Open Quantum Symmetric Simple Exclusion Process

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We present the solution to a model of fermions hopping between neighboring sites on a line with random Brownian amplitudes and open boundary conditions driving the system out of equilibrium. The average dynamics reduces to that of the symmetric simple exclusion process. However, the full distribution encodes for a richer behavior, entailing fluctuating quantum coherences which survive in the steady limit. We determine exactly the steady statistical distribution of the system state. We show that the out-of-equilibrium quantum coherence fluctuations satisfy a large-deviation principle, and we present a method to recursively compute exactly the large-deviation function. As a by-product, our approach gives a solution of the classical symmetric simple exclusion process based on fermion technology. Our results open the route towards the extension of the macroscopic fluctuation theory to many-body quantum systems.

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Introduction.—Nonequilibrium phenomena are ubiquitous in nature. Understanding the fluctuations of the flux of heat or particles through systems is a central question in nonequilibrium statistical mechanics. The last decade has witnessed tremendous conceptual and technical progress in this direction for classical systems, starting from the exact analysis of simple models [1–3], such as the symmetric simple exclusion process (SSEP) [4–7], via the understanding of fluctuation relations [8–10] and their interplay with time reversal [11,12], and culminating in the formulation of macroscopic fluctuation theory (MFT), which is an effective theory adapted to describe transport and its fluctuations in diffusive classical systems [13]. Whether MFT may be extended to the quantum realm is yet unexplored.

In parallel, the study of quantum systems out of equilibrium has received a large amount of attention in recent years [14–17]. Experimentally, unprecedented control of cold atom gases gave access to the observation of many-body quantum systems in inhomogeneous and isolated setups [18–22]. Theoretically, results about closed quantum systems have recently flourished, with a better perception of the roles of integrability and chaos or disorder [23–34]. In critical or integrable models, a good understanding has been obtained with a precise description of entanglement dynamics and quenched dynamics, as well as transport [35–46]. These efforts culminated in the development of a hydrodynamic picture adapted to integrable systems [47,48]. However, these understandings are restricted to closed, predominantly ballistic, systems.

Many quantum transport processes are diffusive rather than ballistic [49] and, to some extents, physical systems are generically in contact with external environments. It is thus crucial to extend the previous studies by developing simple models for fluctuations in open, quantum many-body, locally diffusive, out-of-equilibrium systems. Putting aside the quantum nature of the environments leads us to consider model systems interacting with classical reservoirs or noisy external fields. In the context of quantum many-body systems, and especially quantum spin chains, the study of such models has recently been revitalized [50–57], partly in connection with random quantum circuits [58–64], as a way to get a better understanding of entanglement production or information spreading.

In this work, we introduce and solve an iconic example of such models. It is a stochastic variant of the Heisenberg XX spin chain. It codes for typical features of quantum many-body at scales smaller than the coherence length (to keep interference effects) but larger than the mean free path (to include diffusion). It describes fermions hopping from site to site on a discretized line, but with Brownian hopping amplitudes, and interacting with reservoirs at the chain boundaries. For reasons explained below, we may call this model the *quantum SSEP*. Its average dynamics reduces to the classical SSEP, but the model codes for the fluctuations around this mean behavior. Although decoherence is at play in the mean behavior, fluctuating quantum coherences survive to the noisy interaction. Their magnitudes typically scale proportionally with the inverse of the square root of the system size. We characterize completely the steady measure on the system state which encodes for the fluctuations of the quantum coherences and occupation numbers at large values of time. We also present a recursive method to compute exactly, order by order, the large-deviation function of these fluctuations. These findings open the route towards the extension of the MFT [13] to many-body quantum systems.

The open quantum SSEP.—For an open chain in contact with external reservoirs at their boundaries, the quantum SSEP dynamics results from the interplay between unitary, but stochastic, bulk flows with dissipative, but deterministic, boundary couplings. The bulk flows induce unitary evolutions of the system density matrix ρ_t onto $e^{-idH_t}\rho_t e^{idH_t}$ with Hamiltonian increments

$$dH_t = \sqrt{D}\sum_{j=0}^{L-1}(c_{j+1}^\dagger c_j dW_t^j + c_j^\dagger c_{j+1} d\bar{W}_t^j) \qquad (1)$$

for a chain of length L; where c_j and $c_j^\dagger$ are canonical fermionic operators, one pair for each site of the chain, with $\{c_j, c_k^\dagger\} = \delta_{j;k}$; and W_t^j and $\bar{W}_t^j$ are pairs of complex conjugated Brownian motions, one pair for each edge along the chain, with quadratic variations $dW_t^j d\bar{W}_t^k = \delta^{j;k}dt$. This model was shown to describe the effective dynamics of the stochastic Heisenberg XX spin chain with dephasing noise in the strong noise limit [65]. It codes for a diffusive evolution of the number operators $\hat{n}_j = c_j^\dagger c_j$, with the parameter D being the diffusion constant. This model is one of the simplest models of quantum stochastic diffusion. It shares similarities with that of Ref. [57]. Exact results concerning the statistical mean behavior of this model, and more generally of the dephasing Heisenberg spin chain, were described in Refs. [66–69]. Properties of the closed periodic version of this model were deciphered in Ref. [70] via a mapping to random matrix theory. We set $D = 1$ in the following.

Assuming the interaction between the chain and the reservoirs to be Markovian, the contacts with the external leads can be modeled by Lindblad terms [71]. The resulting equations of motion read

$$d\rho_t = -i[dH_t, \rho_t] - \frac{1}{2}[dH_t, [dH_t, \rho_t]] + \mathcal{L}_{\mathrm{bdry}}(\rho_t)dt, \qquad (2)$$

with dH_t as above and $\mathcal{L}_{\mathrm{bdry}}$ being the boundary Lindbladian. The first two terms result from expanding the unitary increment $\rho_t \to e^{-idH_t}\rho_t e^{idH_t}$ to second order (because the Brownian increments scale as $\sqrt{dt}$). The third term codes for the dissipative boundary dynamics, with $\mathcal{L}_{\mathrm{bdry}} = \alpha_0\mathcal{L}_0^+ + \beta_0\mathcal{L}_0^- + \alpha_L\mathcal{L}_L^+ + \beta_L\mathcal{L}_L^-$ and

$$\mathcal{L}_j^+(\bullet) = c_j^\dagger \bullet c_j - \frac{1}{2}(c_j c_j^\dagger \bullet + \bullet\, c_j c_j^\dagger), \qquad (3)$$

$$\mathcal{L}_j^-(\bullet) = c_j \bullet c_j^\dagger - \frac{1}{2}(c_j^\dagger c_j \bullet + \bullet\, c_j^\dagger c_j), \qquad (4)$$

where the parameters α_j (β_j) are the injection (extraction) rates.

The dynamics being noisy, so is the density matrix, and hence the quantum expectations such as the mean quantum occupation numbers $n_j = \mathrm{Tr}(\hat{n}_j\rho_t)$. Their stochastic averages $\mathbb{E}[n_j]$ evolve according to

$$\partial_t\mathbb{E}[n_j] = \Delta_j^{\mathrm{dis}}\mathbb{E}[n_j] + \sum_{k\in\{0,L\}}\delta_{j;k}(\alpha_k(1-\mathbb{E}[n_k]) - \beta_k\mathbb{E}[n_k]),$$

with Δ_j^{dis} being the discrete Laplacian, $\Delta_j^{\mathrm{dis}}n_j = n_{j+1} - 2n_j + n_{j-1}$, illustrating the diffusive bulk dynamics and the boundary injection or extraction processes. At large values of time, they reach a linear profile,

$$n_j^* \coloneqq \lim_{t\to\infty}\mathbb{E}[n_j] = \frac{n_a(L+b-j) + n_b(j+a)}{L+a+b}, \qquad (5)$$

with $n_a \coloneqq [\alpha_0/(\alpha_0+\beta_0)]$, $n_b \coloneqq [\alpha_L/(\alpha_L+\beta_L)]$, $a \coloneqq [1/(\alpha_0+\beta_0)]$, $b \coloneqq [1/(\alpha_L+\beta_L)]$, associated with a steady mean flow from one reservoir to the other. In the large size limit, $L\to\infty$ at $x = i/L$ fixed, this mean profile, $n^*(x) = n_a + x(n_b - n_a)$, interpolates linearly the two boundary mean occupations n_a and n_b, in agreement with Refs. [57,66–69].

In the mean, the off-diagonal quantum expectations $G_{ji} \coloneqq \mathrm{Tr}(c_i^\dagger c_j\rho_t)$ vanish exponentially fast, $\lim_{t\to\infty}\mathbb{E}[G_{ji}] = 0$ for $j \neq i$, hence reflecting decoherence due to destructive interferences induced by the noise. However, this statement is only valid in the mean as fluctuating coherences survive at subleading orders with a rich statistical structure, with long-range correlations.

The steady state: Fluctuations and coherences.—As exemplified by the above one-point functions, a steady state is attained at large values of time in the sense that the distribution of quantum expectations reaches a stationary value. Equivalently, the limit $\lim_{t\to\infty}\mathbb{E}[F(G_t)]$ exists for any smooth function F of the matrix of two-point quantum expectations G, and this defines an invariant measure $\mathbb{E}_\infty[\bullet]$ of the flow [Eq. (2)], which we shall denote by $[\bullet]$ to simplify the notation. Diagonal elements G_{jj} code for occupation numbers, while the off-diagonal elements G_{ji} code for coherences, and hence $[\bullet]$ codes for their steady statistics.

Amongst the two-point functions $\mathbb{E}[G_{ij}G_{kl}]$, only those with $\{i=j, k=l\}$ and $\{i=l, j=k\}$ survive at large time values; the others decrease exponentially fast. This leaves us with three possible configurations: $[G_{ii}^2]$, $[G_{ii}G_{jj}]$, and $[G_{ij}G_{ji}]$, with $j\neq i$, coding respectively for quantum occupation and coherence fluctuations. They are determined by solving the stationarity equations for the invariant measure (see the Supplemental Material [72]):

$$[G_{ij}G_{ji}]^c = \frac{(\Delta n)^2(i+a)(L-j+b)}{(L+a+b-1)(a+b+L)(a+b+L+1)},$$

$$[G_{ii}G_{jj}]^c = -\frac{(\Delta n)^2(i+a)(L-j+b)}{(L+a+b-1)(a+b+L)^2(a+b+L+1)},$$

$$[G_{ii}^2]^c = \frac{(\Delta n)^2(2(i+a)(L-i+b) - (L+a+b))}{2(a+b+L)^2(a+b+L+1)},$$

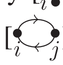
for $i<j$ with $\Delta n = n_b - n_a$ and $[G_{ii}G_{jj}]^c = [G_{ii}G_{jj}] - [G_{ii}][G_{jj}]$. The first lesson is that coherences are present in the large-time steady state, as their

covariances do not vanish exponentially but remain finite. At large size, $L \to \infty$ with $x = i/L$, $y = j/L$ fixed, their second moments behave as

$$[G_{ij}G_{ji}]^c = \frac{1}{L}(\Delta n)^2 x(1-y) + O(L^{-2}), \qquad (6)$$

$$[G_{ii}G_{jj}]^c = -\frac{1}{L^2}(\Delta n)^2 x(1-y) + O(L^{-3}), \qquad (7)$$

for $x < y$, while $[G_{ii}^2]^c = (1/L)(\Delta n)^2 x(1-x) + O(L^{-2})$. The second lesson is, on one hand, that these fluctuating coherences scale as $1/\sqrt{L}$ in the thermodynamic limit, and on the other hand, that the correlations between the quantum occupation numbers n_i and n_j at distinct sites $i \neq j$ scale as $1/L^2$, and hence are subleading. These correlations coincide with those of the statistical mean of the number operator two-point expectations, for reasons explained below, but this coincidence does not hold for higher ($N > 3$) point correlations.

These facts hold for higher-order cumulants $[G_{i_1 j_1} \dots G_{i_N j_N}]^c$ of the matrix of two-point quantum expectations. These cumulants are nonvanishing only if the sets $\{i_1, \dots, i_N\}$ and $\{j_1, \dots, j_N\}$ coincide so that the N-tuple $(j_1, \dots, j_N)$ is a permutation of $(i_1, \dots, i_N)$. With such a product $G_{i_1 j_1} \dots G_{i_N j_N}$, we can associate an oriented graph with a vertex for each point $i_1, \dots, i_N$ and an oriented edge from i to j for each insertion of G_{ji}. These graphs may be disconnected. The condition that the sets $\{i_1, \dots, i_N\}$ and $\{j_1, \dots, j_N\}$ coincide translates into the fact that the number of ongoing edges equals that of outgoing edges, at each vertex. For instance, $[G_{ii}]$ is represented by [], $[G_{ii}G_{jj}]$ for $i \neq j$ by [], $[G_{ij}G_{ji}]$ for $i \neq j$ by [] and $[G_{ii}^2]$ by [].

The claim is that expectations of single-loop diagrams, corresponding to the expectations of cyclic products $[G_{i_1 i_N} \dots G_{i_3 i_2} G_{i_2 i_1}]^c$, are the elementary building blocks in the large-size limit. They scale proportionally to $1/L^{N-1}$ in the thermodynamic limit

$$[G_{i_1 i_N} \dots G_{i_3 i_2} G_{i_2 i_1}]^c = \frac{1}{L^{N-1}} g_N(x_1, \dots, x_N) + O(L^{-N}), \qquad (8)$$

with $x_p = i_p/L$. The expectations g_N depend on which sector the points $\boldsymbol{x} := (x_1, \dots, x_N)$ belong to, with the sectors indexing how the ordering of the points along the chain match or unmatch that along the loop graph. Let us choose to fix an ordering of the points along the chain so that $0 \leq x_1 < \dots < x_N \leq 1$, and let σ be the permutation coding for the ordering of the point vertices around the loop so that by turning around the oriented loop, one successively encounters the vertices labeled by $x_{\sigma(1)}, x_{\sigma(2)}, \dots,$ up to $x_{\sigma(N)}$. There are $(N-1)!/2$ sectors because the ordering around the loop is defined up to cyclic permutations and because reversing the orientation of the loop preserves the expectations. Let us then set $f_N^\sigma(\boldsymbol{x}) := g_N(x_{\sigma(1)}, \dots, x_{\sigma(N)})$.

The f_N^σ's are recursively determined by a set of equations which arise from the stationarity conditions of the invariant measure (see the Supplemental Material [72]). First, stationarity in the bulk imposes that $\Delta_{x_j} f_N^\sigma(\boldsymbol{x}) = 0$ for all j, with Δ_x being the Laplacian with respect to x, as a consequence of the bulk diffusivity. Second, the couplings at the boundaries freeze the fluctuations so that

$$f_N^\sigma(\boldsymbol{x})|_{x_1=0} = f_N^\sigma(\boldsymbol{x})|_{x_N=1} = 0. \qquad (9)$$

Third, contact interactions due to noisy hoppings impose two conditions on expectations at the boundary between the sectors σ and $\pi_{j;j+1}\sigma$, with $\pi_{j;j+1}$ being the permutation transposing j and $j+1$. The ordering of the point vertices in the sectors σ and $\pi_{j;j+1}\sigma$ differs by the exchange of x_j and x_{j+1}, so that $x_{j+1} = x_j$ at these boundaries. The first contact condition is the continuity condition:

$$f_N^\sigma(\boldsymbol{x})|_{x_{j+1}=x_j} = f_N^{\pi_{j;j+1}\sigma}(\boldsymbol{x})|_{x_{j+1}=x_j}. \qquad (10)$$

To write the second contact condition, let us define j_*^- (j_*^+) to be the σ preimage of j ($j+1$), i.e., $j = \sigma(j_*^-)$ and $j+1 = \sigma(j_*^+)$. Since the vertices x_{j+1} and x_j are identified at these sector boundaries, the loop graph splits into two subloops touching at the vertex x_j, one including the circle arc $x_{\sigma(j_*^- - 1)}, x_j, x_{\sigma(j_*^+ + 1)}$, and the other containing the circle arc $x_{\sigma(j_*^+ - 1)}, x_j, x_{\sigma(j_*^- + 1)}$, denoted $\ell_j^{\sigma,-}$ and $\ell_j^{\sigma,+}$, respectively. The second contact condition is the Neumann-like matching condition

$$(\nabla_{x_j} - \nabla_{x_{j+1}})(f_N^\sigma(\boldsymbol{x}) + f_N^{\pi_{j;j+1}\sigma}(\boldsymbol{x}))|_{x_{j+1}=x_j} = 2(\nabla_{x_j}[\mathfrak{R}_j^+ f^\sigma](\boldsymbol{x}))(\nabla_{x_j}[\mathfrak{R}_j^- f^\sigma](\boldsymbol{x})), \qquad (11)$$

with $[\mathfrak{R}_j^\pm f^\sigma]$ being the expectations of the reduced subloops $\ell_j^{\sigma,\pm}$. Equations (9)–(11) allow us to recursively compute the building-block loop expectations [Eq. (8)]. See Fig. 1 for a graphical representation of Eq. (11).

Furthermore, connected expectations of pinched graphs obtained by identifying points in single-loop graphs are obtained by continuity from the expectations of the corresponding parent loop graphs, thanks to Eq. (10).

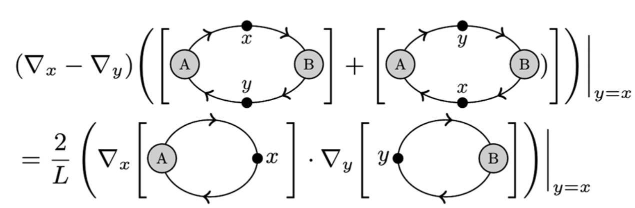


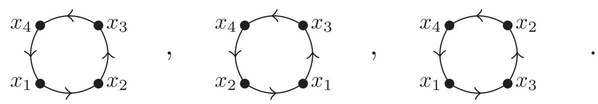
FIG. 1. Graphical representation of the contact relation [Eq. (11)].

They are of order $1/L^{N-1}$ with N being the number of edges in the pinched graph (and hence the number of insertions of matrix elements of G). All other connected expectations of disconnected graphs are subleading in the large-size limit.

The conditions (9), (10), and (11) allow us to determine all leading expectations recursively. For $N=3$, there is only one sector, and $g_3(x,y,z)=(\Delta n)^3 x(1-2y)(1-z)$ for $x<y<z$, so that

$$[G_{ik}G_{kj}G_{ij}]^c = \frac{1}{L^2}(\Delta n)^3 x(1-2y)(1-z) + O(L^{-3}), \quad (12)$$

with $x=i/L$, $y=j/L$, and $z=k/L$ $(i<j<k)$. For $N=4$, there are three sectors, associated with the identity and the transpositions $\pi_{1;2}$ and $\pi_{2;3}$, respectively:



For $x_1<x_2<x_3<x_4$, their expectations are, respectively,

$$\frac{1}{L^3}(\Delta n)^4 x_1(1-3x_2-2x_3+5x_2x_3)(1-x_4),$$
$$\frac{1}{L^3}(\Delta n)^4 x_1(1-3x_2-2x_3+5x_2x_3)(1-x_4),$$
$$\frac{1}{L^3}(\Delta n)^4 x_1(1-4x_2-x_3+5x_2x_3)(1-x_4),$$

up to $O(L^{-4})$ contributions.

The scaling behavior of the single-loop expectations [Eq. (8)] ensures that the fluctuations of the matrix of quantum two-point expectations G satisfy a large-deviation principle, in the sense that their generating function is such that $[e^{\mathrm{Tr}(AG)}] \asymp_{L\to\infty} e^{L\mathfrak{F}(A)}$ for some function $\mathfrak{F}(A)$, called the large-deviation function,

$$\mathfrak{F}(A) = \lim_{L\to\infty}\frac{1}{L}\log[e^{\mathrm{Tr}(AG)}]. \quad (13)$$

This function admits a series expansion, $\mathfrak{F}(A)=\sum_N (1/N!)\mathfrak{F}^{(N)}$, with the $\mathfrak{F}^{(N)}$'s given by the multiple sums $L^{-N}\sum_{i_1,\ldots,i_N}[G_{i_1 i_N}\cdots G_{i_3 i_2}G_{i_2 i_1}]^c(A_{i_1 i_2}\cdots A_{i_N i_1})$, which converge to multiple integrals. To the lowest order,

$$\mathfrak{F}(A) = \int_0^1 dx\, n^*(x)A(x,x) + (\Delta n)^2\int_0^1 dx\int_x^1 dy\, x(1-y)A(x,y)A(y,x) + \cdots. \quad (14)$$

Higher orders can be recursively computed by using Eqs. (9)–(11).

Sketch of proof.—Since both the Hamiltonian increments [Eq. (1)] and the Lindbladians [Eq. (3)] are quadratic in the fermionic creation and annihilation operators, the stochastic evolution [Eq. (2)] preserves Gaussian states of the form $\rho_t = Z_t^{-1}e^{c^\dagger M_t c}$, with M_t being a $L\times L$ matrix and $Z_t=\mathrm{Tr}(e^{c^\dagger M_t c})$. These density matrices are parametrized by M_t or, equivalently, by the matrix of quantum two-point expectations $G_{ij}=\mathrm{Tr}(\rho_t c_j^\dagger c_i)$. One can show that $G_t=[e^{M_t}/(1+e^{M_t})]$. Equation (2) then becomes a stochastic equation for M_t or G_t. For instance, for $0\neq i<j\neq L$,

$$dG_{ij} = -2G_{ij}dt + i(G_{i;j-1}d\bar{W}_t^{j-1} + G_{i;j+1}dW_t^j) - i(G_{i-1;j}dW_t^{i-1} + G_{i+1;j}d\bar{W}_t^i), \quad (15)$$

with similar equations for G_{ii} and at the two chain boundaries. Imposing the stationarity of the measure amounts to demanding that the statistical expectations $[F(G_t)]$ are time independent for any function F. Since the Itô derivatives of polynomials in G_t are polynomials in G_t of the same degrees, the stationarity conditions are sets of linear equations on moments of a given order. There are two types of contributions arising from the Itô derivatives of polynomials: one completing the drift term in Eq. (15) to produce discrete Laplacians acting on products of G_t's, the other producing contact interactions. For instance, $dG_{kj}dG_{l;j+1}|_{\mathrm{contact}} = -(G_{k;j+1}dW^j)(G_{lj}d\bar{W}^j) = -G_{k,j+1}G_{lj}dt$, which implements the transposition of the adjacent points j and $j+1$. As a consequence, the Itô derivatives of graphs coding for products of G_t's with adjacent indices induce a reshuffling of the connections of these graphs. See Fig. 2 for an illustration. Thus, the stationarity conditions yield relations between expectations of reshuffled graphs, from which the relations (9), (10), and (11) can be deduced (see the Supplemental Material [72]). More details will be described elsewhere [73].

Connecting to the classical SSEP.—The mean density matrix $\bar{\rho}_t \coloneqq \mathbb{E}[\rho_t]$ evolves according to the Lindblad equation $\partial_t\bar{\rho}_t = \mathcal{L}_{\mathrm{bulk}}(\bar{\rho}_t)+\mathcal{L}_{\mathrm{bdry}}(\bar{\rho}_t)$, with $\mathcal{L}_{\mathrm{bdry}}$ defined in Eq. (3) and the bulk Lindbladian

$$\mathcal{L}_{\mathrm{bulk}}(\bar{\rho}_t) = -\frac{1}{2}\sum_{j=0}^{L-1}([c_{j+1}^\dagger c_j,[c_j^\dagger c_{j+1},\bar{\rho}_t]] + \mathrm{H.c}). \quad (16)$$

This Lindblad dynamics has been studied in Refs. [66–69]. For density matrices diagonal in the occupation number

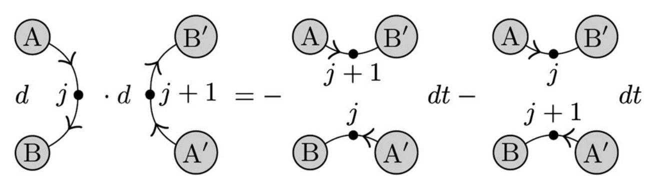


FIG. 2. Graphical representation of the reshuffling relation.

basis, it codes for the time evolution of SSEP. Asymptotically in time, decoherence is effective and the mean density matrix is diagonal, $\bar{\rho}_t = \sum_{[n]} \bar{Q}_t[\mathfrak{n}]\mathbb{P}_{[\mathfrak{n}]}$, where the $\mathbb{P}_{[\mathfrak{n}]}$'s are the projectors on the occupation number eigenstates $|[\mathfrak{n}]\rangle$ and $\bar{Q}_t[\mathfrak{n}]$ the mean populations. The $\mathbb{P}_{[\mathfrak{n}]}$'s are products of projectors $\mathbb{P}_j^{\mathfrak{n}_j}$ on each site of the chain, with $\mathfrak{n}_j = 0$ ($\mathfrak{n}_j = 1$) for empty (full). On adjacent pairs of projectors, the bulk Lindbladian acts as

$$\mathcal{L}_{\rm bulk}(\mathbb{P}_j^1\mathbb{P}_{j+1}^0) = -\mathbb{P}_j^1\mathbb{P}_{j+1}^0 + \mathbb{P}_j^0\mathbb{P}_{j+1}^1,$$
$$\mathcal{L}_{\rm bulk}(\mathbb{P}_j^0\mathbb{P}_{j+1}^1) = -\mathbb{P}_j^0\mathbb{P}_{j+1}^1 + \mathbb{P}_j^1\mathbb{P}_{j+1}^0,$$

whereas $\mathcal{L}_{\rm bulk}(\mathbb{P}_j^0\mathbb{P}_{j+1}^0) = 0$ and $\mathcal{L}_{\rm bulk}(\mathbb{P}_j^1\mathbb{P}_{j+1}^1) = 0$. This is equivalent to the SSEP transition matrix.

As a consequence, the SSEP generating function for the occupation number fluctuations can be identified with the statistical average of the generating function of quantum expectations of the number operators,

$$\langle e^{\sum_j a_j \mathfrak{n}_j}\rangle_{\rm ssep} = {\rm Tr}(\bar{\rho} e^{\sum_i a_i \hat{n}_i}) = [{\rm Tr}(\rho e^{\sum_i a_i \hat{n}_i})],$$

with $\hat{n}_i = c_i^\dagger c_i$. It can be computed using Wick's theorem, so that the SSEP cumulants read

$$\langle \mathfrak{n}_{j_1} \cdots \mathfrak{n}_{j_N}\rangle^c_{\rm ssep} = \frac{(-)^{N-1}}{L^{N-1}} \sum_\sigma f_N^\sigma(\boldsymbol{x}) + O(L^{-N}), \quad (17)$$

with $x_k = j_k/L$ all distinct. The sum is over permutations σ modulo cyclic permutations (see the Supplemental Material [72]). The expectations [Eq. (8)] of the matrix of quantum two-point expectations cannot be reconstructed from the SSEP expectations (for $N \geq 4$), because the latter are symmetric under permutations and hence only involve the sum of the sectors.

Discussion.—We have introduced a quantum extension of the SSEP and outlined how to solve it exactly by characterizing its invariant measure and computing the large-deviation function of the matrix of quantum two-point expectations. The quantum SSEP is a simple, if not the simplest, model coding for diffusive behavior of quantum operators in a many-body fermionic system. In the mean, it reduces to the classical SSEP so that the statistical averages of the quantum expectations of the number operators coincide with those of the classical SSEP.

The quantum SSEP is strictly finer than its classical counterpart and contains much more information, including fluctuations of quantum coherences. Although decoherence is at work on the mean steady state, we have observed and quantified subleading fluctuating coherences which are not visible in the mean behavior [66–68]. In the thermodynamic large-size limit, the system state approaches a self-averaging nonequilibrium state dressed by occupancy and coherence fluctuations whose amplitudes scale proportionally to $1/\sqrt{L}$. We have described how to compute the large-deviation function for these fluctuations, order by order. One simple experimental route to probe these coherences consists in transposing to our system the recently proposed setup [57] to conduct interferometry experiments between two parts of the system. Another possibility to generate echoes of coherence effects in the occupancy number correlations consists in injecting fermions in quantum states which are not eigenstates of the occupancy operators.

As an example of quantum out-of-equilibrium exclusion processes and of fluctuating quantum discrete hydrodynamics, our findings open several new research directions. The first concerns the integrable structure underlying the exact solution we have presented and its connection with the existing solution methods for classical exclusion processes [74–77]. The second concerns the extension of our work to deal with the quantum analogue of the asymmetric simple exclusion process (ASEP). We have already noticed that the appropriate generalization amounts to coupling the fermionic system to quantum noise [78,79]. But the most important ones deal with using the present model, and its generalizations to interacting systems, to formalize the extension of the MFT [13] to many-body quantum systems. Proposing such quantization of the MFT incorporating the fluctuating quantum coherences requires going beyond the statistical mean behavior. It has been observed in Ref. [80] that the additivity principle [81], which applies classically with some degree of generality, also holds for some statistics encoded into the mean system state of diffusive spin chains. How do we extend this principle to keep track of the quantum coherences and their statistical fluctuations? How do we take the continuum limit of those models to provide a quantization of the MFT? We plan to report on these questions in the near future [82].

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[72] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.123.080601 for more details. Part 1 illustrates how to compute exactly low order cumulants, part 2 provides a proof of the stationary conditions introduced in the main text, part 3 explains the connection to the classical symmetric simple exclusion process, part 4 presents some questions asked during the refereeing process and our answers to them.

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