Extendibility Limits the Performance of Quantum Processors

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Resource theories in quantum information science are helpful for the study and quantification of the performance of information-processing tasks that involve quantum systems. These resource theories also find applications in other areas of study; e.g., the resource theories of entanglement and coherence have found use and implications in the study of quantum thermodynamics and memory effects in quantum dynamics. In this paper, we introduce the resource theory of unextendibility, which is associated with the inability of extending quantum entanglement in a given quantum state to multiple parties. The free states in this resource theory are the k-extendible states, and the free channels are k-extendible channels, which preserve the class of k-extendible states. We make use of this resource theory to derive nonasymptotic, upper bounds on the rate at which quantum communication or entanglement preservation is possible by utilizing an arbitrary quantum channel a finite number of times, along with the assistance of k-extendible channels at no cost. We then show that the bounds obtained are significantly tighter than previously known bounds for quantum communication over both the depolarizing and erasure channels.

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Introduction.—Recent years have seen progress in the development of programmable quantum computers and information processing devices; several groups are actively developing superconducting quantum processors [1] and satellite-to-ground quantum key distribution [2]. It is thus pertinent to establish benchmarks on the information-processing capabilities of quantum devices that are able to process a finite number of qubits reliably. Experimentalists can then employ these benchmarks to evaluate how far they are from achieving the fundamental limitations on performance.

In this paper, we first develop a resource theory of unextendibility and then apply it to bound the performance of quantum processors. In particular, the resource theory of unextendibility leads to non-asymptotic upper bounds on the rate at which entanglement can be preserved when using a given quantum channel a finite number of times. We then apply this general bound to depolarizing and erasure channels, which are common models of noise in quantum processors. For these channels, we find that our bounds are significantly tighter than previously known nonasymptotic bounds from Refs. [3,4].

The resource theory of unextendibility can be understood as a relaxation of the well-known resource theory of entanglement [5,6], and it is a relaxation alternative to the resource theory of negative partial transpose states from Refs. [7,8], in which the free states are the positive partial transpose (PPT) states and the free channels are completely PPT-preserving channels. In the resource theory of entanglement, the free states are the separable states, those not having any entanglement at all. Any separable state σ_{AB} can be written as $\sigma_{AB} = \sum_x p(x)\tau_A^x \otimes \omega_B^x$, where $p(x)$ is a probability distribution and $\{\tau_A^x\}_x$ and $\{\omega_B^x\}_x$ are sets of states; the free channels are those that can be performed by local operations and classical communication (LOCC) [5,9]. An LOCC channel $\mathcal{L}_{AB\to A'B'}$ is a separable superoperator (although the converse is not true), and can hence be written as $\mathcal{L}_{AB\to A'B'} = \sum_y \mathcal{E}^y_{A\to A'} \otimes \mathcal{F}^y_{B\to B'}$, where $\{\mathcal{E}^y_{A\to A'}\}_y$ and $\{\mathcal{F}^y_{B\to B'}\}_y$ are sets of completely positive (CP) maps such that $\mathcal{L}_{AB\to A'B'}$ is trace preserving. A special kind of LOCC channel is a one-way (1W-) LOCC channel from A to B, in which Alice performs a quantum instrument, sends the classical outcome to Bob, who then performs a quantum channel conditioned on the classical outcome received from Alice. As such, any 1W-LOCC channel takes the form stated above, except that $\{\mathcal{E}^y_{A\to A'}\}_y$ is a set of CP maps such that the sum map $\sum_y \mathcal{E}^y_{A\to A'}$ is trace preserving, while $\{\mathcal{F}^y_{B\to B'}\}_y$ is a set of quantum channels.

The set of free states in the resource theory of unextendibility is larger than the set of free states in the resource theory of entanglement. By relaxing the resource theory of entanglement in this way, we obtain tighter, nonasymptotic bounds on the entanglement transmission rates of a quantum channel.

Before we begin with our development, we note here that detailed proofs of all statements that follow are given in our companion paper [10].

Resource theory of unextendibility.—In the resource theory of unextendibility, there is implicitly a positive integer $k \geq 2$, with respect to which the theory is defined. The free states in this resource theory are the k-extendible states [11–13], a prominent notion in quantum information and entanglement theory that we recall now. For a positive integer $k \geq 2$, a bipartite state ρ_{AB} is k extendible with respect to system B if (1) *(State extension)* There exists a state $\omega_{AB_1\cdots B_k}$ that extends ρ_{AB}, so that $\mathrm{Tr}_{B_2\cdots B_k}\{\omega_{AB_1\cdots B_k}\} = \rho_{AB}$, with systems B_1 through B_k each isomorphic to system B of ρ_{AB}. (2) *(Permutation invariance)* The extension state $\omega_{AB_1\cdots B_k}$ is invariant with respect to permutations of the B systems, in the sense that $\omega_{AB_1\cdots B_k} = W^{\pi}_{B_1\cdots B_k}\omega_{AB_1\cdots B_k}W^{\pi\dagger}_{B_1\cdots B_k}$, where $W^{\pi}_{B_1\cdots B_k}$ is a unitary representation of the permutation $\pi \in S_k$, with S_k denoting the symmetric group.

To give some physical context to the definition of a k-extendible state, suppose that Alice and Bob share a bipartite state and that Bob subsequently mixes his system and the vacuum state at a 50:50 beam splitter. Then the resulting state of Alice's system and one of the outputs of the beam splitter is a two-extendible state by construction. As a generalization of this, suppose that Bob sends his system through the N-splitter of Ref. [14], [Eq. (10)] with the other input ports set to the vacuum state. Then the state of Alice's system and one of the outputs of the N splitter is N extendible by construction. One could also physically realize k-extendible states in a similar way by means of approximate quantum cloning machines [15].

It is worthwhile to mention that there are free states in the resource theory of unextendibility that are not free in the resource theory of entanglement. For example, if we send one share of the maximally entangled state Φ_{AB} through a 50% erasure channel [16], then the resulting state $\frac{1}{2}(\Phi_{AB} + I_A/2 \otimes |e\rangle\langle e|_B)$ is a two-extendible state, and is thus free in the resource theory of unextendibility for $k = 2$. However, this state has distillable entanglement via LOCC [17], and so it is not free in the resource theory of entanglement.

Let $\mathrm{EXT}_k(A;B)$ denote the set of k-extendible states, where with this notation and as above, we take it as implicit that the system B is being extended. The k-extendible states are a relaxation of the set of separable (unentangled) states, in the sense that a separable state is k extendible for any positive integer $k \geq 2$. Furthermore, if a state ρ_{AB} is entangled, then there exists some k for which ρ_{AB} is not k extendible, and ρ_{AB} is not ℓ extendible for all $\ell > k$ [12,13].

We define the free channels in the resource theory of unextendibility to be bipartite channels that satisfy two constraints generalizing those given above for the free states. Recall that a bipartite channel $\mathcal{N}_{AB\to A'B'}$ has two input systems A and B and two output systems A' and B'. The systems A and A' are held by a single party Alice, and the systems B and B' are held by another party Bob. It could be the case that any of these systems encompass a number of smaller subsystems, and we make use of this in what follows. We define a bipartite channel $\mathcal{N}_{AB\to A'B'}$ to be k extendible if (1) *(Channel extension)* There exists a quantum channel $\mathcal{M}_{AB_1\cdots B_k\to A'B'_1\cdots B'_k}$ that extends $\mathcal{N}_{AB\to A'B'}$, in the sense that the following equality holds for all quantum states $\theta_{AB_1\cdots B_k}$: $\mathrm{Tr}_{B'_2\cdots B'_k}\{\mathcal{M}_{AB_1\cdots B_k\to A'B'_1\cdots B'_k}(\theta_{AB_1\cdots B_k})\} = \mathcal{N}_{AB\to A'B'}(\theta_{AB_1})$, with $B_1\cdots B_k$ each isomorphic to B, and $B'_1\cdots B'_k$ each isomorphic to B'. (2) *(Permutation covariance)* The extension channel $\mathcal{M}_{AB_1\cdots B_k\to A'B'_1\cdots B'_k}$ is covariant with respect to permutations of the input B and output B' systems, in the sense that the following equality holds for all quantum states $\theta_{AB_1\cdots B_k}$: $\mathcal{M}_{AB_1\cdots B_k\to A'B'_1\cdots B'_k}(W^{\pi}_{B_1\cdots B_k}\theta_{AB_1\cdots B_k}W^{\pi\dagger}_{B_1\cdots B_k}) = W^{\pi}_{B'_1\cdots B'_k}\mathcal{M}_{AB_1\cdots B_k\to A'B'_1\cdots B'_k}(\theta_{AB_1\cdots B_k})W^{\pi\dagger}_{B'_1\cdots B'_k}$, where $W^{\pi}_{B_1\cdots B_k}$ and $W^{\pi}_{B'_1\cdots B'_k}$ are unitary representations of the permutation $\pi \in S_k$.

The first condition above can be understood as a no-signaling condition. That is, it implies that it is impossible for the parties controlling the $B_2\cdots B_k$ systems to communicate to the parties holding systems $A'B'_1$.

We advocate that our definition above is a natural channel generalization of state extendibility, since the reduced channel $\mathcal{N}_{AB\to A'B'}$ of the channel extension $\mathcal{M}_{AB_1\cdots B_k\to A'B'_1\cdots B'_k}$ is defined in an unambiguous way only when we impose a no-signaling constraint. Furthermore, the above definition is quite natural in the resource theory of unextendibility developed here, as evidenced by the following theorem:

Theorem 1: Let $\rho_{AB} \in \mathrm{EXT}_k(A;B)$, and let $\mathcal{N}_{AB\to A'B'}$ be a k-extendible channel. Then the output state $\mathcal{N}_{AB\to A'B'}(\rho_{AB})$ is k extendible.

The above theorem is fundamental for the resource theory of unextendibility, indicating that the k-extendible channels are free, as they preserve the free states.

There are several interesting classes of k-extendible channels that we can consider. Even if it might seem trivial, we should mention that a particular kind of k-extendible channel is in fact a k-extendible state, in which the input systems A and B are trivial. Thus, k-extendible channels can generate k-extendible states.

Any 1W-LOCC channel is k extendible for all $k \geq 2$, similar to the way in which any separable state is k extendible for all $k \geq 2$. Thus, a 1W-LOCC channel is free in the resource theory of unextendibility. The fact that a 1W-LOCC channel takes a k-extendible input state to a k-extendible output state had already been observed for the special case $k = 2$ in Ref. [18].

Quantifying unextendibility.—In any resource theory, it is pertinent to quantify the resourcefulness of the resource states and channels. It is desirable for any quantifier to be non-negative, attain its minimum for the free states and channels, and be monotone under the action of a free channel [19]. With this in mind, we define the k-unextendible generalized divergence of an arbitrary density operator ρ_{AB} as follows:

$$\mathbf{E}_k(A;B)_\rho = \inf_{\sigma_{AB}\in \mathrm{EXT}_k(A;B)} \mathbf{D}(\rho_{AB}\|\sigma_{AB}), \quad (1)$$

where $\mathbf{D}(\rho\|\sigma)$ denotes a generalized divergence [20,21], which is any quantifier of the distinguishability of states ρ and σ that is monotone under the action of a quantum channel. Special cases of the quantifier in Eq. (1) were previously defined in Refs. [18,22] (relative entropy to two-extendible states and to k-extendible states, respectively), [23] (best two-extendible approximation, related to max-relative entropy of unextendibility defined here), and [24] (maximum k-extendible fidelity).

Particular examples of generalized divergences between states ρ and σ are the ε-hypothesis-testing divergence $D_h^\varepsilon(\rho\|\sigma)$ [25,26], and the max-relative entropy $D_{\max}(\rho\|\sigma)$ [27,28], where for $\varepsilon\in[0,1]$,

$$D_h^\varepsilon(\rho\|\sigma) \coloneqq -\log_2 \inf_{\Lambda\in[0,I]} \{\mathrm{Tr}\{\Lambda\sigma\} : \mathrm{Tr}\{\Lambda\rho\} \geq 1-\varepsilon\},$$

and $D_{\max}(\rho\|\sigma) \coloneqq \inf\{\lambda : \rho \leq 2^\lambda \sigma\}$ in the case that $\mathrm{supp}(\rho)\subseteq \mathrm{supp}(\sigma)$, and otherwise $D_{\max}(\rho\|\sigma) = +\infty$.

Information-processing tasks.—Now that we have established the free states and channels in the resource theory of unextendibility, we are ready to discuss tasks that can be performed in it. We consider two main tasks here: entanglement distillation and quantum communication with the assistance of k-extendible channels. The goal of these protocols is to use many copies of a bipartite state or many invocations of a quantum channel, along with the free assistance of k-extendible channels, in order to generate a high-fidelity maximally entangled state with as much entanglement as possible. This kind of task was defined and developed in Ref. [29], albeit with the assistance of a particular kind of k-extendible channel and only the case $k=2$ was considered there, generalizing the usual notion of entanglement distillation and quantum communication protocols from Refs. [5,30–36].

Let $n, M \in \mathbb{Z}^+$ and $\varepsilon\in[0,1]$. Let ρ_{AB} be a bipartite state. An (n,M,ε) entanglement distillation protocol assisted by k-extendible channels begins with Alice and Bob sharing n copies of ρ_{AB}, to which they apply a k-extendible channel $\mathcal{K}_{A^nB^n\to M_AM_B}$, where it is understood that this is a bipartite channel with Alice possessing systems A^n and M_A and Bob possessing systems B^n and M_B. The resulting state satisfies the following performance condition:

$$F[\mathcal{K}_{A^nB^n\to M_AM_B}(\rho_{AB}^{\otimes n}), \Phi_{M_AM_B}] \geq 1-\varepsilon, \quad (2)$$

where $\Phi_{M_AM_B} \coloneqq (1/M)\sum_{m,m'} |m\rangle\langle m'|_{M_A} \otimes |m\rangle\langle m'|_{M_B}$ is a maximally entangled state of Schmidt rank M and $F(\omega,\tau) \coloneqq \|\sqrt{\omega}\sqrt{\tau}\|_1^2$ is the quantum fidelity [37]. Let $D^{(k)}(\rho_{AB}, n, \varepsilon)$ denote the nonasymptotic distillable entanglement with the assistance of k-extendible channels; i.e., $D^{(k)}(\rho_{AB}, n, \varepsilon)$ is equal to the maximum value of $(1/n)\log_2 M$ such that there exists an (n,M,ε) protocol for ρ_{AB} satisfying Eq. (2).

We define two different variations of quantum communication, with one simpler and one more involved. Let $\mathcal{N}_{A\to B}$ denote a quantum channel. In the simpler version, an (n,M,ε) entanglement transmission protocol assisted by a k-extendible postprocessing begins with Alice preparing a maximally entangled state $\Phi_{RA'}$ of Schmidt rank M. She applies a quantum channel $\mathcal{E}_{A'\to A^n}$, which serves as an encoding and leads to a state $\rho_{RA^n} \coloneqq \mathcal{E}_{A'\to A^n}(\Phi_{RA'})$. She transmits the systems $A^n \coloneqq A_1\cdots A_n$ using the channel $\mathcal{N}_{A\to B}^{\otimes n}$. Alice and Bob then perform a k-extendible channel $\mathcal{K}_{RB^n\to M_AM_B}$, such that

$$F[\mathcal{K}_{RB^n\to M_AM_B}(\mathcal{N}_{A\to B}^{\otimes n}(\rho_{RA^n})), \Phi_{M_AM_B}] \geq 1-\varepsilon. \quad (3)$$

Let $Q_{\mathrm{I}}^{(k)}(\mathcal{N}_{A\to B}, n, \varepsilon)$ denote the nonasymptotic quantum capacity assisted by a k-extendible postprocessing; i.e., $Q_{\mathrm{I}}^{(k)}(\mathcal{N}_{A\to B}, n, \varepsilon)$ is the maximum value of $(1/n)\log_2 M$ such that there exists an (n,M,ε) protocol for $\mathcal{N}_{A\to B}$ satisfying Eq. (3).

For the cases of entanglement distillation and the simpler version of entanglement transmission, note that an (n,M,ε) entanglement distillation protocol for the state ρ_{AB} is a $(1,M,\varepsilon)$ protocol for the state $\rho_{AB}^{\otimes n}$ and vice versa. Similarly, an (n,M,ε) entanglement transmission protocol for the channel $\mathcal{N}_{A\to B}$ is a $(1,M,\varepsilon)$ protocol for the channel $\mathcal{N}_{A\to B}^{\otimes n}$ and vice versa.

In the more involved version of entanglement transmission, every channel use is interleaved with a k-extendible channel, similar to the protocols considered in Refs. [38–40]. Specifically, the protocol is a special case of one discussed in Ref. [40] for general resource theories. We do not discuss these protocols in detail here, but we simply note that, for an (n,M,ε) quantum communication protocol assisted by k-extendible channels, the performance criterion is that the final state of the protocol should have fidelity $\geq 1-\varepsilon$ to a maximally entangled state $\Phi_{M_AM_B}$ of Schmidt rank M. Let $Q_{\mathrm{II}}^{(k)}(\mathcal{N}_{A\to B}, n, \varepsilon)$ denote the nonasymptotic quantum capacity assisted by k-extendible channels; i.e., $Q_{\mathrm{II}}^{(k)}(\mathcal{N}_{A\to B}, n, \varepsilon)$ is the maximum value of $(1/n)\log_2 M$ such that there exists an (n,M,ε) protocol for $\mathcal{N}_{A\to B}$ as described for the more involved case above.

Theorem 2: The following bound holds for all $k\geq 2$ and for any $(1,M,\varepsilon)$ entanglement transmission protocol that uses a channel $\mathcal{N}$ assisted by a k-extendible postprocessing:

$$-\log_2\left(\frac{1}{M}+\frac{1}{k}-\frac{1}{Mk}\right) \le \sup_{\psi_{RA}} E_k^{\varepsilon}(R;B)_{\tau}, \qquad (4)$$

where $E_k^{\varepsilon}(R;B)_{\rho} \coloneqq \inf_{\sigma_{RB}\in\mathrm{EXT}_k(R;B)} D_h^{\varepsilon}(\rho_{RB}\|\sigma_{RB})$, $\tau_{RB} \coloneqq \mathcal{N}_{A\to B}(\psi_{RA})$, and the optimization is with respect to pure states ψ_{RA} such that $R \simeq A$. The following bound holds for all $k \ge 2$ and for any $(1, M, \varepsilon)$ entanglement distillation protocol that uses a quantum state ρ_{AB} assisted by a k-extendible postprocessing:

$$-\log_2\left(\frac{1}{M}+\frac{1}{k}-\frac{1}{Mk}\right) \le E_k^{\varepsilon}(A;B)_{\rho}. \qquad (5)$$

The proof of the above theorem follows by employing the fact that E_k^{ε} does not increase under the action of a k-extendible channel, because the extendibility of a k-extendible state does not change under the action of $U \otimes U^*$ for a unitary U, and by employing Ref. [41], (Theorem III 8).

Theorem 3: The following bound holds for all $k \ge 2$ and for any (n, M, ε) quantum communication protocol employing n uses of a channel $\mathcal{N}$ interleaved by k-extendible channels:

$$-\log_2\left[\frac{1}{M}+\frac{1}{k}-\frac{1}{Mk}\right] \le nE_k^{\max}(\mathcal{N}) + \log_2\left(\frac{1}{1-\varepsilon}\right),$$

where

$$E_k^{\max}(\mathcal{N}) \coloneqq \sup_{\psi_{RA}} \inf_{\sigma_{RB}\in\mathrm{EXT}_k(R;B)} D_{\max}(\tau_{RB}\|\sigma_{RB}),$$

$\tau_{RB} \coloneqq \mathcal{N}_{A\to B}(\psi_{RA})$, and the optimization is with respect to pure states ψ_{RA} with $|R| = |A|$.

We note here that special cases of the entanglement distillation and quantum communication protocols described above occur when the k-extendible assisting channels are taken to be 1W-LOCC channels. As such, $D^{(k)}(\rho_{AB}, n, \varepsilon)$, $Q_{\mathrm{I}}^{(k)}(\mathcal{N}_{A\to B}, n, \varepsilon)$, and $Q_{\mathrm{II}}^{(k)}(\mathcal{N}_{A\to B}, n, \varepsilon)$ are upper bounds on the nonasymptotic distillable entanglement and capacities when 1W-LOCC channels are available for assistance.

Pretty strong converse for antidegradable channels.— As a direct application of Theorem 3, we revisit the "pretty strong converse" of Ref. [42] for antidegradable channels. Recall that a channel $\mathcal{N}_{A\to B}$ is antidegradable [43,44] if the output state $\mathcal{N}_{A\to B}(\rho_{RA})$ is two extendible for any input state ρ_{RA}. Because of this property, antidegradable channels have zero asymptotic quantum capacity [17,45]. Theorem 3 implies the following bound for the nonasymptotic case:

Corollary 1: Fix $\varepsilon \in [0, 1/2)$. The following bound holds for any (n, M, ε) quantum communication protocol employing n uses of an antidegradable channel $\mathcal{N}$ interleaved by two-extendible channels: $(1/n)\log_2 M \le (1/n)\log_2(1/(1-2\varepsilon))$.

We conclude from the above inequality that, for an antidegradable channel, there is a strong limitation on its ability to generate entanglement whenever the error parameter $\varepsilon < \frac{1}{2}$, as is usually desired for applications in quantum computation. We also remark that the bound above is tighter than related bounds given in Ref. [42], and furthermore, the bound applies to quantum communication protocols assisted by interleaved two-extendible channels, which were not considered in Ref. [42].

Limitations on quantum devices.—In practice, the evolution effected by quantum processors is never a perfect unitary process. There is always some undesirable interaction with the environment, the latter of which is inaccessible to the processor. Furthermore, there are practical limitations on the ability to construct perfect unitary gates [46]. The depolarizing and erasure channels are two classes of noisy models for qubit quantum processors that are widely considered (see Refs. [47–49]).

Both families of channels mentioned above are covariant channels [50]; i.e., these channels are covariant with respect to a group G with representations given by a unitary one design. Thus, these channels can be simulated using 1W-LOCC with the Choi states as the resource states [51], (Sec. VII). Using this symmetry and the monotonicity of the unextendible generalized divergence under 1W-LOCC, we conclude that the optimal input state to a covariant channel $\mathcal{N}$, with respect to the upper bound in Theorem 2, is a maximally entangled state Φ_{RA}. Also, for any (n, M, ε) quantum communication protocol conducted over a covariant channel and assisted by any k-extendible channel, the optimal input state is $\Phi_{RA}^{\otimes n}$ and $Q_{\mathrm{II}}^{(k)}(\mathcal{N}_{A\to B}, n, \varepsilon) = Q_{\mathrm{I}}^{(k)}(\mathcal{N}_{A\to B}, n, \varepsilon)$; i.e., an upper bound on nonasymptotic quantum capacity $Q_{\mathrm{II}}^{(k)}$ is given by Theorem 2.

A qubit depolarizing channel acts on an input state ρ as $\mathcal{D}_{A\to B}^{p}(\rho) = (1-p)\rho + (p/3)(X\rho X + Y\rho Y + Z\rho Z)$, where $p \in [0, 1]$ is the depolarizing parameter, and X, Y, and Z are Pauli operators. The best known upper bound on the aysmptotic quantum capacity of this channel for values of $p \in [0, \frac{1}{4})$ was recently derived in Refs. [52,53], and this channel has zero asymptotic quantum capacity for $p \in [\frac{1}{4}, 1]$ [54,55].

With the goal of bounding the nonasymptotic quantum capacity of $\mathcal{D}^p$, we make a particular choice of the k-extendible state for E_k^{ε} (which need not be optimal) to be a tensor power of the isotropic states $\sigma_{AB}^{(t,2)}$, which is similar to what was done in Ref. [3]. The inequality in Theorem 2 then reduces to

$$\frac{1}{n}\log_2 M \le \frac{1}{n}\log_2\left(1-\frac{1}{k}\right) - \frac{1}{n}\log_2\left(f(\varepsilon, p, t) - \frac{1}{k}\right), \qquad (6)$$

where $f(\varepsilon, p, t) = 2^{-D_h^{\varepsilon}(\{1-p,p\}^{\otimes n}\|\{t,1-t\}^{\otimes n})}$ and $\{1-p, p\}$ denotes a Bernoulli distribution. The optimal measurement

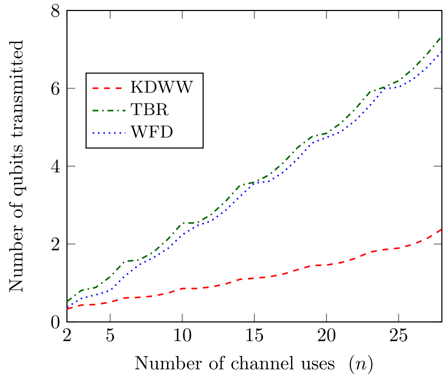

FIG. 1. Upper bounds on the number of qubits that can be reliably transmitted over a depolarizing channel with $p = 0.15$ and $\varepsilon = 0.05$. The red dashed line is from Theorem 2. The green dash-dotted and blue dotted lines are upper bounds from Refs. [3] and [4], respectively.

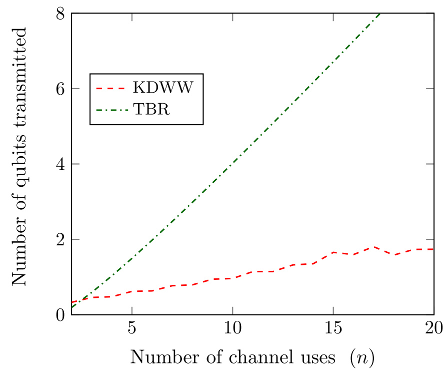

FIG. 2. Upper bounds on the number of qubits that can be reliably transmitted over an erasure channel with $p = 0.35$ and $\varepsilon = 0.05$. The red dashed line is the bound from Theorem 2. The green dash-dotted line is an upper bound from [3].

(Neyman-Pearson test) for the resulting hypothesis testing relative entropy between Bernoulli distributions is then well known [56] (see also Ref. [57]), giving an explicit upper bound on the rate $(1/n)\log_2 M$. Figure 1 compares various upper bounds on the number of qubits that can be reliably transmitted over n uses of the depolarizing channel. The bounds plotted are the ones derived from Theorem 2 (labeled "KDWW"), as well as two other known upper bounds on nonasymptotic quantum capacities [3,4]. The figure demonstrates that the bounds coming from the resource theory of unextendibility are significantly tighter than those from Refs. [3,4]. Note that Eq. (6) converges to the upper bound from Refs. [3,58] in the limit $k \to \infty$.

A qubit erasure channel acts on an input state ρ as $\mathcal{E}^p_{A\to B}(\rho_A) = (1-p)\rho_B + p|e\rangle\langle e|_B$ [16], where $p \in [0,1]$ is the erasure probability, and the erasure state $|e\rangle\langle e|$ is orthogonal to the input Hilbert space. We employ the symmetries of the erasure channel to make a particular choice of the k-extendible state for E^ε_k. Theorem 2 gives upper bounds on the number of qubits that can be reliably transmitted over n uses of the erasure channel. The bounds that we obtain are not necessarily optimal, but they still are significantly tighter than those from Ref. [3]. See Fig. 2.

Discussion.—In this Letter, we developed the resource theory of unextendibility and discussed limits that it places on the performance of finite-sized quantum processors. The free states in this resource theory are k-extendible states, and the free channels are the k-extendible channels. We determined nonasymptotic upper bounds on the rate at which qubits can be transmitted over a finite number of uses of a given quantum channel. The bounds coming from the resource theory of unextendibility are significantly tighter than those in Refs. [3,4] for depolarizing and erasure channels.

It would be interesting to explore the resource theory of unextendibility further. One plausible direction would be to use this resource theory to obtain nonasymptotic converse bounds on the entanglement distillation rate of bipartite quantum interactions and compare with the bounds obtained in Refs. [59,60]. Another direction is to analyze the bounds in Theorem 2 for other noise models that are practically relevant. Finally, it remains open to link the bounds developed here with the open problem of finding a strong converse for the quantum capacity of degradable channels [42]. To solve that problem, recall that one contribution of Ref. [42] was to reduce the question of the strong converse of degradable channels to that of establishing the strong converse for symmetric channels.

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Note added.—Recently we noticed the related work of Ref. [61], which like our work uses extendibility to address entanglement distillation, and which presents results that are complementary to ours.

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