Non-Hermitian Many-Body Localization

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(Received 4 December 2018; published 30 August 2019)

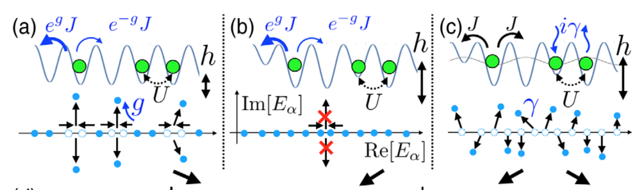
Many-body localization is shown to suppress the imaginary parts of complex eigenenergies for general non-Hermitian Hamiltonians having time-reversal symmetry. We demonstrate that a real-complex transition, which we conjecture occurs upon many-body localization, profoundly affects the dynamical stability of non-Hermitian interacting systems with asymmetric hopping that respects time-reversal symmetry. Moreover, the real-complex transition is shown to be absent in non-Hermitian many-body systems with gain and/or loss that breaks time-reversal symmetry, even though the many-body localization transition still persists.

DOI: 10.1103/PhysRevLett.123.090603

Introduction.—The reality of the eigenenergies of a Hamiltonian is closely related to the dynamical stability. Even without Hermiticity, certain classes of non-Hermitian Hamiltonians have real eigenenergies. A real-complex transition of eigenenergies of non-Hermitian systems featuring parity-time (PT) symmetry [1–15] has attracted growing interest motivated by experimental realization of such systems [16–34]. A real-complex transition also occurs in non-Hermitian systems with disorder and time-reversal symmetry (TRS). Hatano and Nelson [35–38] investigated a single-particle disordered model with asymmetric hopping, and they found that real eigenenergies become complex when localization is destroyed by strong non-Hermiticity. However, it is nontrivial whether a real-complex transition due to localization and TRS in noninteracting models [39–45] persists in non-Hermitian many-body systems. Previous results [36,46] are inconclusive because many-body localization (MBL) [47–68] was not well established. This problem is relevant to the depinning transition in type-II superconductors [35–37].

Non-Hermitian setups are relevant for continuously monitored quantum many-body systems. Indeed, the non-Hermitian dynamics is justified for individual quantum trajectories (i.e., pure states where measurement outcomes are postselected) with no quantum jumps [69]. It is nontrivial whether disorder affects the dynamics of such open systems because the non-Hermitian treatment describes physics differently from the master-equation approach [70–74], where outcomes are averaged out.

In this Letter, we show that localization suppresses imaginary parts of many-body eigenenergies for non-Hermitian interacting Hamiltonians having TRS and can induce a real-complex transition. Investigating disordered interacting particles with asymmetric hopping, we find a real-complex transition at which almost all eigenenergies become real [Figs. 1(a) and 1(b)]. In the real-eigenenergy phase, energy absorption or emission disappears despite non-Hermiticity, demonstrating that quantum states become dynamically stable. We show that a non-Hermitian MBL occurs close to the real-complex phase transition point. We conjecture that these two transitions occur at the same point in the thermodynamic limit on the basis of an analytical discussion about the stability of eigenstates. We also demonstrate that real-complex transitions are absent in non-Hermitian systems with gain and/or loss that break TRS, although non-Hermitian MBL still survives [see Fig. 1(c)]. These results are summarized in Fig. 1(d).



(d)

	asymmetric hopping		gain and/or loss	
disorder	weak	strong	weak	strong
time reversal	yes	yes	no	no
eigenenergy statistics	complex random matrix	real Poisson	complex random matrix	complex Poisson
entanglement entropy	volume law	area law	volume law	area law

FIG. 1. (a) Weakly disordered models with asymmetric hopping. Two eigenenergies on the real axis coalesce by a non-Hermitian perturbation and become complex-conjugate pairs. (b) Strongly disordered models with asymmetric hopping. Coalescence of eigenenergies due to perturbation is prohibited by localization. (c) Disordered models with gain and/or loss. Eigenenergies acquire nonzero imaginary parts without coalescence due to the absence of TRS, irrespective of the presence of localization. (d) Eigenenergy statistics (reality and level-spacing statistics) and entanglement entropy of eigenstates for an asymmetric-hopping model [Eq. (1)] and a gain-loss model [Eq. (2)] with varying disorder strength.

Localization suppresses complex eigenenergies.—We first show that localization suppresses imaginary parts of many-body eigenenergies for generic non-Hermitian Hamiltonians having TRS. Let us decompose a local Hamiltonian into an unperturbed part and a non-Hermitian perturbation as $\hat{H} = \hat{H}_0 + \hat{V}_{\rm NH}$ ($\hat{H}_0$ can be non-Hermitian). We consider a set of real eigenenergies $\{\mathcal{E}_a\}$ of $\hat{H}_0$ and the corresponding right (left) eigenstates $|\mathcal{E}_a^r\rangle(|\mathcal{E}_a^l\rangle)$, which satisfy $\langle\mathcal{E}_a^l|\mathcal{E}_b^r\rangle = \delta_{ab}$ [75].

To see why TRS is crucial for the reality of eigenenergies, we consider the first-order energy shift $\langle\mathcal{E}_a^l|\hat{V}_{\rm NH}|\mathcal{E}_a^r\rangle$, which is, in general, complex [Fig. 1(c)] but becomes real when TRS is imposed [76]. On the other hand, even with TRS, eigenenergies can coalesce and acquire imaginary parts [1] for large non-Hermiticity $\hat{V}_{\rm NH}$. The coalescence occurs due to the mixing of eigenstates, resulting from higher-order perturbations. As detailed later, whereas the mixing of two adjacent eigenstates occurs for delocalized eigenstates, it does not for MBL eigenstates [77]. In fact, for delocalized phases, many complex eigenenergies appear via coalescence of excited eigenstates [78] [Fig. 1(a)], whereas such coalescence is suppressed for MBL phases [Fig. 1(b)].

Model.—We consider hard-core bosons with asymmetric hoppings on a one-dimensional lattice subject to the periodic boundary condition:

$$\hat{H} = \sum_{i=1}^{L}[-J(e^{-g}\hat{b}_{i+1}^\dagger\hat{b}_i + e^{g}\hat{b}_i^\dagger\hat{b}_{i+1}) + U\hat{n}_i\hat{n}_{i+1} + h_i\hat{n}_i]. \quad (1)$$

Here, $\hat{n}_i = \hat{b}_i^\dagger\hat{b}_i$ is the particle-number operator at site i with the annihilation operator $\hat{b}_i$ of a hard-core boson, g controls non-Hermiticity, and h_i is randomly chosen from $[-h, h]$. This model has TRS (complex conjugation) and can be realized experimentally in continuously measured ultracold atomic systems, where strong disorder has been realized [65] and asymmetric hopping can be implemented [80]. Below, we assume $J = 1$, $U = 2$, $g = 0.1$, and a fixed particle-number subspace with $M = L/2$ (half-filling). See the Supplemental Material [79] for other parameters and an asymmetric-hopping Bose-Hubbard model [46].

Real-complex transition.—Figure 2(a) shows the eigenenergies of the Hamiltonian [Eq. (1)]. The spectrum is symmetric around the real axis due to TRS. Eigenenergies with nonzero imaginary parts decrease with increasing h.

We define $f_{\rm Im} = \overline{D_{\rm Im}/D}$, where $D_{\rm Im}$ is the number of eigenenergies with nonzero imaginary parts, D is the total number of eigenenergies, and the overline denotes the disorder average [86]. This quantity measures the fraction of eigenenergies with nonzero imaginary parts.

Figure 2(b) shows the h dependence of $f_{\rm Im}$ for different values of L. As the system size increases, $f_{\rm Im}$ increases for $h \lesssim h_c^R \simeq 8$ and decreases for $h \gtrsim h_c^R$. We also confirm the critical scaling collapse of $f_{\rm Im}$ as a function of $(h - h_c^R)L^{1/\nu}$, indicating a real-complex phase transition of many-body

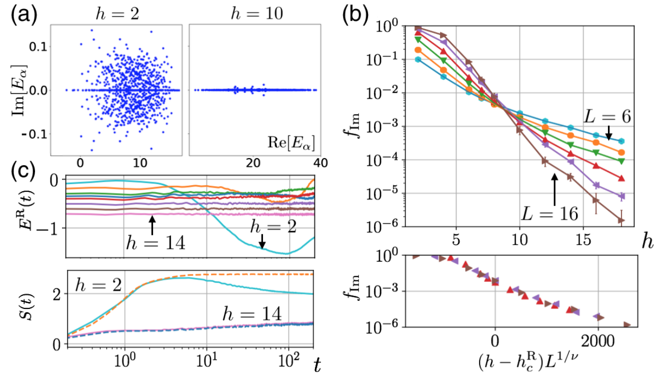


FIG. 2. (a) Eigenenergies of the non-Hermitian Hamiltonian [Eq. (1)] with $L = 12$ for two values of h. As h increases, the number of complex eigenenergies decreases. (b) (top) Dependence of $f_{\rm Im}$ on h for $L = 6$, 8, 10, 12, 14, and 16 [85]. As L increases, $f_{\rm Im}$ increases for $h \lesssim h_c^R(\simeq 8)$ and decreases for $h \gtrsim h_c^R$. (bottom) Critical scaling collapse of $f_{\rm Im}$ as a function of $(h - h_c^R)L^{1/\nu}$, where we find $h_c^R = 8.0$ and $\nu = 0.5$. The value of $f_{\rm Im}$ is identified to be zero if it is below the cutoff of imaginary part $C = 10^{-13}$; thus complex eigenvalues ($|\mathrm{Im}E_\alpha| \gg C$) and machine errors ($|\mathrm{Im}E_\alpha| \ll C$) are clearly separated. (c) (top) Time evolution of the real part of energy of the system for $h = 2$, 4, 6, 7, 8, 10, 12, and 14, where the initial energy decreases with increasing h. The energy changes in time for $h \lesssim h_c^R$ but stays almost constant for $h \gtrsim h_c^R$. (bottom) Dynamics of half-chain entanglement entropy for $g = 0.1$ (solid) and $g = 0$ (dotted) with different h. For $h = 2$, $S(t)$ first exhibits a linear growth for both values of g but decreases for $t \simeq 5$ only for $g = 0.1$. For $h = 14$, $S(t)$ exhibits a logarithmic growth for both $g = 0$ and 0.1. We take $|\psi_0\rangle = |1010\cdots\rangle$ as an initial state.

eigenenergies at $h = h_c^R$ in the thermodynamic limit ($L \to \infty$): almost all eigenenergies are complex for $h < h_c^R$ and real for $h > h_c^R$. This real-complex transition is identified by the statistics of the spectrum (i.e., the average over disorder and eigenstates) in the thermodynamic limit (the maximum imaginary part among all eigenenergies also exhibits the same transition [79]). This is in contrast to the conventional PT transition, which is identified to be the point where eigenstates coalesce without an average over eigenstates.

Our results show that highly excited eigenenergies suddenly become almost real with increasing disorder. This change significantly affects the dynamical stability of the system. Figure 2(c) shows the evolution of the real part of energy $E^R(t) = \mathrm{Re}[\overline{\langle\psi(t)|\hat{H}|\psi(t)\rangle}]$. Here,

$$|\psi(t)\rangle = \frac{e^{-i\hat{H}t}|\psi_0\rangle}{||e^{-i\hat{H}t}|\psi_0\rangle||}$$

describes a quantum trajectory with no quantum jumps, which is microscopically justified for continuously measured systems [69]. Although the energy is conserved for any h in the Hermitian Hamiltonian, it is sensitive to small non-Hermiticity for $h \lesssim h_c^R$. This sensitivity results from the delocalized eigenstates with nonzero imaginary parts and signals the dynamical instability. In contrast, the system is stable for $h \gtrsim h_c^R$, where the energy is conserved, except for negligibly small oscillations because almost all eigenenergies are real. The real-complex transition is relevant for the dynamics of other quantities [79], e.g., the half-chain entanglement entropy $S(t)$ as shown in Fig. 2(c).

Although the non-Hermitian dynamics is different from the Hermitian ones, our results indicate that the dynamics and stationary states are distinct in the two phases. This provides a first step toward understanding the statistical mechanics of such open quantum systems. For example, recurrence does not occur in the complex-eigenenergy phase, but it does in the real-eigenenergy phase [87].

Non-Hermitian many-body localization.—We next discuss a localization-delocalization transition. Although it is nontrivial how to characterize MBL in non-Hermitian systems, we show that some of the known machinery to characterize Hermitian MBL can be generalized to the non-Hermitian regime. We first consider the nearest-level-spacing distribution of (unfolded) eigenenergies [88] on the complex plane for small h and on the real axis for large h in Fig. 3(a) [89]. Here, nearest-level spacings (before unfolding) for an eigenenergy E_α on the complex plane are given by the minimum distance of $\min_\beta |E_\alpha - E_\beta|$. For a weak disorder, we numerically find that the distribution is a Ginibre distribution $P^{\mathrm{C}}_{\mathrm{Gin}}(s) = cp(cs)$, which describes an ensemble for non-Hermitian Gaussian random matrices [88,90]. Here,

$$p(s) = \lim_{N\to\infty}\left[\prod_{n=1}^{N-1} e_n(s^2)e^{-s^2}\right]\sum_{n=1}^{N-1}\frac{2s^{2n+1}}{n!e_n(s^2)}$$

with

$$e_n(x) = \sum_{m=0}^{n}\frac{x^m}{m!}$$

and

$$c = \int_0^\infty ds s p(s) = 1.1429\cdots$$

[42,88,91]. Our result demonstrates a non-Hermitian generalization of the conjecture [49,88,92] that level-spacing statistics of delocalized Hermitian systems obey the Wigner-Dyson distribution. This phase is a novel delocalized phase that, we conjecture, emerges for generic non-Hermitian interacting systems. For large h, the level-spacing distribution becomes Poissonian on the real axis $P^R_{\mathrm{Po}}(s) = e^{-s}$.

We also consider the half-chain entanglement entropy for the right eigenstates $|E^r_\alpha\rangle$ [93]: $S_\alpha = \mathrm{Tr}_{L/2}[|E^r_\alpha\rangle\langle E^r_\alpha|]$, where

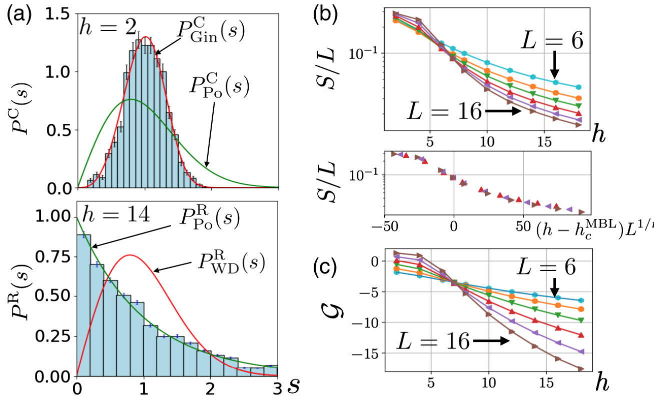


FIG. 3. (a) Nearest-level-spacing distribution of (unfolded) eigenenergies on the complex plane for $h = 2$ and that of eigenenergies on the real axis for $h = 14$ (a single disorder realization with $L = 16$). For $h = 2$, the distribution is a Ginibre distribution $P^{\mathrm{C}}_{\mathrm{Gin}}(s)$ rather than a Poisson distribution $P^{\mathrm{C}}_{\mathrm{Po}}(s) = \pi s/2 e^{-(\pi/4)s^2}$ on the complex plane [88]. For $h = 14$, the distribution is a Poisson distribution on the real axis $P^R_{\mathrm{Po}}(s) = e^{-s}$ rather than the Wigner-Dyson distribution $P^R_{\mathrm{WD}}(s) = \pi s/2e^{-(\pi/4)s^2}$. Statistics are taken from eigenenergies lying within $\pm 10\%$ of the real and imaginary parts from the middle of the spectrum. (b) (top) Half-chain entanglement entropy S/L obtained by averaging S_α/L over disorder [85] and eigenstates for which the eigenenergies lie within $\pm 2\%$ from the middle of the real part of the spectrum. With increasing h, S/L shows a crossover from the volume to area law. (bottom) The critical scaling collapse is confirmed as a function of $(h - h_c^{\mathrm{MBL}})L^{1/\nu}$, where we use $h_c^{\mathrm{MBL}} = 7.1$ and $\nu = 1.3$. (c) Stability of eigenstates $\mathcal{G}$ for different values of L and with $\hat{V}_{\mathrm{NH}} = \hat{b}_i^\dagger \hat{b}_{i+1}$ [85]. With increasing h, $\mathcal{G}$ changes from $\sim \alpha L$ to $\sim -\beta L$ $(\alpha, \beta > 0)$ at $h_c^{\mathrm{MBL}} \simeq 7 \pm 1$.

$|E^r_\alpha\rangle$ is normalized to unity, i.e., $\langle E^r_\alpha | E^r_\alpha\rangle = 1$ [94]. Figure 3(b) shows the L dependence of S_α/L averaged over the eigenstates around the middle of the spectrum [95], and that the entanglement entropy exhibits a crossover from the volume to area law as the MBL phase sets in around $h \simeq h_c^{\mathrm{MBL}}$. We confirm the critical scaling collapse as a function of $(h - h_c^{\mathrm{MBL}})L^{1/\nu}$. These results show that the delocalized and MBL phases can be distinguished by the entanglement entropy, even in non-Hermitian systems, in a manner similar to the Hermitian case [62,96]. For the delocalized phase, eigenstates of Hermitian or non-Hermitian systems are well described by those of Hermitian or non-Hermitian random matrices, which are regarded as random eigenvectors satisfying the volume law. For the localized phase, eigenstates are characterized by quasilocal conserved quantities for both Hermitian and non-Hermitian cases (because non-Hermiticity does not significantly affect eigenstates for the localized phase, as will be discussed below), and the area law holds.

As discussed above, the stability of the eigenstates of $\hat{H}_0$ under local perturbations $\hat{V}_{\mathrm{NH}}$ is important for the suppression of complex eigenenergies. We consider

$$\mathcal{G} = \overline{\ln \frac{|\langle \mathcal{E}^l_{a+1}|\hat{V}_{\rm NH}|\mathcal{E}^r_a\rangle|}{|\mathcal{E}'_{a+1} - \mathcal{E}'_a|}}$$

as a measure of the stability of the eigenstates. The factor in the logarithm is nothing but the convergence factor in the standard perturbation theory [79]. Here, we only consider $\mathcal{E}'_a = \mathcal{E}_a + \langle \mathcal{E}^l_a|\hat{V}_{\rm NH}|\mathcal{E}^r_a\rangle$ that stays real, and the labels of eigenstates a are taken such that $\mathcal{E}'_1 \leq \mathcal{E}'_2 \leq \cdots$. This is a non-Hermitian generalization of the Hermitian counterpart introduced in Ref. [77], where $\partial\mathcal{G}/\partial L > 0$ for the delocalized phase because level spacings (denominator) become much smaller than typical off-diagonal matrix elements (numerator) [97–101], and $\partial\mathcal{G}/\partial L < 0$ for the localized phase because quasilocal conserved quantities strongly suppress the off-diagonal matrix elements [54,55,66,67].

Figure 3(c) shows the h dependence of $\mathcal{G}$ for the non-Hermitian setting. We find $\mathcal{G} \sim \alpha L (\alpha > 0)$ for the delocalized phase and $\mathcal{G} \sim -\beta L (\beta > 0)$ for the localized phase, which is similar to the Hermitian case because of the same reason as the entanglement entropy. From the point at which $\mathcal{G}$ is independent of L, the non-Hermitian MBL transition point is determined as $h_c^{\rm MBL} \simeq 7 \pm 1$.

We conjecture that the real-complex transition point h_c^R and the MBL transition point $h_c^{\rm MBL}$ coincide in the thermodynamic limit. Indeed, from the above discussions of stability $\mathcal{G}$, the coalescence of adjacent eigenstates is suppressed due to the non-Hermitian MBL. Furthermore, for our model in Eq. (1), we can show that the coalescence process is suppressed even for nonadjacent eigenstates, and hence the entirely real spectra are realized, as detailed in the Supplemental Material [79]. Note that, in our numerics up to $L = 16$, the two transition points are close ($h_c^R \simeq 8 \pm 1$ and $h_c^{\rm MBL} \simeq 7 \pm 1$ for $g = 0.1$) but slightly different. On the basis of the analytical discussions of the stability, we conjecture that the small deviation is attributed to the finite-size effects.

Disordered model with gain and loss.—Finally, we show that, without TRS, the real-complex transition does not occur upon the non-Hermitian MBL transition. We study the model with gain and loss, which is experimentally feasible [102,103]:

$$\hat{H} = \sum_{i=1}^{L} \left[-J(\hat{b}^\dagger_{i+1}\hat{b}_i + \text{H.c.}) + U\hat{n}_i\hat{n}_{i+1} + (h_i - i\gamma(-1)^i)\hat{n}_i \right]. \tag{2}$$

In this model with broken TRS, the particle gain (loss) at odd (even) sites ($+i\gamma\hat{n}_i / -i\gamma\hat{n}_i$) tends to decrease (increase) the number of particles at the corresponding site. Figure 4(a) shows that the eigenenergies have nonzero imaginary parts, irrespective of h. We confirm $f_{\rm Im} = 1$ for any h and L (data not shown).

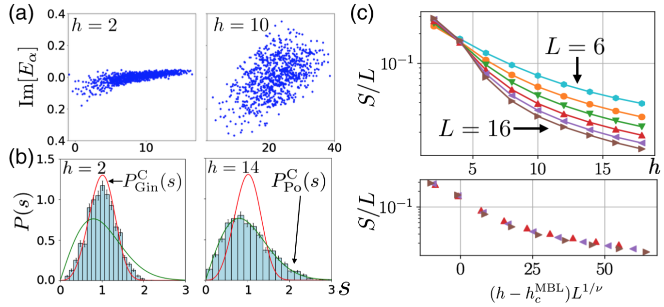


FIG. 4. (a) Complex spectrum of the non-Hermitian Hamiltonian in Eq. (2) with $\gamma = 0.1$ for a single disorder realization. Eigenenergies have nonzero imaginary parts, irrespective of disorder strength h. (b) Nearest-level-spacing distributions on the complex plane for eigenenergies having nonzero imaginary parts. The distribution exhibits random-matrix universality slightly different from the Ginibre distribution $P^{\rm C}_{\rm Gin}(s)$ [104] for $h = 2$ and a Poisson distribution of $P^{\rm C}_{\rm Po}(s)$ for $h = 14$. Statistics are taken from eigenstates of a single disorder realization ($L = 16$) for which the eigenenergies lie within $\pm 10\%$ of the real and imaginary parts from the middle of the spectrum. (c) (top) System-size dependence of S/L averaged over eigenstates from the middle ($\pm 2\%$) of the real part of the spectrum for different values of L with $\gamma = 0.1$ [85]. Entanglement entropy decreases with increasing L when the many-body localization sets in. (bottom) The critical scaling collapse of S/L is found as a function of $(h - h_c^{\rm MBL})L^{1/\nu}$ with $h_c^{\rm MBL} = 4.2$ and $\nu = 1.8$.

Although the real-complex transition is absent, the delocalization-localization transition occurs. Figure 4(b) shows nearest-level-spacing distributions on the complex plane for different values of h. For $h \lesssim h_c^{\rm MBL}$ ($h_c^{\rm MBL} \gtrsim 4$), the distribution is a non-Hermitian random-matrix distribution with transposition symmetry [104], which is slightly different from $P^{\rm C}_{\rm Gin}(s)$; for $h \gtrsim h_c^{\rm MBL}$, it is a Poisson distribution $P^{\rm C}_{\rm Po}(s)$. This change in the level statistics indicates the non-Hermitian MBL transition. Although we find that the transition is a crossover for $L = 16$ (data not shown), it is a future challenge to investigate whether it becomes sharp at the MBL transition in the thermodynamic limit. The entanglement entropy of the eigenstates shows the volume law for $h \lesssim h_c^{\rm MBL}$ and the area law for $h \gtrsim h_c^{\rm MBL}$ [see Fig. 4(c)].

Conclusion.—We have shown that non-Hermitian MBL suppresses complex eigenenergies for generic non-Hermitian interacting Hamiltonians having TRS and that a real-complex transition, which occurs upon MBL, profoundly affects the dynamical stability of interacting systems with asymmetric hopping. We have demonstrated that real-complex transitions are absent in systems with gain and/or loss that break TRS, although the non-Hermitian MBL persists.

The real-complex transition found here is conceptually new, in that it never occurs in isolated, few-body, or clean systems. It is interesting to investigate other properties of the non-Hermitian MBL, such as critical phenomena (implied by our critical scaling collapse), where all of the non-Hermiticity, disorder, and interaction come into play. Our work is also relevant to quantum chaos [92,105,106] in open interacting systems described by non-Hermitian Hamiltonians, as indicated by its random-matrix level statistics. The properties of non-Hermitian MBL, such as the entanglement entropy, may offer a new approach to understanding the interaction effect in the depinning transition of type-II superconductors [35].

We are grateful to Zongping Gong, Yuto Ashida, Masaya Nakagawa, and Naomichi Hatano for fruitful discussions. This work was supported by KAKENHI Grant No. JP18H01145 and a Grant-in-Aid for Scientific Research on Innovative Areas "Topological Materials Science" (KAKENHI Grant No. JP15H05855) from the Japan Society for the Promotion of Science (JSPS). R. H. and K. K. were supported by the JSPS through the Program for Leading Graduate Schools (ALPS) and KAKENHI Grants No. JP17J03189 and No. JP19J21927.

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[86] Although the density of states depends on energy, we here consider the average over eigenenergies for the entire energy range for simplicity. Thus, for sufficiently large systems, we essentially consider the energy range corresponding to the infinite temperature, where the density of states is maximal.
[87] No well-defined definitions for ergodicity and thermalization exist in non-Hermitian systems because stationary states are not thermal. On the other hand, although the initial information of the state is lost for the complex-eigenenergy phase, it is retained in the real-eigenenergy phase. The latter case is reminiscent of the nonergodicity in the Hermitian MBL. As another example, the entanglement dynamics suggests that the long-time behavior of entanglement is larger for the delocalized phase than the MBL phase, just like Hermitian thermalization. This can be related to the entanglement entropy of eigenstates, which we find exhibits the volume (area) law for the delocalized (MBL) phase, even for non-Hermitian systems.
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