σ Models on Quantum Computers

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We formulate a discretization of σ models suitable for simulation by quantum computers. Space is substituted with a lattice, as usually done in lattice field theory, while the target space (a sphere) is replaced by the "fuzzy sphere", a construction well known from noncommutative geometry. Contrary to more naive discretizations of the sphere, in this construction the exact O(3) symmetry is maintained, which suggests that the discretized model is in the same universality class as the continuum model. That would allow for continuum results to be obtained for very rough discretizations of the target space as long as the space discretization is made fine enough. The cost of performing time evolution, measured as the number of controlled-NOT operations necessary, is $12LT/\Delta t$, where L is the number of spatial sites, T the maximum time extent, and Δt the time spacing.

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Introduction.—The advent of quantum computers opens up a new method to attack several physics problems which have, up to now, remained intractable. Perhaps the most interesting of those is the numerical treatment of many-body or field theory with sign problems. In particular, the nonperturbative calculation of real time observables, where very little progress has been made up to now [1–3], is an obvious target for quantum computation. Of course, the hope of attacking these problems hinges on being able to formulate quantum field theories in a way suitable for quantum computers. This topic is still in its infancy. The naive expectation is that fermionic fields can be more easily implemented in quantum computers, as a qubit can encode the presence or absence of a fermion in a given state. This is borne out by the few existing calculations that have been performed on quantum computers [4–6]. Bosonic fields are not so simply implemented. The attempts made up to now involve either eliminating the bosonic fields using some special property of the model or truncating the occupation number at any given site [7–18]. The situation is analogous to the early days of (classical) computing in field theory. Classical bits also seem more amenable to describing fermionic fields than bosonic ones, as the cost of storing and manipulating reasonable approximations to real numbers was too high to be practical in the early days. There were at the time several attempts made at substituting bosonic continuous field values with a finite set of values [19–23]. In all of these schemes, the symmetry of the model is reduced by the discretization of the bosonic fields.

When discretization reduces the symmetry, it is unclear whether, in the spacetime continuum limit, the original model is recovered. For instance, the nonlinear σ model in one spatial dimension with fields taking values on a sphere was studied in the approximation where the sphere is substituted by the vertices of a platonic solid. It seems to still be controversial whether the dodecahedron model is in the same universality class as the original spherical model [24–29]. In the case of gauge theories, the question, at least for Abelian theories, was settled long ago: Abelian gauge theories with any finite discrete group $\mathbb{Z}_N$ are not in the same universality class as the U(1) model and do not approach the U(1) gauge theory as the spacetime continuum limit is taken [30–32].

This suggests that, to obtain the right continuum limit, we should construct a scheme where all of the symmetries of the original model are maintained while discretizing the field variables in order to make the Hilbert space have a finite (and hopefully small) dimension. This Letter presents such a formulation based on a well-known construction in noncommutative geometry (the "fuzzy sphere" [33,34]) used before in the study of (super-)membranes. The resulting system can be simulated on a quantum computer with two qubits per spatial site. We implement our scheme on a simulated quantum computer and verify its correctness. The number of gates required is of the order of $\sim 12L(T/\Delta t)$, where L is the number of spatial sites, T the maximum time extent, and Δt the time discretization step.

σ model on the fuzzy sphere.—The O(3) σ model is defined on a discretized space by the Hamiltonian

$$\mathcal{H} = \sum_r \left(\frac{g^2}{2} \boldsymbol{\pi}(r)^2 + \frac{1}{2g^2 \Delta x^2} [\mathbf{n}(r+1) - \mathbf{n}(r)]^2 \right) = \sum_r \left(\frac{g^2}{2} \boldsymbol{\pi}(r)^2 + \frac{1}{g^2 \Delta x^2} [1 - \mathbf{n}(r+1) \cdot \mathbf{n}(r)] \right), \quad (1)$$

where $\mathbf{n}$ is a unit three-dimensional vector, $\boldsymbol{\pi}^2$ is the Laplace-Beltrami operator on S^2, and the sum runs over the L spatial lattice sites. The global symmetry $\mathbf{n}(r) \to O \cdot \mathbf{n}(r)$, where O is an orthogonal matrix, is evident. This model is asymptotically free in one spatial dimension [35].

In the Hamiltonian formalism, the wave function is a function of L copies of the sphere S^2, $\psi(\mathbf{n}_1, \ldots, \mathbf{n}_L)$. The Hilbert space is infinite dimensional even for $L = 1$. We will approximate this model by substituting the target space (the sphere) by the fuzzy sphere [33,34]. The fuzzy sphere is not a subset of points of the sphere; instead, it is the functions on the sphere that are substituted by elements of a finite-dimensional Hilbert space. Let us demonstrate the construction first in the $L = 1$ case, where $\psi(\mathbf{n})$ can be expanded as

$$\psi(\mathbf{n}) = \psi_0 + \psi_i n_i + \frac{1}{2} \psi_{ij} n_i n_j + \cdots, \quad (2)$$

constrained by $n_i n_i = 1$. The fuzzy sphere regularization substitutes this Hilbert space with the space of matrices

$$\Psi = \psi_0 \mathbb{1} + \psi_i \mathbb{J}_i + \frac{1}{2} \psi_{ij} \mathbb{J}_i \mathbb{J}_j + \cdots, \quad (3)$$

where $\mathbb{J}_i$, $i = 1, 2, 3$ are generators of SU(2) in a given representation j (normalized such that $\sum_{i=1}^3 \mathbb{J}_i^2 = \mathbb{1}$). The important difference between Eqs. (2) and (3) is that Eq. (3) terminates (for instance, if $j = 1/2$, $\mathbb{J}_i^2 \sim \sigma_i^2 = \mathbb{1}$). Thus, the infinite-dimensional Hilbert space is substituted by a space of dimension $(2j+1)^2$. Since the space is finite dimensional, it can be informally thought of as being the space of functions defined on a space with a finite number of points, the fuzzy sphere. In the $j = 1/2$ case, the space of matrices Ψ is four-dimensional, and the fuzzy sphere can be thought of as a four-point discretization of the sphere. However, the "points" of the fuzzy sphere are "spread out" and preserve rotational symmetry. Notice that the fuzzy sphere is not defined as a subset of points of the sphere. It is the algebra of functions on the sphere that is deformed into a (noncommutative) algebra given by matrix multiplication. Still, the fuzzy sphere is an approximation to the sphere in the sense that an operation defined on the sphere can be approximated by equivalent constructions on the fuzzy sphere. For instance, the norms on the sphere and on the fuzzy sphere satisfy

$$\frac{1}{2j+1} \mathrm{tr}(\Psi^\dagger \Psi) \underset{j \to \infty}{\to} \int_{S^2} \frac{d\Omega}{4\pi} |\psi|^2. \quad (4)$$

We refer to Ref. [34] for a discussion of the geometrical constructs of the sphere framed in terms of the algebra of functions (and their extensions to the fuzzy sphere). In particular, the Hamiltonian of the σ model with one spatial site is simply the Laplacian on the sphere (i.e., the quantum mechanics of a free particle on the sphere)

$$-\frac{g^2}{2} \nabla^2 \psi \to H^0 \Psi = \kappa \frac{g^2}{2} \sum_{k=1}^3 [\mathbb{J}_k, [\mathbb{J}_k, \Psi]], \quad (5)$$

with κ being a normalization factor. The eigenvalues of the Laplacian operator on the sphere are $l(l+1)$ for $l = 0, 1, \ldots$ with multiplicities $2l+1$. When $\kappa = j(j+1)$, the spectrum of H^0 is *exactly* the same but is truncated to its lowest $(2j+1)^2$ values. This is in contrast to other discretizations where the lowest eigenvalues are reproduced only *approximately*. Notice also that, as stressed before and contrary to other discretizations, H^0 has an exact O(3) invariance $\Psi \to U(g)^\dagger \Psi U(g)$, where U is the representation of the rotation g.

From now on, we will work with $j = 1/2$, so the dimension of the Hilbert space at each site is 4. A convenient basis for this space is $\mathbb{T}_0 = i\mathbb{1}/\sqrt{2}$ and $\mathbb{T}_i = \sqrt{3/2}\mathbb{J}_i$, which satisfies $tr(\mathbb{T}_a^\dagger \mathbb{T}_b) = \delta_{ab}$. In a system with $L > 1$ spatial sites, the Hilbert space of the system is the tensor product of L single-site Hilbert spaces. The generic wave function can be written as

$$\Psi = \sum_{a_0=0}^3 \cdots \sum_{a_{L-1}=0}^3 \psi_{a_{L-1}, \ldots, a_0} |a_{L-1}, \ldots, a_0\rangle, \quad \text{with} \quad |a_{L-1}, \ldots, a_0\rangle \equiv \mathbb{T}_{a_{L-1}} \otimes \cdots \otimes \mathbb{T}_{a_0}. \quad (6)$$

The kinetic term $\mathcal{H}^0 = \sum_n H^0(r)$ of the Hamiltonian is the sum of Eq. (5) acting on the Hilbert space of each site, and it is diagonal in the basis $|a_{L-1}, \ldots, a_0\rangle$ [for a single-site operator A, $A(r) \equiv \mathbb{1}^{\otimes L-r-1} \otimes A \otimes \mathbb{1}^{\otimes r-1}$ denotes the operator acting on site r]. In the $\mathbb{T}$ basis, the kinetic term is represented by a sum of similar tensor products of 4×4 matrices with

$$h^0_{ij} = \langle \mathbb{T}_i | H^0 | \mathbb{T}_j \rangle = \mathrm{tr} \mathbb{T}_i^\dagger H^0 \mathbb{T}_j = \frac{k g^2}{2} \sum_{k=1}^3 \mathrm{tr} \mathbb{T}_i^\dagger [\mathbb{J}_k, [\mathbb{J}_k, \mathbb{T}_j]],$$

$$\text{and thus } h^0 = g^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (7)$$

The eigenvalues of $\mathcal{H}^0$ are $E^0_{a_{L-1},\ldots,a_0} = g^2 \sum_r 1 - \delta_{a0}$; i.e., the kinetic energy of site r is 0 if $a_r = 0$, and g^2 otherwise.

The "interaction" term arises from expanding the nearest-neighbor interaction term $-\mathbf{n}(r+1)\cdot\mathbf{n}(r)$ in Eq. (1): $\mathcal{H}^I = \sum_r \sum_{k=1}^3 H^{Ik}(r+1,r)$ with $H^{Ik}(r+1,r) = -(\kappa/g^2\Delta x^2)(\mathbb{J}_k)_{r+1}(\mathbb{J}_k)_r$. They involve only two neighbor sites at a time (one link). In the $\mathbb{T}$ basis, the $\mathbb{J}_k$ operators are represented by 4×4 matrices $(j_k)_{ij} \equiv \langle \mathbb{T}_i|\mathbb{J}_k|\mathbb{T}_j\rangle$:

$$j_1 = \mathbb{1}\otimes\sigma_2/\sqrt{3}, \quad j_2 = \sigma_2\otimes\sigma_3/\sqrt{3}, \quad j_3 = \sigma_2\otimes\sigma_1/\sqrt{3}, \tag{8}$$

where the σ's are Pauli matrices. The interaction term $H^{Ik}(r+1,r)$ in the $\mathbb{T}$ basis is the matrix $h^{Ik}(r+1,r) = -(\kappa/g^2\Delta x^2)(j_k)_{r+1}(j_k)_r$. By this we mean that the element $\langle a_{L-1},\ldots,a_0|H^{Ik}|a'_{L-1},\ldots,a'_0\rangle$ is $h^{Ik}(r+1,r)_{i,j}$, where i and j are numbers with representation in basis 4 $a_{L-1},\ldots,a_0$ and $a'_{L-1},\ldots,a'_0$, respectively (the matrix indices run from 0 to 4^L-1).

Implementation of time evolution and estimate of resources.—The implementation of the time evolution of the model in terms of quantum gates starts by splitting the time evolution over a number of smaller steps $\Delta t = t/N$, and each time step using the Suzuki-Trotter formula. Each time step is further split into the evolution due to the four parts of $H = H^0 + H^{I1} + H^{I2} + H^{I3}$:

$$e^{-iHt} \approx (e^{-iH^{I3}\Delta t}e^{-iH^{I2}\Delta t}e^{-iH^{I1}\Delta t}e^{-iH^0\Delta t})^N. \tag{9}$$

The state of the system is time evolved by first applying the kinetic term $e^{-i\Delta t\mathcal{H}^0}$ site by site; the site order does not matter since the $H^0(r)$ for different r commute. We follow the kinetic term with the first interaction term $e^{-i\Delta t\mathcal{H}^{I1}}$. This evolution is done link by link, and again, the order does not matter, as all $H^{Ik}(r,r+1)$ commute with each other. We use periodic boundary conditions, so the evolution of the link from $r = L-1$ to $r = L$ is followed by evolution of a link from $r = L$ to $r = 1$. The link-by-link evolution is repeated for $e^{-i\Delta t\mathcal{H}^{I2}}$ and $e^{-i\Delta t\mathcal{H}^{I3}}$. This process is then repeated N times.

The local four-dimensional Hilbert space is encoded by two qubits, so

$$|a_{L-1},\ldots,a_0\rangle \leftrightarrow |\underbrace{q_{2L-1},q_{2L-2}}_{\text{site}L-1},\ldots,\underbrace{q_1,q_0}_{\text{site}0}\rangle, \tag{10}$$

with the pair of qubits being the binary digits of the corresponding index a. For instance, $q_1 = q_0 = 0$ corresponds to $a_0 = 0$, and $q_1 = q_0 = 1$ corresponds to $a_0 = 3$. In this basis, the kinetic term evolution at each site is

$$e^{-i\Delta t h_0} = e^{-i\Delta t}\begin{pmatrix} e^{i\Delta t} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

where from now on we set $g^2 = 1$ and $\kappa/g^2\Delta x^2 = 1$ for simplicity. The circuit implementing $e^{-i\Delta t\mathcal{H}^0}$ is depicted on the left side of Fig. 1. Since $\mathtt{HS}^\dagger\sigma_2\mathtt{SH} = \sigma_3$, the interacting term $H^{I1} = \frac{1}{3}\mathbb{1}\otimes\sigma_2\otimes\mathbb{1}\otimes\sigma_2$ is related by a similarity transformation to $\frac{1}{3}\mathbb{1}\otimes\sigma_3\otimes\mathbb{1}\otimes\sigma_3$, with the change of basis given by single qubit operations (here H is the Hadamard and S the phase one-qubit gate). Similarly, since $\mathtt{H}\sigma_1\mathtt{H} = \sigma_3$, H^{I2} and H^{I3} are related to $\frac{1}{3}\sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_3$. We now use the facts that a controlled-NOT (CNOT) gate implements a similarity transformation that takes $\sigma_3\otimes\sigma_3$ into $\sigma_3\otimes\mathbb{1}$, and that $\exp[i\theta\sigma_3\otimes\mathbb{1}]$ is simply a rotation on the left qubit. For H^{I1}, we apply this for the q_2 and q_0 qubit pair, whereas for the other two terms we have to apply the CNOT transformation on the (q_1,q_0) and (q_3,q_1) pairs to reduce $\sigma_3\otimes\sigma_3\otimes\sigma_3\otimes\sigma_3$ to $\sigma_3\otimes\mathbb{1}\otimes\sigma_3\otimes\mathbb{1}$, and then a CNOT tranformation on the (q_3,q_1) pair to reduce the exponentiation to a single qubit rotation. The quantum circuit implementations are found in Fig. 1.

The counting of quantum gates is dependent on the instruction set available in the hardware. However, the difficulty in the hardware implementation of two-qubit gates makes it unlikely that any two-qubit gate besides the CNOT gate will be available in the hardware. CNOT gates are also, by far, the ones most likely to generate decoherence. Thus, we will count only the number of CNOT gates required for our implementation. The kinetic term circuit has two CNOT gates per site (notice that a controlled-U_1 operation requires two CNOT gates to be implemented). The $\mathcal{H}^{I2}$ and $\mathcal{H}^{I3}$ link terms seem to require six CNOT gates per

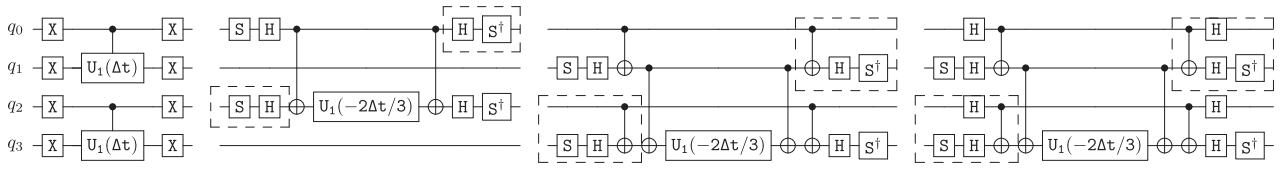

FIG. 1. Circuit implementing the time evolution. Starting from the left, we display the kinetic term $\exp[-i\Delta t\mathcal{H}^0]$ (for two sites) and the link terms: $\exp[-i\Delta t H^{I1}(1,0)]$, $\exp[-i\Delta t H^{I2}(1,0)]$, and $\exp[-i\Delta t H^{I3}(1,0)]$. The notation for the gates used here is standard in the quantum computing literature [36].

site. However, since we apply $e^{i\delta t H^{I2}}$ on every link, the gates shown in the dashed box in Fig. 1 cancel between adjacent links and do not have to be applied. The result is that only four CNOT gates per link are required. A similar thing happens to the $\mathcal{H}^{I3}$ links. Finally, $\mathcal{H}^{I1}$ requires two CNOT gates, for a total of 12 CNOT gates per site (for periodic boundary conditions, where there are as many links as sites). Since $T/\Delta t$ steps are needed, $36(T/\Delta t)$ CNOT gates are required to implement $U(T)$ in our three-site model. This gate depth renders the model inaccessible to the current generation of processors. As a proof of principle, however, we run our algorithm in a quantum computer simulator (QISKIT [37,38]) for $L = 3$, and results for $\Delta t = 0.2$, $\Delta x = 1$, $g^2 = 1$ are shown in Fig. 2. The results are, for illustrative purposes, the time-dependent probabilities of finding the states $|000000\rangle$, $|000001\rangle$, and $|111111\rangle$, starting with the initial state with equal amplitudes of all elements of the basis $|q_5 q_4 q_3 q_2 q_1 q_0\rangle$ obtained by applying a Hadamard transformation on the state $|000000\rangle$. In the same figure, we show the exact time evolution and Suzuki-Trotter-formula-mediated evolution by multiplying the appropriate $4^3 \times 4^3$ matrices. The error bars reflect the expected variance from the quantum mechanical measurement.

It is important to stress that every step in our construction can be easily carried out in much bigger lattices, and even in more spatial dimensions. The size of the blocks of time evolution—involving at most four qubits at a time—are independent of the system size. Also, the time evolution due to the kinetic term can be applied simultaneously to all sites, and the evolution due to the hopping term can be applied simultaneously to half of the links at once. The method is essentially unchanged as the number of spatial dimensions is increased. Unfortunately, in the implementation in terms of the quantum circuits we found, the number of CNOT gates is a little too large for current quantum computers. Our attempts at running it on the IBM's ibmqx4 machine have resulted in mostly noise.

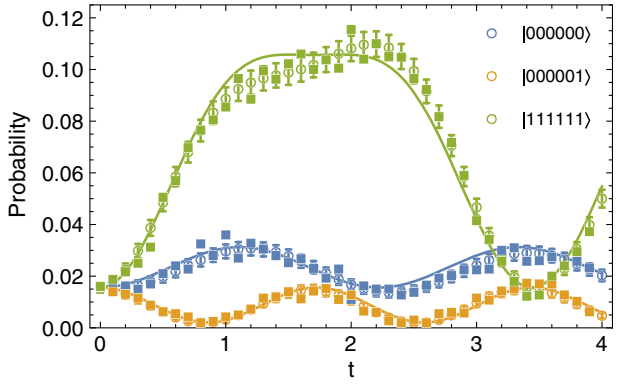

FIG. 2. Probabilities of states $|000000\rangle$, $|000001\rangle$, and $|111111\rangle$ starting from $(|000000\rangle + |000001\rangle + \cdots + |111111\rangle)/\sqrt{64}$. The solid lines are the exact result obtained by diagonalization of the Hamiltonian. The empty circles are from the Suzuki-Trotter formula, and their error bars the expected uncertainty following the binomial distribution. The filled points are from the quantum simulator. Each data point from the simulator is an average of 4000 measurements.

Conclusions and prospects.—One issue to be faced on the road to using quantum computers in quantum field theory is the presence of bosonic fields. The Hilbert space of a bosonic theory has an infinite number of dimensions *per spatial site*. In contrast, the Hilbert space describing a quantum computer with a finite number of registers is finite dimensional. Thus, even after discretizing space, some further truncation of the field space is required [5,7]. We propose a method to accomplish which preserves the theory's O(3) symmetry.

There are two ways in which our fuzzy sphere model approximates the continuum O(3) σ model. First of all, by increasing the dimension $2j + 1$ of the representation of O(3), the fuzzy sphere approaches the O(3) σ model defined by Eq. (1). Perhaps more interesting is the fact that the fuzzy model, defined by $H = H^0 + H^I$ (generalized to a large number of spatial sites), and the σ model defined by the lattice Hamiltonian, Eq. (1), are likely to approach the same continuum limit as $\Delta t, \Delta x \to 0$. In fact, the continuum limit of the σ model is obtained by tuning $\Delta t, \Delta x \to 0$, and g^2 is such a way as to keep physical quantities (mass gap, scattering amplitudes) fixed in physical units [perturbation theory indicates that the model is asymptotically free, so the correct scaling is $g^2 \sim -1/\log(\Delta x)$ [35]]. In this limit, details of the Hamiltonian become irrelevant, and any other Hamiltonian with the same field content and symmetries, on account of universality, gives rise to the same continuum limit [39]. More precisely, any other operators, consistent with the O(3) symmetry, are of higher dimension and, presumably, are irrelevant in the continuum limit. The reasonable assumption of universality can be checked in a classical calculation. The Suzuki-Trotter-formula-mediated time evolution operator [Eq. (9)] corresponds to an action discretized in both time and space. A Monte Carlo calculation using this action (analytically continued to imaginary time) can demonstrate whether the fuzzy model is in the same universality class as the σ model and thus has the same $\Delta t, \Delta x \to 0$ limit.

Our model shares features with quantum link models [40–42], in particular, the use of quantum spins as dynamical variables. They differ, however, in that quantum link models are formulated in one extra dimension, and the $(1+1)$D O(3) model arises through dimensional reduction. Since the spatial part of the $(1+2)$D theory survives this reduction, quantum link models access Euclidean correlators of the $(1+1)$D theory, while our model accesses the real time correlators. Our truncation of the Hilbert space is also similar to Ref. [43], which makes the O(3) model amenable to tensor networks. Finally, our

approach should not be confused with theories with noncommutative *spacetime* [44–47], which are motivated by string theory and serve as an alternative to lattice regularization [48].

Among theories of physical significance, bosonic fields also appear in principal chiral models (as, for instance, in low energy QCD) and gauge theories. These bosonic fields take values on group manifolds. A slight modification of the scheme proposed in this Letter can also be used in these cases, but it is somewhat more involved. A full account of these extensions will appear separately.

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Note added.—Recently, numerical evidence was presented for a similar digitization of the O(3) model in the same universality class as the $(1+1)$D O(3) model [49].

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