\geq 2/(\langle w\rangle/D)$, and this inequality saturates when $\beta T\to 0$ [46]. This result induces us to conjecture that the FTUR of Eq. (12) has this tighter bound for general continuous-time systems with equilibrium initial distributions and time-symmetric external protocols.

We also tested the FTUR for a discrete-time random walk on a ring with an observable counting the number of laps, and we confirmed that the FTUR is satisfied for this system (see [46]).

Conclusion.—In the present Letter, we have derived the FTUR solely from the fluctuation theorem with respect to the total entropy production and the observable, which is antisymmetric under time reversal. Although the bound of the FTUR is weaker than that of the conventional TUR, the FTUR is general in the sense that it can handle systems that have not been covered by the previously reported TURs. Since the fluctuation theorem is the central relation in nonequilibrium thermodynamics, the present study can be a basis for obtaining other thermodynamic bounds.

We would like to thank Amos Maritan for providing valuable comments. The present study was supported by Ministry of Education, Culture, Sports, Science and Technology (MEXT) KAKENHI Grants No. JP16K00325 and No. JP19K12153.

*hasegawa@biom.t.u-tokyo.ac.jp
†tan@biom.t.u-tokyo.ac.jp

[1] F. Ritort, in *Advances in Chemical Physics*, Vol. 137, edited by S. A. Rice (Wiley, New York, 2008), pp. 31–123.
[2] U. Seifert, Rep. Prog. Phys. **75**, 126001 (2012).
[3] C. Van den Broeck and M. Esposito, Physica (Amsterdam) **418A**, 6 (2015).
[4] D. J. Evans, E. G. D. Cohen, and G. P. Morriss, Phys. Rev. Lett. **71**, 2401 (1993).
[5] G. Gallavotti and E. G. D. Cohen, Phys. Rev. Lett. **74**, 2694 (1995).
[6] D. J. Searles and D. J. Evans, Phys. Rev. E **60**, 159 (1999).
[7] J. Kurchan, J. Phys. A **31**, 3719 (1998).
[8] J. L. Lebowitz and H. Spohn, J. Stat. Phys. **95**, 333 (1999).
[9] G. E. Crooks, Phys. Rev. E **60**, 2721 (1999).
[10] D. J. Searles and D. J. Evans, J. Chem. Phys. **113**, 3503 (2000).
[11] C. Jarzynski, J. Stat. Phys. **98**, 77 (2000).
[12] P. Gaspard, J. Chem. Phys. **120**, 8898 (2004).
[13] R. J. Harris and G. M. Schüz, J. Stat. Mech. (2007) P07020.
[14] R. Spinney and I. Ford, Fluctuation relations: A pedagogical overview, in *Nonequilibrium Statistical Physics of Small Systems* (John Wiley & Sons, New York, 2013), Chap. 1, pp. 3–56.
[15] D. J. Searles and D. J. Evans, J. Chem. Phys. **112**, 9727 (2000).
[16] C. Jarzynski, Phys. Rev. Lett. **78**, 2690 (1997).
[17] A. C. Barato and U. Seifert, Phys. Rev. Lett. **114**, 158101 (2015).
[18] A. C. Barato and U. Seifert, J. Phys. Chem. B **119**, 6555 (2015).
[19] T. R. Gingrich, J. M. Horowitz, N. Perunov, and J. L. England, Phys. Rev. Lett. **116**, 120601 (2016).
[20] M. Polettini, A. Lazarescu, and M. Esposito, Phys. Rev. E **94**, 052104 (2016).
[21] P. Pietzonka, A. C. Barato, and U. Seifert, Phys. Rev. E **93**, 052145 (2016).
[22] C. Hyeon and W. Hwang, Phys. Rev. E **96**, 012156 (2017).
[23] J. M. Horowitz and T. R. Gingrich, Phys. Rev. E **96**, 020103 (R) (2017).
[24] K. Proesmans and C. V. den Broeck, Europhys. Lett. **119**, 20001 (2017).
[25] P. Pietzonka, F. Ritort, and U. Seifert, Phys. Rev. E **96**, 012101 (2017).
[26] S. Pigolotti, I. Neri, E. Roldán, and F. Jülicher, Phys. Rev. Lett. **119**, 140604 (2017).
[27] J. P. Garrahan, Phys. Rev. E **95**, 032134 (2017).
[28] A. Dechant and S.-i. Sasa, J. Stat. Mech. (2018) 063209.
[29] A. C. Barato, R. Chetrite, A. Faggionato, and D. Gabrielli, New J. Phys. **20**, 103023 (2018).

[30] T. Koyuk, U. Seifert, and P. Pietzonka, J. Phys. A **52**, 02LT02 (2019).
[31] W. Hwang and C. Hyeon, J. Phys. Chem. Lett. **9**, 513 (2018).
[32] H. Vroylandt, D. Lacoste, and G. Verley, J. Stat. Mech. (2019), 054002.
[33] A. Dechant and S.-i. Sasa, arXiv:1804.08250.
[34] I. D. Terlizzi and M. Baiesi, J. Phys. A **52**, 02LT03 (2019).
[35] Y. Hasegawa and T. Van Vu, Phys. Rev. E **99**, 062126 (2019).
[36] A. Dechant, J. Phys. A **52**, 035001 (2019).
[37] S. Ito and A. Dechant,arXiv:1810.06832.
[38] T. V. Vu and Y. Hasegawa, arXiv:1901.05715.
[39] K. Macieszczak, K. Brandner, and J. P. Garrahan, Phys. Rev. Lett. **121**, 130601 (2018).
[40] U. Seifert, Phys. Rev. Lett. **95**, 040602 (2005).
[41] R. García-García, D. Domínguez, V. Lecomte, and A. B. Kolton, Phys. Rev. E **82**, 030104(R) (2010).
[42] N. Merhav and Y. Kafri, J. Stat. Mech. (2010) P12022.
[43] N. Shiraishi, arXiv:1706.00892.
[44] G. E. Crooks, Phys. Rev. E **61**, 2361 (2000).
[45] D. J. Evans and G. Morriss, *Statistical Mechanics of Nonequilibrium Liquids* (Cambridge University Press, Cambridge, England, 2008).
[46] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.123.110602 for details of calculations, which includes Refs. [47,48].
[47] G. P. Morriss and C. P. Dettmann, Chaos **8**, 321 (1998).
[48] H. C. Tuckwell and J. Jost, Physica (Amsterdam) **388A**, 4115 (2009).
[49] T. Speck and U. Seifert, Europhys. Lett. **74**, 391 (2006).
[50] C. Van den Broeck and M. Esposito, Phys. Rev. E **82**, 011144 (2010).
[51] H. Risken, *The Fokker–Planck Equation: Methods of Solution and Applications*, 2nd ed. (Springer, New York, 1989).
[52] G. M. Wang, E. M. Sevick, E. Mittag, D. J. Searles, and D. J. Evans, Phys. Rev. Lett. **89**, 050601 (2002).
[53] R. van Zon and E. G. D. Cohen, Phys. Rev. E **67**, 046102 (2003).