Relativistic Quantum Reference Frames: The Operational Meaning of Spin

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The spin is the prime example of a qubit. Encoding and decoding information in the spin qubit is operationally well defined through the Stern-Gerlach setup in the nonrelativistic (i.e., low velocity) limit. However, an operational definition of the spin in the relativistic regime is missing. The origin of this difficulty lies in the fact that, on the one hand, the spin gets entangled with the momentum in Lorentz-boosted reference frames, and on the other hand, for a particle moving in a superposition of velocities, it is impossible to "jump" to its rest frame, where spin is unambiguously defined. Here, we find a quantum reference frame transformation corresponding to a "superposition of Lorentz boosts," allowing us to transform to the rest frame of a particle that is in a superposition of relativistic momenta with respect to the laboratory frame. This enables us to first move to the particle's rest frame, define the spin measurements there (via the Stern-Gerlach experimental procedure), and then move back to the laboratory frame. In this way, we find a set of "relativistic Stern-Gerlach measurements" in the laboratory frame, and a set of observables satisfying the spin $\mathfrak{su}(2)$ algebra. This operational procedure offers a concrete way of testing the relativistic features of the spin, and opens up the possibility of devising quantum information protocols for spin in the special-relativistic regime.

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Introduction.—The description of physical systems is standardly given in terms of coordinates as defined by reference frames. Thanks to the principle of covariance, stating the equivalence of all descriptions regardless of the choice of the reference frame, it is possible to choose the reference frame where the relevant dynamical quantities can be most conveniently described. For example, it is typically easier to describe the dynamics of a system from the point of view of its rest frame, because only internal degrees of freedom (d.o.f.) contribute to the dynamics in the rest frame.

When the external d.o.f. (momentum) of the system are in a quantum superposition from the perspective of the laboratory, no classical reference frame transformation can map the description of physics from the laboratory to the rest frame. However, this can be achieved via a quantum reference frame (QRF) transformation between two frames moving in a superposition of velocities relative to one another. In order to achieve such change of quantum reference frame in the nonrelativistic regime, a formalism was introduced in Ref. [1] to change the description to a reference frame which is in a quantum relationship with the initial one. This QRF transformation only depends on relational quantities, and it has also been derived starting from a gravity inspired symmetry principle in a perspective neutral model [2,3]. An immediate consequence of the formalism is that entanglement and superposition are QRF-dependent features. This formalism naturally leads to the possibility of identifying the rest frame of a quantum system in an operational way.

Here, we further develop this approach in the case of a relativistic quantum particle with spin, with the goal of finding an operational description of the spin in a special-relativistic setting. Spin is operationally defined in the rest frame of a particle (or, to a good approximation, for slow velocities) via the Stern-Gerlach experiment. When the particle has relativistic velocities, the spin d.o.f. transforms in a momentum-dependent way. If a standard Stern-Gerlach measurement is performed on a particle in a pure quantum state moving in a superposition of relativistic velocities, the operational identification of the spin fails, because no orientation of the Stern-Gerlach apparatus returns an outcome with unit probability. This happens because, as shown in Ref. [4], the reduced density matrix of the spin d.o.f. is mixed when a Lorentz boost is performed and the momentum is traced out. The question arises whether it is possible to find "covariant measurements" of the spin and possibly momentum, which predict invariant probabilities in different Lorentzian reference frames also for the case of a quantum relativistic particle moving in a superposition of velocities. In this case, it would be possible to map the unambiguous description of spin in the rest frame of the particle to the frame of the laboratory, and therefore derive the corresponding observables to be measured in the laboratory frame to verify spin with probability one.

The question of finding such covariant measurements is motivated by the ubiquitous applications where the spin d.o.f. is used as a qubit, to encode and transmit quantum information. Such protocols are no longer valid in a relativistic context, thus limiting the range of applicability of techniques involving spin as a quantum information carrier. It is then important to explore possible alternative methods which could overcome this limitation. In the context of relativistic quantum information, this question has been extensively discussed [4–17] in relation to Wigner rotations [18–20] and has been related to the problem of identifying a covariant spin operator. The problem of identifying such a covariant spin operator had arisen long before the birth of relativistic quantum information, and dates back to the early times of quantum mechanics [21–23]. Since then, a multitude of relativistic spin operators have been proposed [24], such as the Frenkel [22], the Pauli-Lubański [25–27], the Pryce [28], the Foldy-Wouthuysen [29,30], the Czachor [31], the Fleming [32], the Chakrabarti [33], and the Fradkin-Good [34] spin operators. A comparative description of spin observables can be found in Ref. [17].

Here, we introduce "superposition of Lorentz boosts" which allow us to "jump" into the rest frame of a relativistic quantum particle even if the particle is *not* in a momentum eigenstate. In the rest frame, the spin observables fulfill the spin $\mathfrak{su}(2)$ algebra (the algebra of a qubit) and are operationally defined through the Stern-Gerlach experiment. We transform the set of spin observables in the rest frame back to an isomorphic set of observables in the laboratory frame. The transformed observables are in general entangled in the spin and momentum d.o.f. The set fulfills the $\mathfrak{su}(2)$ algebra and is operationally defined through a "relativistic Stern-Gerlach experiment": we construct the interaction and the measurement between the spin-momentum d.o.f. and the electromagnetic field in the laboratory frame which gives the same probabilities as the Stern-Gerlach experiment in the rest frame. This set of observables in the laboratory frame allows us to partition the total Hilbert space into two (highly degenerate) subspaces corresponding to the two outcomes "spin up" and "spin down." Hence, with QRFs techniques the relativistic spin can effectively be described as a qubit in an operationally well-defined way.

A relativistic Stern-Gerlach experiment.—In the following, we build a QRF transformation between the reference frame of a laboratory C of mass m_C and the rest frame of the external d.o.f. A of a relativistic quantum particle of mass $m_A > 0$ with spin d.o.f. $\tilde{A}$, as illustrated in Fig. 1. We allow the particle to have any quantum state, and in particular to move in a superposition of momenta. This implies that there is a nonclassical relationship between the initial and the final reference frame, i.e., that the rest frame A and the laboratory frame C are not related by a standard boost transformation. We show in this section how to generalize the boost transformation to this case. Formally, the situation we consider can be described by taking the one-particle sector of the positive-energy solutions of the Dirac equation in the Foldy-Wouthuysen representation [29]. (For simplicity, we only consider spin-1/2 particles, but the method can be straightforwardly applied to arbitrary spin).

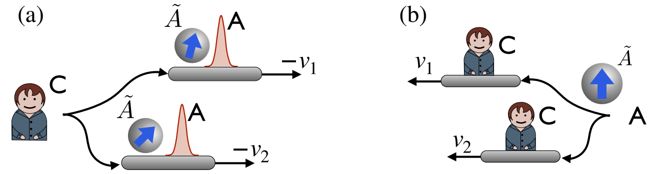

FIG. 1. (a) The state of a Dirac particle A with spin $\tilde{A}$ as seen from the laboratory perspective (C). When the state is in a superposition of relativistic velocities $-v_1$ and $-v_2$, the spin d.o.f. and the momentum d.o.f. are no longer separable. (b) The state of the spin $\tilde{A}$ and of the laboratory C as seen in the rest frame of the quantum particle A. In this quantum reference frame, the spin is operationally defined by means of the Stern-Gerlach experiment.

Following Ref. [1] (see Supplemental Material [35] for a review of the original formalism), when we "stand" in the rest frame of a particle, we describe all the systems external to the particle, but not the external d.o.f. (i.e., the momentum d.o.f.) of the particle itself. Hence, the quantum state describes the relational information in a given reference frame. In the reference frame in which A is at rest, the quantum state is assigned to the internal d.o.f. $\tilde{A}$ and the laboratory C. For simplicity, we consider that the particle and the laboratory are moving with constant, yet not necessarily well-defined, relative velocity and define the x axis along the direction of the relative motion. The total state of the spin and the laboratory is assumed to be

$$|\Psi\rangle_{\tilde{A}C}^{(A)} = |\vec{\sigma}\rangle_{\tilde{A}}|\psi\rangle_C, \tag{1}$$

where $|\vec{\sigma}\rangle_{\tilde{A}}$ is any vector representing the state of the spin in the rest frame A. In the rest frame, the spin state can in principle be tomographically verified by performing a series of standard Stern-Gerlach measurements. The state of the laboratory has a momentum-basis representation along the x direction (at this stage, we neglect the quantum state in the y and z direction) $|\psi\rangle_C = \int d\mu_C(\pi_C)\psi(\pi_C)|\pi_C\rangle_C$, where $d\mu_C(\pi_C) = d\pi_C/[(2\pi)^{1/2}\sqrt{2(m_C^2c^2 + \pi_C^2)}]$ is the Lorentz-covariant integration measure.

We now construct the transformation corresponding to the "superposition of Lorentz boosts" to the QRF of the laboratory. The unitary operator to boost to the QRF C is

$$\hat{S}_L = \mathcal{P}_{CA}^{(v)} U_{\tilde{A}}(\hat{\pi}_C), \tag{2}$$

where $U_{\tilde{A}}(\hat{\pi}_C)$ is a unitary transformation acting on the total Hilbert space $\mathcal{H}_{\tilde{A}} \otimes \mathcal{H}_C$ (notice that $\hat{\pi}_C$ is an operator), and $\mathcal{P}^{(v)}_{AC}$ is the "generalized parity operator" introduced in Ref. [1], whose explicit expression is $\mathcal{P}^{(v)}_{CA} = P_{AC} \exp[(i/\hbar)\log\sqrt{m_A/m_C}(\hat{q}_C\hat{\pi}_C + \hat{\pi}_C\hat{q}_C)]$, where P_{AC} is the parity-swap operator mapping $\hat{x}_A \to -\hat{q}_C$ and $\hat{p}_A \to -\hat{\pi}_C$ (and viceversa), where $\hat{q}_C$, $\hat{\pi}_C$ are canonically conjugated one-particle operators of C in the reference frame of A and $\hat{x}_A$, $\hat{p}_A$ are canonically conjugated one-particle operators of A in the reference frame of C. Additionally to the action of P_{AC}, the operator $\mathcal{P}^{(v)}_{AC}$ rescales the momentum of A by the ratio of the masses of A and C, i.e., $\mathcal{P}^{(v)}_{AC}\hat{p}_A\mathcal{P}^{(v)\dagger}_{AC} = -(m_A/m_C)\hat{\pi}_C$. This enforces the physical condition that the velocity of A is mapped to the opposite of the velocity of C via the transformation. (For a relativistic particle the relation between the i-th velocity component and the momentum is $v_i = (p_i/m_i)[1 + (|\vec{p}|^2/m_i^2c^2)]^{-1/2}$, where $|\vec{p}|^2$ is the norm of the spatial momentum. Therefore, only the ratio between momentum and mass determines the velocity.) The operator $\hat{S}_L$ can be defined via its action on a basis of the total Hilbert space of the spin and the laboratory $\hat{S}_L|\vec{\sigma}\rangle_{\tilde{A}}|\pi\rangle_C = |-(m_A/m_C)\pi;\Sigma_\pi\rangle_{A\tilde{A}}$, where the state $|p;\Sigma_p\rangle_{A\tilde{A}}$ is defined via a standard Lorentz boost $\hat{U}(L_p)$ from the rest frame as $|p;\Sigma_p\rangle_{A\tilde{A}} = \hat{U}(L_p)|k;\vec{\sigma}\rangle_{A\tilde{A}}$ and $k = (mc,\vec{0})$ is the momentum in the rest frame. In the Supplemental Material [35] we derive the transformation $\hat{S}_L$ in terms of standard Lorentz boosts connecting two relativistic reference frames where the parameter of the boost transformation is promoted to an operator.

The state of A and $\tilde{A}$ expressed in the laboratory frame is $|\Psi\rangle^{(C)}_{A\tilde{A}} = \hat{S}_L|\Psi\rangle^{(A)}_{\tilde{A}C}$, and is explicitly written as

$$|\Psi\rangle^{(C)}_{A\tilde{A}} = \int d\mu_A(p_A)\psi\left(-\frac{m_C}{m_A}p_A\right)|p_A;\Sigma_{p_A}\rangle_{A\tilde{A}}, \quad (3)$$

where $d\mu_A(p_A) = dp_A/[(2\pi)^{1/2}\sqrt{2(m_A^2c^2 + p_A^2)}]$ and the spin d.o.f. cannot be separated anymore from the momentum d.o.f., which means that the state is not a product state in the laboratory frame. Notice that the effect of the $\hat{S}_L$ transformation is to apply the usual boost transformation conditional on C's momentum d.o.f. In the laboratory frame C, unless particle A is in a sharp momentum state, no spin measurement in a standard Stern-Gerlach experiment would give a result with probability one, because of two reasons: the spin and momentum are no longer separable, and the relation between the laboratory and the rest frame is not a standard (classical) reference frame transformation. Our goal is to devise a different measurement in the laboratory reference frame, possibly involving both the spin and momentum d.o.f., which gives the same probability distribution as a standard Stern-Gerlach experiment would give, if performed in the rest frame.

In order to devise such measurement we note that, in the laboratory frame, it is possible to define the observables corresponding to the spin operators in the rest frame by transforming the spin, as defined in the rest frame, with a QRF transformation

$$\hat{\Xi}_i = \hat{S}_L(\hat{\sigma}_i \otimes \mathbb{1}_C)\hat{S}_L^\dagger, \qquad i = x, y, z. \quad (4)$$

In terms of the momenta and of the manifestly covariant Pauli-Lubański operator $\hat{\Sigma}_{\hat{p}_A} = (\hat{\Sigma}^0_{\hat{p}_A}, \hat{\vec{\Sigma}}_{\hat{p}_A})$, the operators $\hat{\Xi}_i$ are expressed as (see Supplemental Material [35]) $\hat{\vec{\Xi}} = \hat{\vec{\Sigma}}_{\hat{p}_A} - \hat{\gamma}_A/(\hat{\gamma}_A+1)(\hat{\vec{\Sigma}}_{\hat{p}_A}\cdot\hat{\vec{\beta}}_A)\hat{\vec{\beta}}_A$, where $\hat{\gamma}_A = \sqrt{1+\hat{p}_A^2/(m_A^2c^2)}$ and $\hat{\vec{\beta}}_A = (\hat{\beta}^x_A, \hat{\beta}^y_A, \hat{\beta}^z_A)$, where each component is $\hat{\beta}^i_A = (\hat{p}^i_A/\sqrt{m_A^2c^2 + \hat{\vec{p}}_A^2})$ with $i = x, y, z$. The operators $\hat{\Xi}_i$ are equivalent to the Foldy-Wouthuysen [29] or Pryce spin operator [28]. By definition, these operators satisfy the $\mathfrak{su}(2)$ algebra $[\hat{\Xi}_i, \hat{\Xi}_j] = i\epsilon_{ijk}\hat{\Xi}_k$, and have the same eigenvalues as the Pauli operators $\hat{\sigma}_i$, $i = x, y, z$. This last property can be easily checked by choosing an eigenvector $|\lambda_i\rangle$ of the operator $\hat{\sigma}_i$ in the rest frame A, such that $\hat{\sigma}_i|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$ and by noting that $\hat{\Xi}_i\hat{S}_L|\lambda_i\rangle_{\tilde{A}}|\psi\rangle_C = \lambda_i\hat{S}_L|\lambda_i\rangle_{\tilde{A}}|\psi\rangle_C$. Hence, it is possible to partition the total Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_{\tilde{A}}$ into two equivalence classes, defined as

$$\mathcal{H}_0 = \{|\Psi\rangle_{A\tilde{A}} \in \mathcal{H}_A \otimes \mathcal{H}_{\tilde{A}} \text{ such that } |\Psi\rangle_{A\tilde{A}} \sim \hat{S}_L|0\rangle_{\tilde{A}}|\psi\rangle_C,\ \forall\, |\psi\rangle_C \in \mathcal{H}_C\}, \quad (5a)$$

$$\mathcal{H}_1 = \{|\Phi\rangle_{A\tilde{A}} \in \mathcal{H}_A \otimes \mathcal{H}_{\tilde{A}} \text{ suchthat } |\Phi\rangle_{A\tilde{A}} \sim \hat{S}_L|1\rangle_{\tilde{A}}|\phi\rangle_C,\ \forall\, |\phi\rangle_C \in \mathcal{H}_C\}, \quad (5b)$$

where $|0\rangle_{\tilde{A}}$ and $|1\rangle_{\tilde{A}}$ are the eigenvectors of $\hat{\sigma}_z$ and two states are said to be equivalent, i.e., $|\Psi\rangle_{A\tilde{A}} \sim \hat{S}_L|i\rangle_{\tilde{A}}|\psi\rangle_C$, with $i = 0, 1$, if they are both eigenvectors of the $\hat{\Xi}_z$ operator with the same eigenvalue. (Notice that we could have chosen any other Pauli operator to define this partition.) We can then build a partition of the Hilbert space into two highly degenerate subspaces, one corresponding to the "spin up" and the other to the "spin down" eigenvalue, and on which it is possible to define a set of operators satisfying the $\mathfrak{su}(2)$ algebra, which can be used to encode or decode information of a single qubit.

The operators $\hat{\Xi}$ in general act on both the external and the internal d.o.f. of the particle. Operationally, they can be defined via a "relativistic Stern-Gerlach experiment," illustrated in Fig. 2. Traditionally, in a Stern-Gerlach experiment, the spin measurement is performed by applying a magnetic field, which interacts with the spin as $\vec{B}\cdot\vec{\sigma}$ and is inhomogeneous along the direction of its orientation, i.e., $\vec{B} = B(\vec{r}\cdot\vec{n})\vec{n}$, where $\vec{n}$ gives the direction and $\vec{r} = (x, y, z)$. If the magnetic field is aligned precisely in the direction in which the spin state is prepared, the

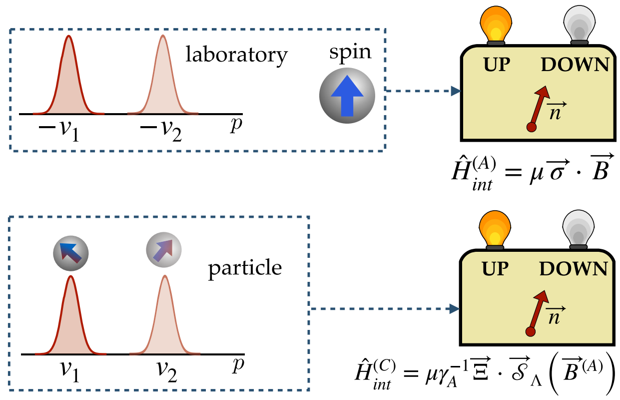

FIG. 2. The relativistic Stern-Gerlach experiment as seen from the QRF A (above) and from the QRF C (below). In the rest frame of particle A, the spin is operationally defined via the Stern-Gerlach experiment. To measure spin along direction $\vec{n}$ the spin (Pauli operator) $\vec{\sigma}$ is coupled to an inhomogeneous magnetic field oriented along $\vec{n}$. The particle is then deflected towards the direction $\vec{n}$ and $-\vec{n}$ corresponding to outcome spin up and spin down, respectively. When transforming to the laboratory frame C, the magnetic field and the spin transform with a superposition of Lorentz boosts for v_1 and v_2. The interaction Hamiltonian is also transformed, giving rise to a coupling between the transformed vector $\vec{\mathcal{S}}_\Lambda(\vec{B}^{(A)}) = \hat{\gamma}_A[\vec{B}^{(C)} - \hat{\gamma}_A/(\hat{\gamma}_A+1)(\hat{\vec{\beta}}_A \cdot \vec{B}^{(C)})\hat{\vec{\beta}}_A + (\hat{\vec{\beta}}_A \times \vec{E}^{(C)})]$ aligned in the same direction $\vec{n}$ as the magnetic field in the rest frame, and the transformed spin operator $\hat{\vec{\Xi}}$. The particle is again deflected either to $\vec{n}$ or $-\vec{n}$ corresponding to the outcome "spin up" and "spin down", respectively. The probability of detecting the outcomes "spin up" and "spin down" is preserved under change of QRF.

outcome is obtained with certainty. However, if the particle carrying the spin is moving in a superposition of relativistic velocities, no measurement of the spin alone in the laboratory frame will return the result with probability one in general. To treat such a case we set up a hypothetical Stern-Gerlach experiment in the rest frame of the particle, where the interaction Hamiltonian is $H^{(A)}_{\mathrm{int}} = \mu \vec{B}^{(A)} \cdot \vec{\sigma}$ and μ is a coupling constant. We assume that the direction in which the magnetic field is aligned $\vec{n}$ is orthogonal to the direction of the boost x. Formally, this geometric configuration requires us to enlarge the Hilbert space of the laboratory to the z direction, which we identify with the direction $\vec{n}$ of deflection, and modify our previous definition of the state in Eq. (1) as $|\psi\rangle_C = |\psi_x\rangle_C |\psi_z\rangle_C$, where $|\psi_x\rangle_C$ transforms with $\hat{S}_L$ and $|\psi_z\rangle_C$ is left invariant by the transformation $\hat{S}_L$, except for the fact that the label is changed from C to A, i.e., $\hat{S}_L |\psi_z\rangle_C = |\psi_z\rangle_A$. Additionally, we assume that the motion in the z direction is nonrelativistic. We then transform the Hamiltonian to the laboratory frame via the QRF transformation $\hat{S}_L$. Knowing that the magnetic field transforms under superposition of Lorentz boosts as $\hat{\vec{\mathcal{S}}}_\Lambda(\vec{B}^{(A)}) = \hat{\gamma}_A[\vec{B}^{(C)} - \hat{\gamma}_A/(\hat{\gamma}_A+1)(\hat{\vec{\beta}}_A \cdot \vec{B}^{(C)})\hat{\vec{\beta}}_A + (\hat{\vec{\beta}}_A \times \vec{E}^{(C)})]$, we find that the interaction Hamiltonian $H^{(A)}_{\mathrm{int}}$ is transformed to

$$H^{(C)}_{\mathrm{int}} = \mu \hat{\gamma}_A^{-1} \hat{\vec{\mathcal{S}}}_\Lambda(\vec{B}^{(A)}) \cdot \hat{\vec{\Xi}}. \tag{6}$$

It is straightforward to check that the direction of $\hat{\vec{\mathcal{S}}}_\Lambda(\vec{B}^{(A)})$ is also $\vec{n}$; therefore the deflection of the particle in the laboratory frame happens in the same direction as in the rest frame. Notice that, since both the quantum state and the observables transform unitarily, probabilities are automatically conserved after the change of QRF. In particular, if in the rest frame of the particle A the Stern-Gerlach measurement detects that the spin is "up" with probability one, the "relativistic Stern-Gerlach" experiment in the laboratory frame with the interaction Hamiltonian of Eq. (6) will also detect "spin up" with probability one. Note that the specific form of the electromagnetic field in Eq. (6) is not crucial to our result, but we can design the coupling between the particle and the electromagnetic field according to our experimental capabilities in each reference frame. However, it is crucial that the electromagnetic field couples to the operator $\hat{\vec{\Xi}}$, unlike in the standard Stern-Gerlach experiment. In the Supplemental Material [35], we set up a different experiment, where we couple an inhomogenous magnetic field in the laboratory frame to give an explicit analysis of a relativistic Stern-Gerlach experiment.

It is worth noting that the interaction Hamiltonian of Eq. (6) is covariant, because the quantity $H^0 \coloneqq \hat{\gamma}_A H^{(C)}_{\mathrm{int}}$ transforms like the zero component of a 4-vector. Therefore, the Schrödinger equation in the reference frame of A, $i\hbar(d/dt_A)|\psi\rangle^{(A)}_{\tilde{A}C} = H^{(A)}_{\mathrm{int}}|\psi\rangle^{(A)}_{\tilde{A}C}$, where t_A is the proper time in the rest frame of A, is mapped to $i\hbar(d/dt_C)|\psi\rangle^{(C)}_{\tilde{A}A} = H^{(C)}_{\mathrm{int}}|\psi\rangle^{(C)}_{\tilde{A}A}$, where t_C is the proper time in the rest frame of C and the relation $t_C = \hat{\gamma}_A t_A$ holds. The general, manifestly covariant expression of H^0 is

$$H^0 = \frac{1}{2}\eta^{0\rho}\epsilon_{\rho\mu\nu\lambda}\hat{\Sigma}^{\mu}_{p_A}F^{\nu\lambda}, \tag{7}$$

where $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric, $F^{\nu\lambda}$ is the electromagnetic tensor, and $\epsilon_{\rho\mu\nu\lambda}$ is the totally antisymmetric tensor such that $\epsilon_{0123} = 1$.

In order to complete the measurement, we now have to project the position of the particle along the z direction. Formally, this is achieved by defining the two operators $\hat{\Pi}^{(A)}_{+} = \int_0^{+\infty} dz_c |z_C\rangle_C\langle z_C|$ and $\hat{\Pi}^{(A)}_{-} = \int_{-\infty}^{0} dz_c |z_C\rangle_C\langle z_C|$, distinguishing whether the particle is, respectively, deflected upwards or downwards. For a thorough analysis of a concrete detection of spin via the "relativistic Stern-Gerlach experiment" proposed here and more details on the measurement, see Supplemental Material [35].

The QRF transformation provides the description of the same experiment from the point of view of two different QRFs, which move in a superposition of velocities relative to each other. This treatment of the relativistic Stern-Gerlach experiment makes it possible to associate an operational meaning to the spin of a relativistic quantum particle, thus solving the problem of encoding quantum information in a particle with spin d.o.f. as in a qubit.

Conclusions.—In this Letter, we have provided an operational description of the spin of a special-relativistic quantum particle. Such an operational description is hard to obtain with standard methods due to the combined effect of special relativity, which makes the spin and momentum not separable, and quantum mechanics, which makes it impossible to jump to the rest frame with a standard reference frame transformation. We have introduced the "superposition of Lorentz boosts" transformation to the rest frame of a quantum particle, moving in a superposition of relativistic velocities from the point of view of the laboratory. We have found how the state transforms under such a quantum reference frame transformation and identified a set of observables in the laboratory frame which satisfies the $\mathfrak{su}(2)$ algebra and has the same eigenvalues as the spin in its rest frame. In addition, this set complies with the desiderata for a relativistic spin operator in Ref. [24]: it commutes with the free Dirac Hamiltonian, it satisfies the $\mathfrak{su}(2)$ algebra, and it has the same eigenvalues as the spin in its rest frame. In addition, it has the correct nonrelativistic limit. It can be easily shown, in fact, that our operator $\vec{\hat{\Xi}}$ coincides with the Foldy-Wouthuysen spin operator [29,30]. Thanks to the unitarity of the transformation, probabilities are the same in the rest frame and in the laboratory frame. Finally, we have generalized the Stern-Gerlach experiment to the special-relativistic regime by means of a transformation of the interaction Hamiltonian from the rest frame to the laboratory frame. Such generalization opens up the possibility of performing quantum information protocols with spin in the special-relativistic regime.

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