Resource Theory of Coherence Based on Positive-Operator-Valued Measures

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Quantum coherence is a fundamental feature of quantum mechanics and an underlying requirement for most quantum information tasks. In the resource theory of coherence, incoherent states are diagonal with respect to a fixed orthonormal basis; i.e., they can be seen as arising from a von Neumann measurement. Here, we introduce and study a generalization to a resource theory of coherence defined with respect to the most general quantum measurements, i.e., to arbitrary positive-operator-valued measures (POVMs). We establish POVM-based coherence measures and POVM-incoherent operations that coincide for the case of von Neumann measurements with their counterparts in standard coherence theory. We provide a semidefinite program that allows us to characterize interconversion properties of resource states and exemplify our framework by means of the qubit trine POVM, for which we also show analytical results.

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Quantum resource theories (QRTs) [1–3] provide a structured framework in which quantum properties such as entanglement, coherence, and purity are described in a quantitative way. Every QRT is based on the notions of free states (which do not contain the resource) and free operations (which cannot create the resource). Building on these basic constituents, QRTs allow us to determine the resource content in quantum states, the optimal distillation of the resource, and the possibility of interconversion between resource states via free operations.

In recent years, the resource theory of quantum coherence has received much attention [4–7]. In the standard resource theory of coherence, the free states or incoherent states are states that are diagonal in a fixed orthonormal basis of a d-dimensional Hilbert space $\mathcal{H}$. Incoherent states ρ_I can thus also be seen as arising from a von Neumann measurement $\mathbf{P} = \{P_i\}$ in this basis, i.e., $\rho_I = \sum_i^d P_i \sigma P_i$ for some state $\sigma \in \mathcal{S}$, where $\mathcal{S}$ denotes the set of quantum states on $\mathcal{H}$, and the measurement operators P_i are mutually orthogonal rank-one projectors that form a complete set, $\sum_i^d P_i = \mathbb{1}$. Coherent states are those that are not of the above form. This notion of coherence has been generalized in two directions. In [8–10], a resource theory of superposition was studied, where the requirement of orthogonality of the basis was lifted. In [11], Åberg proposed a framework that can be seen as the definition of coherence with respect to a general projective measurement, where the orthogonal measurement operators P_i may be of higher rank. In this generalized resource theory of coherence, the free states are block diagonal.

It is an important question whether the notion of coherence as an intrinsic quantum property of states can be further extended and formulated with respect to the most general quantum measurements, i.e., positive-operator-valued measures (POVMs). In this Letter, we answer this question in the affirmative by introducing a resource theory of quantum state coherence based on arbitrary POVMs. More precisely, we establish a *family* of POVM-based resource theories of coherence, as each POVM leads to a different resource theory. In the special case of rank-one orthogonal projective measurements, our theory coincides with standard coherence theory. Note that our approach is distinct from the mentioned previous generalizations [8–10] in terms of free states and operations. A motivation for our work is the fact that POVMs are generally advantageous compared to projective measurements, see [12] for a survey. In addition, we show in [13] that coherence of a state with respect to a POVM can be interpreted as the cryptographic randomness generated by measuring the POVM on the state. That is, the amount of POVM coherence in a state is equal to the unpredictability of measurement outcomes relative to an eavesdropper with maximal information about the state, generalizing results from [14].

For a POVM-based coherence theory, the first challenge is to identify a meaningful notion of free, POVM-incoherent states. This is achieved via the Naimark theorem [15,16], which states that any POVM can be extended to a projective measurement in a larger space. Our concept of POVM coherence of states in $\mathcal{S}$ is linked to a generalized resource theory of coherence from [11] in the extended (Naimark) space, for which we denote the set of states as $\mathcal{S}'$. POVM coherence can be interpreted as the coherence resource that is required to implement the POVM on a given state via the canonical Naimark extension. The latter is realized by coupling the state to a probe, performing a global unitary, and measuring the probe. This is relevant, as POVMs are usually implemented in this way in experiments [17–19]. If one views the probe as a measurement apparatus, POVM-based coherence is the bipartite coherence generated in the global state by this process.

Conceptually, our Letter describes a novel way to construct resource theories. Quantum states and operations from the system space are embedded into a larger space that is equipped with a resource theory, providing a derivated resource theory on the original space. For this reason, our work does not follow the standard construction method for a resource theory: Our starting point is the definition of a POVM-based coherence measure, from which we construct free states and operations. We then provide a semidefinite program that characterizes all POVM-incoherent operations, making them accessible for efficient numerical computation. Finally, we apply our framework to the example of the qubit trine POVM, for which we study the coherence measure and characterize all incoherent unitaries.

In the following, we present our main results and their interpretation. Technical details and proofs from every section of the main text are provided in the corresponding section of the Supplemental Material [20], which includes Refs. [21–33].

POVM and Naimark extension.—A POVM on $\mathcal{H}$ with n outcomes is a set $\mathbf{E} = \{E_i\}_{i=1}^n$ of positive semidefinite operators $E_i \geq 0$, called effects, which satisfy $\sum_i^n E_i = \mathbb{1}$. The probability to obtain the ith outcome when measuring ρ is given by $p_i(\rho) = \mathrm{tr}[E_i\rho]$. We denote by $\{A_i\}$ a set of measurement operators of $\mathbf{E}$, i.e., $E_i = A_i^\dagger A_i$. Each measurement operator A_i is only fixed up to a unitary U_i, as the transformation $A_i \mapsto U_i A_i$ leaves E_i invariant. The ith postmeasurement state for a given A_i is $\rho_i = (1/p_i)A_i\rho A_i^\dagger$.

Let us remind the reader that, according to the Naimark theorem [15,16], every POVM $\mathbf{E} = \{E_i\}_{i=1}^n$ on $\mathcal{H}$, if embedded in a larger Hilbert space, the Naimark space $\mathcal{H}'$ of dimension $d' \geq d$, can be extended to a projective measurement $\mathbf{P} = \{P_i\}_{i=1}^n$ on $\mathcal{H}'$. The most general way to embed the original Hilbert space $\mathcal{H}$ into $\mathcal{H}'$ is via a direct sum, requiring

$$\mathrm{tr}[E_i\rho] = \mathrm{tr}[P_i(\rho \oplus 0)] \tag{1}$$

to hold for all states ρ in $\mathcal{S}$, where $\oplus$ denotes the orthogonal direct sum, and 0 is the zero matrix of dimension $d' - d$. We call any projective measurement $\mathbf{P}$ that fulfills Eq. (1) a Naimark extension of $\mathbf{E}$.

The embedding into a larger-dimensional space can also be performed via the so-called canonical Naimark extension [34,35]: one attaches an ancilla or probe, initially in a fixed state $|1\rangle\langle 1|$, via a tensor product. We denote the map that performs the embedding by $\mathcal{E}[\rho] = \rho \otimes |1\rangle\langle 1|$ and the space of embedded states by $\mathcal{S}_\mathcal{E} = \mathcal{E}[\mathcal{S}]$. A suitable global unitary V describes the interaction between system and probe such that the resulting state is $\rho' = V(\rho \otimes |1\rangle\langle 1|)V^\dagger$. A von Neumann measurement on the probe leads to the same probabilities p_i as the POVM if

$$\mathrm{tr}[E_i\rho] = \mathrm{tr}[(\mathbb{1} \otimes |i\rangle\langle i|)\rho'] = \mathrm{tr}[P_i(\rho \otimes |1\rangle\langle 1|)] \tag{2}$$

holds for all states ρ in $\mathcal{S}$. Here we have included the unitary V into the projective measurement, i.e., $P_i \coloneqq V^\dagger(\mathbb{1} \otimes |i\rangle\langle i|)V$. Thus, P_i has rank d. This type of Naimark extension is not optimal in terms of smallest additionally required dimension [36], but its structure allows for a simpler derivation of general results and directly describes the possibility to implement a POVM in an experiment. Both described types of Naimark extensions are not unique.

Resource theory of block coherence.—Åberg [11] introduced general measures for the degree of superposition in a mixed quantum state with respect to orthogonal decompositions of the underlying Hilbert space, thus pioneering the resource theory of coherence. Here we translate his work into the present-day language of resource theories and refer to it as the resource theory of block coherence.

The set $\mathcal{I}$ of block-incoherent (or free) states ρ_{BI} arises via a projective measurement $\mathbf{P} = \{P_i\}_{i=1}^n$ on the set of quantum states $\mathcal{S}$, namely [11],

$$\rho_{\mathrm{BI}} = \sum_i P_i\sigma P_i = \Delta[\sigma], \qquad \sigma \in \mathcal{S}, \tag{3}$$

where the rank of the orthogonal projectors P_i is arbitrary, and we have defined the block-dephasing map Δ. In this framework, coherence is not "visible" within a subspace given by the range of P_i, but only across different subspaces. If all P_i have rank one, the standard resource theory of coherence is recovered. Note that here we have intentionally chosen the same symbol P_i as in Eq. (2), as we shortly identify the two.

We refer to the largest class of (free) operations that cannot create block coherence as (maximally) block-incoherent (MBI) operations. A channel Λ_{MBI} on $\mathcal{S}$ is element of this class if and only if it maps any block-incoherent state to a block-incoherent state, i.e.,

$$\Lambda_{\mathrm{MBI}}[\mathcal{I}] \subseteq \mathcal{I}, \tag{4}$$

or equivalently, $\Lambda_{\mathrm{MBI}} \circ \Delta = \Delta \circ \Lambda_{\mathrm{MBI}} \circ \Delta$. In standard coherence theory, this class is referred to as maximally incoherent operations (MIO).

The amount of block coherence contained in a state ρ with respect to a projective measurement $\mathbf{P}$ can be quantified by suitable measures. We call a real-valued positive function $C(\rho, \mathbf{P}) \geq 0$ a block-coherence measure if and only if it fulfills the following: (i) *Faithfulness:* $C(\rho, \mathbf{P}) = 0 \Leftrightarrow \rho \in \mathcal{I}$. (ii) *Monotonicity:* $C(\Lambda_{\mathrm{MBI}}[\rho], \mathbf{P}) \leq C(\rho, \mathbf{P})$ for all Λ_{MBI}. (iii) *Convexity:* $C(\sum_i p_i\rho_i, \mathbf{P}) \leq \sum_i p_i C(\rho_i, \mathbf{P})$ for all $\{\rho_i\}$, $p_i \geq 0$, $\sum_i p_i = 1$. Several block-coherence measures were introduced in [11], and a general class of measures can be derived from distances that are contractive under quantum operations [20]. An important example is the relative entropy of block coherence, defined as

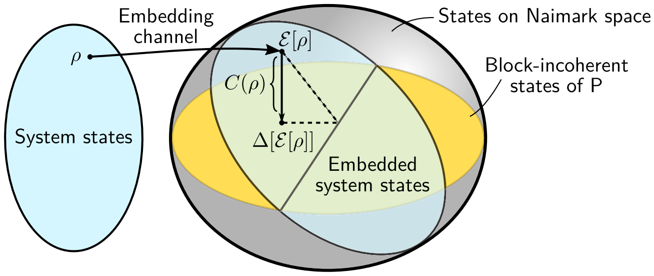

FIG. 1. We introduce a resource theory of POVM-based coherence by making use of the Naimark construction. Quantum states ρ are embedded as $\mathcal{E}[\rho] = \rho \otimes |1\rangle\langle 1|$ to act on a higher-dimensional Hilbert space (Naimark space). The POVM $\mathbf{E}$ is extended to a projective measurement $\mathbf{P}$ on the Naimark space, which defines a set of block-incoherent states $\mathcal{I}$. The POVM-coherence measure $C(\rho, \mathbf{E})$ is the distance between $\mathcal{E}[\rho]$ and its projection $\Delta[\mathcal{E}[\rho]]$ onto block-incoherent states.

$$C_{\mathrm{rel}}(\rho, \mathbf{P}) = \min_{\sigma\in\mathcal{I}} S(\rho\|\sigma) = S(\Delta[\rho]) - S(\rho), \qquad (5)$$

where $S(\rho\|\sigma) = \mathrm{tr}[\rho \log \rho - \rho \log \sigma]$ denotes the quantum relative entropy and $S(\rho) = -S(\rho\|\mathbb{1})$ is the von Neumann entropy. In standard coherence theory, the relative entropy of coherence has several important operational meanings [5,14,37]; e.g., it quantifies the distillable coherence and coherence cost under the class MIO [6].

POVM-based coherence measures.—The main idea of our approach is to define the coherence of a state ρ with respect to the POVM $\mathbf{E}$ via its canonical Naimark extension. This concept is visualized in Fig. 1.

Definition 1: POVM-based coherence measure.—Let $C(\rho', \mathbf{P})$ be a unitarily invariant block-coherence measure on $\mathcal{S}'$. The POVM-based coherence measure $C(\rho, \mathbf{E})$ for a state ρ in $\mathcal{S}$ is defined as the block coherence of the embedded state $\mathcal{E}[\rho] = \rho \otimes |1\rangle\langle 1|$ with respect to a canonical Naimark extension $\mathbf{P}$ of the POVM $\mathbf{E}$, namely,

$$C(\rho, \mathbf{E}) \coloneqq C(\mathcal{E}[\rho], \mathbf{P}), \qquad (6)$$

where the constraint in Eq. (2) has to hold. It is straightforward to generalize this definition to the most general Naimark extension from Eq. (1).

The convexity of the underlying block-coherence measure directly implies that $C(\rho, \mathbf{E})$ is convex in ρ. Here, unitarily invariant means that $C(\rho', \mathbf{P}) = C(U\rho' U^\dagger, U\mathbf{P}U^\dagger)$ holds for all unitaries U on $\mathcal{H}'$. This property ensures that $C(\rho, \mathbf{E})$ is invariant under a change of measurement operators $A_i \mapsto U_i A_i$, with unitary U_i [20]. Note that the right side of Eq. (6) should also remain invariant if we employ a more general Naimark extension of $\mathbf{E}$ regarding dimension and form. We call measures with this property well defined.

In this Letter, we focus on the relative-entropy-based measure for which one can straightforwardly show that it is well defined [20,38]. See [13] for many further well-defined POVM-coherence measures.

Lemma 1: Analytical form of a POVM-based coherence measure.—The relative entropy of POVM-based coherence $C_{\mathrm{rel}}(\rho, \mathbf{E})$ is convex and independent of the choice of Naimark extension for its definition. It admits the following form:

$$C_{\mathrm{rel}}(\rho, \mathbf{E}) = H[\{p_i(\rho)\}] + \sum_i p_i(\rho) S(\rho_i) - S(\rho), \qquad (7)$$

with $p_i(\rho) = \mathrm{tr}[E_i \rho]$, $\rho_i = (1/p_i) A_i \rho A_i^\dagger$, and the Shannon entropy $H[\{p_i(\rho)\}] = -\sum_i p_i \log p_i$. In the special case of $\mathbf{E}$ being a von Neumann measurement, i.e., $E_i = |i\rangle\langle i|$, $C_{\mathrm{rel}}(\rho, \mathbf{E})$ equals the standard relative entropy of coherence.

The independence property holds because the eigenvalues of $\Delta[\mathcal{E}[\rho]]$ are the same for any two Naimark extensions used to define Δ and because the von Neumann entropy solely depends on the eigenvalues of its argument [20].

Minimal and maximal POVM-based coherence.—We show in [20] that, for an n-outcome POVM $\mathbf{E}$, the bounds $0 \le C_{\mathrm{rel}}(\rho, \mathbf{E}) \le \log n$ hold. However, there exist POVMs for which one or both of these bounds cannot be attained for any quantum state. First, let us discuss maximal coherence: the convexity of C_{rel} implies that its maxima are attained by the pure states that lead to the highest entropy of measurement outcomes, see [39] for a partial characterization.

Now, we address the lower bound. We can characterize POVM-incoherent states (i.e., states with zero POVM coherence) as follows.

Lemma 2: Characterization of POVM-incoherent states.—Let $\mathbf{E} = \{E_i\}_{i=1}^n$ be a POVM and let $\bar{E}_i$ denote the projective part of E_i, i.e., the projector onto the eigenvalue-one eigenspace of E_i. A state $\rho_{\mathrm{PI}} \in \mathcal{S}$ is POVM incoherent with respect to $\mathbf{E}$ if and only if

$$\sum_i \bar{E}_i \rho_{\mathrm{PI}} \bar{E}_i = \rho_{\mathrm{PI}}. \qquad (8)$$

By employing the canonical Naimark extension, one can show that ρ is POVM incoherent if and only if $E_i \rho E_j = 0$ holds for all $i \neq j$, generalizing the requirement of vanishing off-diagonal elements for standard incoherent states. From this, Lemma 2 can be obtained [20], which implies that for particular POVMs the set of incoherent states $\mathcal{I}_{\mathrm{POVM}}$ is empty since no effect has a nonzero projective part. This includes any informationally complete POVM and the trine POVM, which we discuss in detail below. The set $\mathcal{I}_{\mathrm{POVM}}$ may be empty because we describe a derivated resource theory, i.e., a part of an encompassing framework in which free states exist. A resource theory where every object contains some resource is meaningful, since different objects can possess very different amounts of resource and are thus of different usefulness. In the following paragraph, we introduce the set of POVM-incoherent operations that is nonempty, as it is defined via the Naimark extension. The generalization of $\mathcal{I}_{\rm POVM}$ is the set $\mathcal{M}$ of *minimally* POVM-coherent states that has similar properties as the standard incoherent set: it is nonempty, convex, and closed under POVM-incoherent operations. Interestingly, the maximally mixed state $\rho = (\mathbb{1}/d)$ is not necessarily contained in $\mathcal{M}$ [13].

POVM-incoherent operations.—The final main ingredient of our resource theory are quantum operations that cannot create POVM-based coherence, i.e., free operations. We denote maps acting on the larger space $\mathcal{S}'$ as Λ', while maps acting on the original system $\mathcal{S}$ are called Λ.

Definition 2: POVM-incoherent operations.—Let $\mathbf{E}$ be a POVM and $\mathbf{P}$ any Naimark extension of it. Let Λ' be a completely positive trace-preserving map on $\mathcal{S}'$ that satisfies the following: (i) *Block incoherent:* Λ' is block incoherent (MBI) with respect to $\mathbf{P}$, see Eq. (4). (ii) *Subspace preserving:* $\Lambda'[\mathcal{S}_\mathcal{E}] \subseteq \mathcal{S}_\mathcal{E}$ for the subset $\mathcal{S}_\mathcal{E} \subseteq \mathcal{S}'$ of embedded system states. We call the channel $\Lambda_{\rm MPI} \coloneqq \mathcal{E}^{-1} \circ \Lambda' \circ \mathcal{E}$ on $\mathcal{S}$ a (maximally) POVM-incoherent (MPI) operation.

While this definition seems to be involved, it merely formalizes the feature that any MPI operation can be extended to an MBI map on a larger space. The second requirement in Definition 2 is necessary so that the POVM-incoherent channel only contains degrees of freedom of the original space $\mathcal{H}$.

Lemma 3: Operations from Definition 2 cannot increase POVM-based coherence.—Let $\Lambda_{\rm MPI}$ be a POVM-incoherent operation of the POVM $\mathbf{E}$. Then, for any well-defined POVM-based coherence measure $C(\rho, \mathbf{E})$ it holds that

$$C(\Lambda_{\rm MPI}[\rho], \mathbf{E}) \leq C(\rho, \mathbf{E}). \tag{9}$$

For any measurement, we can characterize the set of POVM-incoherent operations by a semidefinite program (SDP), since these operations are defined solely by linear conditions (i, ii, and trace preservation) and semidefinite conditions (complete positivity).

Theorem 1: Characterization of POVM-incoherent operations.—The set MPI of POVM-incoherent operations is independent of the chosen Naimark extension and can be characterized by a semidefinite feasibility problem (SDP). In the special case of von Neumann measurements, MPI operations are equivalent to MIO maps of the standard coherence theory.

The independence property holds because, for every two Naimark extensions of a POVM, any block-incoherent map on the larger Naimark space can be identified with a block-incoherent map on the smaller Naimark space that leads to the same (local) POVM-incoherent map [20].

Regarding the interconversion of resource states in our POVM-based coherence theory, we can employ the SDP characterization of POVM-incoherent operations $\Lambda_{\rm MPI}$ for a POVM $\mathbf{E}$ to determine numerically the maximally achievable fidelity $F_{\max}(\rho, \sigma) = \max_{\Lambda_{\rm MPI}} F(\Lambda_{\rm MPI}[\rho], \sigma)$ between a target state σ and $\Lambda_{\rm MPI}[\rho]$, see the Supplemental Material [20]. The fidelity $F(\rho, \sigma) = \mathrm{tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$ quantifies how close two quantum states ρ, σ are.

Example: Qubit trine POVM.—As an example, we analyze the case of the qubit trine POVM $\mathbf{E}^{\rm trine} = \{\frac{2}{3}|\phi_k\rangle\langle\phi_k|\}_{k=1}^3$, with measurement directions $|\phi_k\rangle = 1/\sqrt{2}(|0\rangle + \omega^{k-1}|1\rangle)$, where $\omega = \exp(2\pi i/3)$. The corresponding POVM-based coherence of *pure* states is illustrated in Fig. 2 (left). For the qubit trine POVM there are two states with maximal POVM coherence $C_{\rm rel}^{\max} = \log 3$, namely, $|\Psi_{\max}\rangle \in \{|0\rangle, |1\rangle\}$. The set $\mathcal{M}$ of states with minimal POVM-based coherence $C_{\rm rel}^{\min} = \log 3 - 1$ contains solely the maximally mixed state: $\mathcal{M} = \{\mathbb{1}/2\}$.

Regarding POVM-incoherent (free) operations, the free *unitary* operations can be fully characterized: Up to a global phase, there exist exactly six POVM-incoherent unitaries $U_i^{\rm trine}$. They correspond to the rotations on the Bloch sphere that map the trine star to itself, i.e., the symmetry group of the equilateral triangle. In standard coherence theory, the measurement map $\rho \mapsto \Delta[\rho]$ is incoherent. However, for a general POVM the measurement map $\rho \mapsto \sum_i \sqrt{E_i}\rho\sqrt{E_i}$ is not necessarily POVM incoherent with respect to $\mathbf{E}$ as one can find POVMs for which the map increases the coherence of a state [13]. Notably, for the trine POVM $\mathbf{E}^{\rm trine}$, the SDP from Theorem 1 verifies that the measurement map is indeed POVM incoherent. As to conversion properties, every qubit

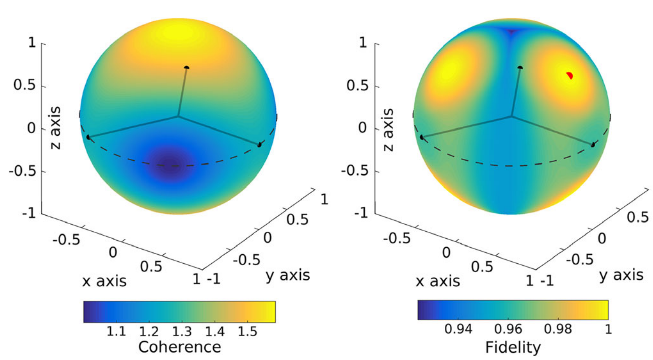

FIG. 2. POVM-based coherence theory for qubit states with respect to the trine POVM $\mathbf{E}^{\rm trine}$ in the Bloch sphere representation. Gray lines indicate the three measurement directions. (Left) POVM-based coherence of pure qubits (surface of sphere). The states $|0\rangle$ and $|1\rangle$ have maximal coherence of $C = \log 3$. The Bloch vectors of the three states with the lowest pure-state coherence $C = 1$ are antipodal to the measurement directions. (Right) Maximally achievable conversion fidelity $F_{\max}(\rho, \sigma) = \max_{\Lambda_{\rm MPI}} F(\Lambda_{\rm MPI}[\rho], \sigma)$ between a pure initial state ρ (red dot) subjected to POVM-incoherent operations $\Lambda_{\rm MPI}$ and a pure target state σ on the sphere surface. Here, $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle$. Only states in the orbit of $|\psi\rangle$ under the six POVM-incoherent unitaries can be reached with unit fidelity, as depicted by the yellow spots.

state ρ can be obtained deterministically by applying some POVM-incoherent operation to a maximally coherent state $|\Psi_{\max}\rangle \in \{|0\rangle, |1\rangle\}$. By applying the SDP, we have numerical evidence that given a state $|\psi\rangle \neq |\Psi_{\max}\rangle$, the only pure states that can be obtained from it with certainty via free operations are in the orbit $\{U_i^{\mathrm{trine}}|\psi\rangle\}$ under the trine-incoherent unitaries. An example for the conversion fidelity when starting from an initial state with less than maximal resource is shown in Fig. 2 (right).

Conclusion and Outlook.—We have introduced a family of resource theories that quantify the coherence of a quantum state with respect to any given POVM. These resource theories are derived from the resource theory of block coherence [11] via the Naimark extension on a higher-dimensional space. The restriction to the embedded original space led to the characterization of free states, free operations, and resulting conversion properties within the POVM-based resource theories. For the case of von Neumann measurements, POVM-coherence measures and POVM-incoherent operations reduce to their counterparts in standard coherence theory.

Our approach has identified the coherence resource that is necessary to implement experimentally a general measurement on a given state via the Naimark extension. Also note other works that elucidate the role of quantum resources in the Naimark extension [40,41].

Several open questions should be addressed in the future. First, it is not clear whether a characterization of POVM-incoherent operations without reference to the Naimark space is possible. A necessary condition is given by $\Lambda_{\mathrm{MPI}}[\mathcal{M}] \subseteq \mathcal{M}$, where $\mathcal{M}$ is the set of states with minimal POVM-based coherence. For projective measurements, this property is also sufficient as $\mathcal{M} = \mathcal{I}$. However, in general, this property is not sufficient: for the trine POVM, $\mathcal{M} = \{\mathbb{1}/2\}$; thus the condition is equivalent to unitality, but there are unital maps that can increase the POVM-based coherence [20].

We expect that further consistent POVM-coherence measures can be introduced which have operational interpretations that generalize the results from standard coherence theory [42–47]. Finally, one can introduce the subclass of *selective* POVM-incoherent operations and study the corresponding conversion properties [13].

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