Classification of Exceptional Points and Non-Hermitian Topological Semimetals

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Exceptional points are universal level degeneracies induced by non-Hermiticity. Whereas past decades witnessed their new physics, the unified understanding has yet to be obtained. Here we present the complete classification of generic topologically stable exceptional points according to two types of complex-energy gaps and fundamental symmetries of charge conjugation, parity, and time reversal. This classification reveals unique non-Hermitian gapless structures with no Hermitian analogs and systematically predicts unknown non-Hermitian semimetals and nodal superconductors; a *topological dumbbell* of exceptional points in three dimensions is constructed as an illustration. Our work paves the way toward richer phenomena and functionalities of exceptional points and non-Hermitian topological semimetals.

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Topology plays a pivotal role in the understanding of phases of matter [1]. In gapless systems such as semimetals and nodal superconductors, topology guarantees stable degeneracies accompanying distinctive excitations [2–7]. A prime example is the Weyl semimetal in three dimensions, where each gapless point is topologically protected by the Chern number defined on the enclosing surface. Symmetry further brings about diverse types of topological semimetals. Their unified understanding is developed as the classification theory according to fundamental symmetries such as $\mathcal{PT}$ and $\mathcal{CP}$ symmetries [8–15].

Recently, the interplay between topology and non-Hermiticity has attracted widespread interest in a non-Hermitian extension of topological insulators [16–53] and semimetals [54–72]. Non-Hermiticity ubiquitously appears, for instance, in nonequilibrium open systems [73] and correlated electron systems [68], leading to unusual properties with no Hermitian counterparts. One of their salient characteristics is the emergence of exceptional points [74–76], i.e., universal non-Hermitian level degeneracies at which eigenstates coalesce with and linearly depend on each other [77]. The past decade has witnessed a plethora of rich phenomena and functionalities induced by exceptional points [73], including unidirectional invisibility [78–81], chiral transport [82–86], enhanced sensitivity [87–92], and unusual quantum criticality [93–99].

Such exceptional points also alter the nodal structures of topological semimetals in a fundamental manner [54–72]. Notably, non-Hermiticity deforms a Weyl point and spawns a ring of exceptional points [75]. This Weyl exceptional ring is characterized by two topological charges [56], a Chern number and a quantized Berry phase, and such a multiple topological structure has no analogs to Weyl and Dirac points in Hermitian systems [2–15]. Its experimental observation has also been reported in an optical waveguide array [69]. Moreover, pseudo-Hermiticity [63], $\mathcal{PT}$ symmetry [62,65], and chiral symmetry [67] enable an exceptional ring (surface) in two (three) dimensions. A symmetry-protected exceptional ring has been experimentally observed in a two-dimensional photonic crystal slab with $\mathcal{PT}$ symmetry [54]. Whereas these unconventional nodal structures imply a fundamental change in the existing classification for Hermitian topological semimetals, its non-Hermitian counterpart has yet to be established. The non-Hermitian topological classification is crucial not only because it provides a general theoretical framework but also because it predicts novel non-Hermitian topological materials and serves as a benchmark for experiments.

This Letter presents the general theory that completely classifies topologically stable exceptional points at generic momentum points in non-Hermitian topological semimetals according to fundamental 38-fold symmetry and two types of complex-energy gaps (Table I [100]). Our classification is based on the observation that non-Hermiticity enables a unique gapless structure with no Hermitian analogs: only one type of the complex-energy gaps is open around an exceptional point [Fig. 1(d)], which is sharply contrasted with conventional Weyl and Dirac points in Hermitian systems around which both types are open [Fig. 1(c)]. We also elucidate that exceptional points generally possess multiple topological structures due to the two types of complex-energy gaps. Our theory provides the unified understanding of non-Hermitian topological semimetals studied in previous works [54–72]. Furthermore, it systematically predicts novel non-Hermitian topological materials unnoticed in the literature; we construct as an illustration a topological dumbbell of exceptional points [Fig. 2(b)], i.e., a three-dimensional variant of the bulk Fermi arcs [60,68].

Non-Hermitian gapless structures.—A unique characteristic of non-Hermitian systems is complex-valued

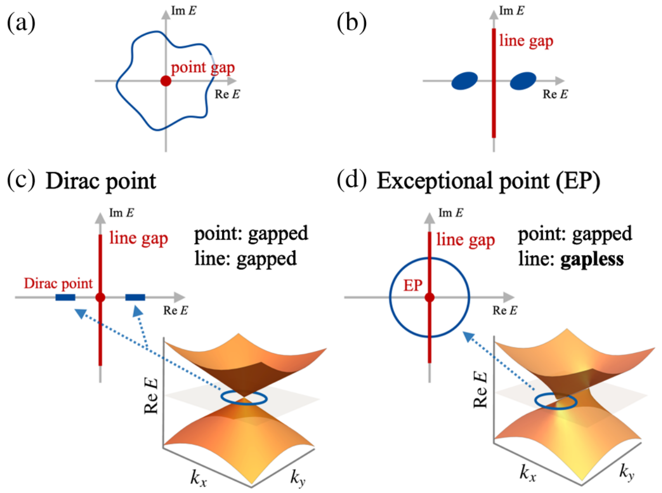

FIG. 1. Non-Hermitian gapless structures. Complex spectra (blue regions) in non-Hermitian systems may host two types of energy gaps: (a) point gap and (b) line gap. A point (line) gap is open when the complex spectrum does not cross a reference point (line) in the complex-energy plane. (c) Hermitian gapless point (Dirac point). On a region (blue circle) around it, both point and line gaps are open. (d) Exceptional point. On a region (blue circle) around it, a point gap is open but a line gap is closed.

eigenenergy, from which two different energy gaps are defined, a point gap [Fig. 1(a)] and a line gap [Fig. 1(b)] [45]. In the presence of a point (line) gap, complex-energy bands do not cross a reference point (line) in the complex-energy plane. A point gap is physically relevant to the localization transition in one-dimensional non-Hermitian systems [33,53,101], whereas topologically protected edge states [18,19,22,24–28] are understood by a line gap for the real part of the complex spectrum. A line gap for the imaginary part also has a significant influence on the nonequilibrium dynamics [38]. If symmetry exists, both complex-energy gaps should be invariant under the symmetry. Without loss of generality, the reference point is supposed to be placed on the reference line. Then, a line gap is always closed when a point gap is closed. However, the converse is not necessarily true; a point gap can be open even when a line gap is closed.

This nature of complex-energy gaps enables two distinct types of non-Hermitian gapless structures. We encircle a gapless region (point, line, surface, and so on) in momentum space by a $(p-1)$-dimensional sphere S^{p-1} $(p \geq 1)$, where S^0, S^1, and S^2 denote a pair of points, a circle, and a surface, respectively. The system has a complex-energy gap on S^{p-1}, but two different situations may happen: (i) both point and line gaps are open [Fig. 1(c)] or (ii) only a point gap is open [Fig. 1(d)]. In the former case, the non-Hermitian Hamiltonian on S^{p-1} can be continuously deformed into a Hermitian (or an anti-Hermitian) one, as rigorously proven in Ref. [45]. Thus, in a manner analogous to the Chern number for the conventional Weyl point, the gapless region hosts a topological charge essentially identical to that in the Hermitian case. In the latter case, by contrast, we need to assign a different topological charge to the gapless region on the basis of the point gap on S^{p-1}. It should be noted that the latter is intrinsic to non-Hermitian systems and impossible in Hermitian ones, since there is no distinction between point and line gaps for Hermitian Hamiltonians.

We find that exceptional points realize the latter unique gapless structure as shown in Fig. 1(d) and are characterized by topological charges for point gaps. A distinctive property of an exceptional point is swapping of eigenenergies and eigenstates upon its encirclement [60,73,76,82–86]. For illustration, let us consider a two-dimensional system with no symmetry that has an exceptional point $\boldsymbol{k} = \boldsymbol{k}_{\mathrm{EP}}$ at which two complex bands $E_{\pm}(\boldsymbol{k})$ coalesce. A representative model is given as $H(\boldsymbol{k}) = k_x \sigma_x + (k_y + i\gamma)\sigma_y$ with Pauli matrices $\sigma_{x,y}$ and the degree of non-Hermiticity γ. The eigenenergies are $E_{\pm}(\boldsymbol{k}) = \pm\sqrt{\boldsymbol{k}^2 - \gamma^2 + 2i\gamma k_y}$, with the square root singularity around the exceptional points $\boldsymbol{k}_{\mathrm{EP}} = (\pm\gamma, 0)$. As a direct result of this singularity, a branch cut and a self-intersecting Riemann surface appear in the complex-energy plane, and thus $E_+(\boldsymbol{k})$ and $E_-(\boldsymbol{k})$ are swapped when we go around a loop S^1 in momentum space that encircles one of the exceptional points. Importantly, a point gap for the reference point $E(\boldsymbol{k}_{\mathrm{EP}})$ is open but a line gap is closed on S^1. As a result, the determinant of $H(\boldsymbol{k}) - E(\boldsymbol{k}_{\mathrm{EP}})$ does not vanish on S^1, which enables us to define the winding number for a point gap as

$$W := \oint_{S^1} \frac{d\boldsymbol{k}}{2\pi i} \cdot \nabla_{\boldsymbol{k}} \log \det\left[H(\boldsymbol{k}) - E(\boldsymbol{k}_{\mathrm{EP}})\right]. \quad (1)$$

An exceptional point with $W \neq 0$ is topologically stable. For the above representative model, the exceptional points $\boldsymbol{k}_{\mathrm{EP}} = (\pm\gamma, 0)$ are characterized by $W = \pm 1$, respectively. We note that a similar topological invariant ν called vorticity is introduced in Refs. [23,28]: $\nu := -(2\pi)^{-1} \oint_{S^1} d\boldsymbol{k} \cdot \nabla_{\boldsymbol{k}} \arg\left[E_+(\boldsymbol{k}) - E_-(\boldsymbol{k})\right]$. Whereas the two invariants are equivalent to each other for the above simple two-band model ($\nu = -W/2$), only W is straightforwardly applicable to general models at which more than two complex bands coalesce.

It is shown that exceptional points that accompany spontaneous $\mathcal{PT}$-symmetry breaking [73,76,102] also possess a similar gapless structure (see Sec. SV in the Supplemental Material [100]): only a point gap is open, and a line gap is closed around them.

Symmetry.—To classify topologically stable exceptional points at generic momentum points for both complex-energy gaps in a general manner, we consider fundamental symmetries that keep all momenta invariant. The prime examples are $\mathcal{PT}$ and $\mathcal{CP}$ symmetries defined by

$$(\mathcal{PT})H^*(\boldsymbol{k})(\mathcal{PT})^{-1}=H(\boldsymbol{k}),\quad (\mathcal{PT})(\mathcal{PT})^*=\pm 1, \tag{2}$$

$$(\mathcal{CP})H^T(\boldsymbol{k})(\mathcal{CP})^{-1}=-H(\boldsymbol{k}),\quad (\mathcal{CP})(\mathcal{CP})^*=\pm 1, \tag{3}$$

where $\mathcal{PT}$ and $\mathcal{CP}$ are unitary operators. Here $\mathcal{PT}$ symmetry emerges in open systems with balanced gain and loss [73], while $\mathcal{CP}$ symmetry is respected in non-Hermitian superconductors and superfluids with inversion symmetry [103]. Whereas both space inversion $\mathcal{P}$ and time reversal $\mathcal{T}$ (charge conjugation $\mathcal{C}$) may be broken individually, the combined symmetry $\mathcal{PT}$ ($\mathcal{CP}$) is required for the topological stability away from high-symmetry points.

Another relevant symmetry is chiral symmetry, which is a combination of $\mathcal{PT}$ and $\mathcal{CP}$ symmetries and hence is defined by

$$\Gamma H^\dagger(\boldsymbol{k})\Gamma^{-1}=-H(\boldsymbol{k}),\qquad \Gamma^2=1, \tag{4}$$

with a unitary operator Γ. It is notable that chiral symmetry is equivalent to pseudo-Hermiticity, defined by $\eta H^\dagger(\boldsymbol{k})\eta^{-1}=H(\boldsymbol{k})$ with a unitary and Hermitian operator η [104]. In fact, when $H(\boldsymbol{k})$ respects chiral symmetry, $iH(\boldsymbol{k})$ respects pseudo-Hermiticity [38]. Moreover, chiral symmetry is distinct from sublattice symmetry due to $H(\boldsymbol{k})\neq H^\dagger(\boldsymbol{k})$ [45], defined by

$$\mathcal{S}H(\boldsymbol{k})\mathcal{S}^{-1}=-H(\boldsymbol{k}),\qquad \mathcal{S}^2=1, \tag{5}$$

with a unitary operator $\mathcal{S}$. The above symmetries constitute the fundamental 38-fold symmetry class for the non-Hermitian gapless phases [100] in a similar manner to the gapped ones in Ref. [45], which updates Bernard-LeClair symmetry [18,32,46,63,105,106].

Topological classification.—Complex-energy gaps are closed at an exceptional point. Near such a gapless point, $H(\boldsymbol{k})$ is expressed as [107]

$$H(\boldsymbol{k})=\sum_{i=1}^{p} v_i\delta k_i\gamma_i+PJP^{-1},\qquad p\coloneqq d-d_{\mathrm{EP}}, \tag{6}$$

where v_i's are complex constants, δk_i's are momentum deviations from the exceptional point, and γ_i's are non-Hermitian Dirac matrices subject to certain symmetries. The gapless point $\delta\boldsymbol{k}=0$ in general takes a Jordan canonical form PJP^{-1} with a Jordan matrix J and an invertible matrix P, which is defective unless J is diagonal [74]. In addition, p is the codimension of the exceptional point with the spatial dimension d and the dimension d_{EP} of the gapless region; exceptional points, lines, and surfaces are described by $d_{\mathrm{EP}}=0$, 1, 2, respectively. Whereas a complex-energy gap is closed at the gapless point $\delta\boldsymbol{k}=0$, it may be open, and a topological invariant can be defined on a $(p-1)$-dimensional surface S^{p-1} that encloses the gapless point, in a similar manner to Fermi surfaces in Hermitian systems [11,12,15,108]. Thus, topology of the exceptional point is determined by classifying the gapped topological phases on S^{p-1}.

As discussed above, only a point gap is open for a sufficiently small S^{p-1}, but a line gap can also be open for a larger $S^{p'-1}$ [p' can be different from p; for instance, $p=2$

TABLE I. Classification table of topologically stable exceptional points at generic momentum points. The codimension p is defined as $p\coloneqq d-d_{\mathrm{EP}}$, with the spatial dimension d and the dimension d_{EP} of the gapless region; exceptional points, lines, and surfaces are described by $d_{\mathrm{EP}}=0$, 1, 2, respectively. Complex-energy gaps have two distinct types, a point (P) or line (L) gap, and the subscript of L specifies a line gap for the real or imaginary part of the complex spectrum. The sign of $\mathcal{PT}$ ($\mathcal{CP}$) symmetry means $(\mathcal{PT})(\mathcal{PT})^*$ [$(\mathcal{CP})(\mathcal{CP})^*$].

Symmetry	Gap	Classifying space	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$
No	P	$\mathcal{C}_p$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
	L	$\mathcal{C}_{p+1}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
Chiral	P	$\mathcal{C}_{p+1}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	L_r	$\mathcal{C}_p$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
	L_i	$\mathcal{C}_{p+1}\times\mathcal{C}_{p+1}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$
Sublattice	P	$\mathcal{C}_p\times\mathcal{C}_p$	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0
	L	$\mathcal{C}_p$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$\mathcal{PT}$, $+1$	P	$\mathcal{R}_p$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$2\mathbb{Z}$	0	0	0
	L_r	$\mathcal{R}_{p+7}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$2\mathbb{Z}$	0	0
	L_i	$\mathcal{R}_{p+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$2\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\mathcal{PT}$, -1	P	$\mathcal{R}_{p+4}$	$2\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
	L_r	$\mathcal{R}_{p+3}$	0	$2\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	L_i	$\mathcal{R}_{p+5}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$2\mathbb{Z}$
$\mathcal{CP}$, $+1$	P	$\mathcal{R}_{p+2}$	$\mathbb{Z}_2$	0	$2\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
	L	$\mathcal{R}_{p+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$2\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\mathcal{CP}$, -1	P	$\mathcal{R}_{p+6}$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$2\mathbb{Z}$	0
	L	$\mathcal{R}_{p+5}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$2\mathbb{Z}$

and $p' = 3$ in Fig. 2(a)]. Depending on types of the complex-energy gaps, an exceptional point may support multiple topological invariants. Taking into account both possibilities, we classify topologically stable exceptional points for each type of complex-energy gaps and all of the 38 symmetry classes [100]. Our results are summarized in Table I and Tables S2–S7 in the Supplemental Material [100]. These periodic tables specify exceptional points and non-Hermitian topological semimetals in a general manner and describe their unconventional nodal structures. In fact, they corroborate previous works [54–72,100]. For example, stable exceptional points in two dimensions [28,60,68,72] are explained by the $\mathbb{Z}$ index in Table I with no symmetry, $p = 2$, and point (P) gap; symmetry-protected exceptional rings in two dimensions [54,62,63,65,67] are explained by the $\mathbb{Z}$ or $\mathbb{Z}_2$ index in Table I with chiral or $\mathcal{PT}$ symmetry, $p = 1$, and P gap. These unique nodal structures, as well as the consequent physical phenomena, are topologically protected against symmetry-preserving perturbations.

Multiple topological structure.—A remarkable feature of Table I is the multiple topological indices for each symmetry class due to the two types of complex-energy gaps. As a result, exceptional points can be characterized by a couple of independent topological charges. A prime example is the Weyl exceptional ring in three-dimensional systems with no symmetry [Fig. 2(a)] [56,75]. For the Hermitian case, the topological stability of a Weyl point in three dimensions is ensured by the Chern number defined on an enclosing surface [2,7]. In the presence of non-Hermiticity, such a Weyl point morphs into an exceptional ring. A representative model is given as $H(\boldsymbol{k}) = k_x\sigma_x + k_y\sigma_y + (k_z + i\gamma)\sigma_z$ with a Weyl exceptional ring at $k_x^2 + k_y^2 = \gamma^2$, $k_z = 0$ [56,75]. On a sphere [S^2 in Fig. 2(a)] that encloses the ring, a line gap is open and the Chern number remains to be well defined unless it annihilates with another ring, which corresponds to the $\mathbb{Z}$ index for a line gap with $p = 3$ (see Table I with no symmetry). Moreover, on S^1 across the ring, a point gap is open and the winding number in Eq. (1) is well defined, which corresponds to the $\mathbb{Z}$ index for a point gap with $p = 2$. Notably, the two topological charges are independent of each other and individually ensure the topological stability of the Weyl exceptional ring.

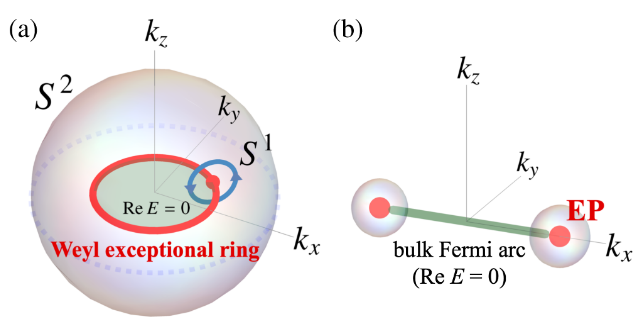

FIG. 2. Non-Hermitian topological semimetals. (a) Weyl exceptional ring (red ring) with the two independent topological charges, the Chern number defined on the surface S^2 that encloses the ring ($p = 3$, line gap) and the winding number defined on the loop S^1 across the ring ($p = 2$, point gap). (b) Topological dumbbell of exceptional points in three dimensions. The bulk Fermi arc with Re$E = 0$ connects a pair of exceptional points (EPs, red points).

Such a multiple topological structure is a general hallmark of non-Hermitian topological semimetals. For instance, according to our classification tables, symmetry-protected exceptional rings in two dimensions [54,62,63,65,67] may host different topological charges for each $p \leq 2$ and each complex-energy gap. Whereas the unique nodal structures were discussed in Refs. [54,62,63,65,67], their multiple topology has not been revealed until this Letter.

Exceptional point in three dimensions.—Our classification also predicts unknown non-Hermitian topological semimetals and nodal superconductors. As an illustrative example, we consider an exceptional point in three dimensions ($p = 3$) protected by chiral symmetry. Whereas exceptional rings [56,75] and surfaces [62,65] were discussed before, such an exceptional point in three dimensions has not hitherto been known. A representative model is systematically constructed in the following manner. We begin with a Hermitian gapless system in four dimensions $H(\boldsymbol{k}) = k_x\sigma_x\tau_x + k_y\sigma_x\tau_y + k_z\sigma_x\tau_z + k_w\sigma_y$ with Pauli matrices σ_i and τ_i. Corresponding to the $\mathbb{Z}$ index for a real line gap with $p = 4$ (see Table I with chiral symmetry), it possesses a topologically stable gapless point at $\boldsymbol{k} = 0$, around which the three-dimensional winding number [100, 109–111] is defined due to chiral symmetry in Eq. (4) with $\Gamma = \sigma_z$. Now we add a non-Hermitian perturbation $i\gamma\sigma_z\tau_x$ to this Hermitian model. As with the Weyl exceptional ring, this non-Hermiticity spawns an exceptional ring. In fact, the complex spectrum is obtained as

$$E(\boldsymbol{k}) = \pm\sqrt{k_x^2 + k_w^2 + \left(\sqrt{k_y^2 + k_z^2} \pm i\gamma\right)^2}, \tag{7}$$

and an exceptional ring appears at $k_x^2 + k_w^2 = \gamma^2$, $k_y = k_z = 0$. Finally, we take $k_w = 0$ and regard this four-dimensional model as a three-dimensional one,

$$H(\boldsymbol{k}) = k_x\sigma_x\tau_x + k_y\sigma_x\tau_y + k_z\sigma_x\tau_z + i\gamma\sigma_z\tau_x, \tag{8}$$

which has a pair of exceptional points at $\boldsymbol{k}_{\mathrm{EP}} = (\pm\gamma, 0, 0)$. As illustrated in Fig. 2(b), these exceptional points are connected by a Fermi arc with Re$E = 0$, forming a topologically stable dumbbell configuration. The topological stability of the exceptional points is ensured by the $\mathbb{Z}$ index in Table I with chiral symmetry, $p = 3$, and point gap, which is given as the Chern number ± 1 of the Hermitian matrix $iH(\boldsymbol{k})\Gamma$ defined on a surface that encloses each of the exceptional points [100].

We stress that topologically stable gapless points are absent in Hermitian three-dimensional systems with chiral symmetry. These gapless points are thus unique to non-Hermitian systems. Furthermore, the above recipe is widely applicable to different symmetry classes and spatial dimensions and unknown non-Hermitian topological semimetals can be systematically predicted according to our classification.

Discussion.—The emergence of exceptional points is one of the most striking and universal characteristics in non-Hermitian physics [73,75,76]. In this Letter, we have formulated the general classification theory on non-Hermitian topology of exceptional points. Our theory is widely applicable to non-Hermitian topological semimetals, providing the unified understanding of their unique nodal structures and predicting novel non-Hermitian topological materials. Since the consequent non-Hermitian topological phenomena and functionalities have yet to be explored, it merits further research to investigate such richer physics on the basis of our theory.

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[1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010); X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011); C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Rev. Mod. Phys. **88**, 035005 (2016).
[2] H. B. Nielsen and M. Ninomiya, Phys. Lett. **130B**, 389 (1983).
[3] S. Murakami, New J. Phys. **9**, 356 (2007).
[4] X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Phys. Rev. B **83**, 205101 (2011).
[5] A. A. Burkov and L. Balents, Phys. Rev. Lett. **107**, 127205 (2011); A. A. Burkov, M. D. Hook, and L. Balents, Phys. Rev. B **84**, 235126 (2011); G. B. Halász and L. Balents, Phys. Rev. B **85**, 035103 (2012); T. Meng and L. Balents, Phys. Rev. B **86**, 054504 (2012).
[6] S. M. Young, S. Zaheer, J. C. Y. Teo, C. L. Kane, E. J. Mele, and A. M. Rappe, Phys. Rev. Lett. **108**, 140405 (2012).
[7] N. P. Armitage, E. J. Mele, and A. Vishwanath, Rev. Mod. Phys. **90**, 015001 (2018).
[8] P. Hořava, Phys. Rev. Lett. **95**, 016405 (2005).
[9] S. Matsuura, P.-Y. Chang, A. P. Schnyder, and S. Ryu, New J. Phys. **15**, 065001 (2013).
[10] Y. X. Zhao and Z. D. Wang, Phys. Rev. Lett. **110**, 240404 (2013); Phys. Rev. B **89**, 075111 (2014).
[11] T. Morimoto and A. Furusaki, Phys. Rev. B **89**, 235127 (2014).
[12] S. Kobayashi, K. Shiozaki, Y. Tanaka, and M. Sato, Phys. Rev. B **90**, 024516 (2014).
[13] K. Shiozaki and M. Sato, Phys. Rev. B **90**, 165114 (2014).
[14] C.-K. Chiu and A. P. Schnyder, Phys. Rev. B **90**, 205136 (2014).
[15] Y. X. Zhao, A. P. Schnyder, and Z. D. Wang, Phys. Rev. Lett. **116**, 156402 (2016); Y. X. Zhao and Y. Lu, Phys. Rev. Lett. **118**, 056401 (2017).
[16] M. S. Rudner and L. S. Levitov, Phys. Rev. Lett. **102**, 065703 (2009); J. M. Zeuner, M. C. Rechtsman, Y. Plotnik, Y. Lumer, S. Nolte, M. S. Rudner, M. Segev, and A. Szameit, Phys. Rev. Lett. **115**, 040402 (2015).
[17] Y. C. Hu and T. L. Hughes, Phys. Rev. B **84**, 153101 (2011).
[18] K. Esaki, M. Sato, K. Hasebe, and M. Kohmoto, Phys. Rev. B **84**, 205128 (2011); M. Sato, K. Hasebe, K. Esaki, and M. Kohmoto, Prog. Theor. Phys. **127**, 937 (2012).
[19] C. Poli, M. Bellec, U. Kuhl, F. Mortessagne, and H. Schomerus, Nat. Commun. **6**, 6710 (2015).
[20] S. Malzard, C. Poli, and H. Schomerus, Phys. Rev. Lett. **115**, 200402 (2015).
[21] T. E. Lee, Phys. Rev. Lett. **116**, 133903 (2016).
[22] S. Weimann, M. Kremer, Y. Plotnik, Y. Lumer, S. Nolte, K. G. Makris, M. Segev, M. C. Rechtsman, and A. Szameit, Nat. Mater. **16**, 433 (2017).
[23] D. Leykam, K. Y. Bliokh, C. Huang, Y. D. Chong, and F. Nori, Phys. Rev. Lett. **118**, 040401 (2017).
[24] D. Kim, K. Mochizuki, N. Kawakami, and H. Obuse, arXiv:1609.09650; L. Xiao, X. Zhan, Z. H. Bian, K. K. Wang, X. Zhang, X. P. Wang, J. Li, K. Mochizuki, D. Kim, N. Kawakami, W. Yi, H. Obuse, B. C. Sanders, and P. Xue, Nat. Phys. **13**, 1117 (2017).
[25] P. St-Jean, V. Goblot, E. Galopin, A. Lemaître, T. Ozawa, L. L. Gratiet, I. Sagnes, J. Bloch, and A. Amo, Nat. Photon. **11**, 651 (2017).
[26] B. Bahari, A. Ndao, F. Vallini, A. E. Amili, Y. Fainman, and B. Kanté, Science **358**, 636 (2017).
[27] G. Harari, M. A. Bandres, Y. Lumer, M. C. Rechtsman, Y. D. Chong, M. Khajavikhan, D. N. Christodoulides, and M. Segev, Science **359**, eaar4003 (2018); M. A. Bandres, S. Wittek, G. Harari, M. Parto, J. Ren, M. Segev, D. Christodoulides, and M. Khajavikhan, Science **359**, eaar4005 (2018).
[28] H. Shen, B. Zhen, and L. Fu, Phys. Rev. Lett. **120**, 146402 (2018).
[29] F. K. Kunst, E. Edvardsson, J. C. Budich, and E. J. Bergholtz, Phys. Rev. Lett. **121**, 026808 (2018).
[30] K. Kawabata, Y. Ashida, H. Katsura, and M. Ueda, Phys. Rev. B **98**, 085116 (2018).

[31] S. Yao and Z. Wang, Phys. Rev. Lett. **121**, 086803 (2018); S. Yao, F. Song, and Z. Wang, Phys. Rev. Lett. **121**, 136802 (2018).
[32] S. Lieu, Phys. Rev. B **98**, 115135 (2018).
[33] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, Phys. Rev. X **8**, 031079 (2018).
[34] T. M. Philip, M. R. Hirsbrunner, and M. J. Gilbert, Phys. Rev. B **98**, 155430 (2018); M. R. Hirsbrunner, T. M. Philip, and M. J. Gilbert, arXiv:1901.09961.
[35] K. Kawabata, K. Shiozaki, and M. Ueda, Phys. Rev. B **98**, 165148 (2018).
[36] K. Takata and M. Notomi, Phys. Rev. Lett. **121**, 213902 (2018).
[37] A. McDonald, T. Pereg-Barnea, and A. A. Clerk, Phys. Rev. X **8**, 041031 (2018).
[38] K. Kawabata, S. Higashikawa, Z. Gong, Y. Ashida, and M. Ueda, Nat. Commun. **10**, 297 (2019).
[39] K. Y. Bliokh, D. Leykam, M. Lein, and F. Nori, Nat. Commun. **10**, 580 (2019).
[40] T. Liu, Y.-R. Zhang, Q. Ai, Z. Gong, K. Kawabata, M. Ueda, and F. Nori, Phys. Rev. Lett. **122**, 076801 (2019).
[41] E. Edvardsson, F. K. Kunst, and E. J. Bergholtz, Phys. Rev. B **99**, 081302(R) (2019).
[42] C. H. Lee and R. Thomale, Phys. Rev. B **99**, 201103(R) (2019).
[43] L. Herviou, J. H. Bardarson, and N. Regnault, Phys. Rev. A **99**, 052118 (2019).
[44] F. K. Kunst and V. Dwivedi, Phys. Rev. B **99**, 245116 (2019).
[45] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, arXiv:1812.09133.
[46] H. Zhou and J. Y. Lee, Phys. Rev. B **99**, 235112 (2019).
[47] Q.-B. Zeng, Y.-B. Yang, and Y. Xu, arXiv:1901.08060.
[48] H.-G. Zirnstein, G. Refael, and B. Rosenow, arXiv: 1901.11241.
[49] W. B. Rui, Y. X. Zhao, and A. P. Schnyder, Phys. Rev. B **99**, 241110 (2019).
[50] D. S. Borgnia, A. J. Kruchkov, and R.-J. Slager, arXiv: 1902.07217.
[51] K. Yokomizo and S. Murakami, arXiv:1902.10958.
[52] P. A. McClarty and J. G. Rau, arXiv:1904.02160.
[53] S. Longhi, Phys. Rev. Lett. **122**, 237601 (2019).
[54] B. Zhen, C. W. Hsu, Y. Igarashi, L. Lu, I. Kaminer, A. Pick, S.-L. Chua, J. D. Joannopoulos, and M. Soljačić, Nature (London) **525**, 354 (2015).
[55] J. González and R. A. Molina, Phys. Rev. Lett. **116**, 156803 (2016); Phys. Rev. B **96**, 045437 (2017); R. A. Molina and J. González, Phys. Rev. Lett. **120**, 146601 (2018).
[56] Y. Xu, S.-T. Wang, and L.-M. Duan, Phys. Rev. Lett. **118**, 045701 (2017).
[57] M. N. Chernodub, J. Phys. A **50**, 385001 (2017); M. N. Chernodub and A. Cortijo, arXiv:1901.06167.
[58] A. A. Zyuzin and A. Y. Zyuzin, Phys. Rev. B **97**, 041203(R) (2018); K. Moors, A. A. Zyuzin, A. Y. Zyuzin, R. P. Tiwari, and T. L. Schmidt, Phys. Rev. B **99**, 041116(R) (2019); A. A. Zyuzin and P. Simon, Phys. Rev. B **99**, 165145 (2019).
[59] A. Cerjan, M. Xiao, L. Yuan, and S. Fan, Phys. Rev. B **97**, 075128 (2018).
[60] H. Zhou, C. Peng, Y. Yoon, C. W. Hsu, K. A. Nelson, L. Fu, J. D. Joannopoulos, M. Soljačić, and B. Zhen, Science **359**, 1009 (2018).
[61] J. Carlström and E. J. Bergholtz, Phys. Rev. A **98**, 042114 (2018); J. Carlström, M. Stålhammar, J. C. Budich, and E. J. Bergholtz, Phys. Rev. B **99**, 161115(R) (2019).
[62] R. Okugawa and T. Yokoyama, Phys. Rev. B **99**, 041202 (R) (2019).
[63] J. C. Budich, J. Carlström, F. K. Kunst, and E. J. Bergholtz, Phys. Rev. B **99**, 041406(R) (2019).
[64] Z. Yang and J. Hu, Phys. Rev. B **99**, 081102(R) (2019).
[65] H. Zhou, J. Y. Lee, S. Liu, and B. Zhen, Optica **6**, 190 (2019).
[66] H. Wang, J. Ruan, and H. Zhang, Phys. Rev. B **99**, 075130 (2019).
[67] T. Yoshida, R. Peters, N. Kawakami, and Y. Hatsugai, Phys. Rev. B **99**, 121101(R) (2019).
[68] V. Kozii and L. Fu, arXiv:1708.05841; M. Papaj, H. Isobe, and L. Fu, Phys. Rev. B **99**, 201107(R) (2019); H. Shen and L. Fu, Phys. Rev. Lett. **121**, 026403 (2018).
[69] A. Cerjan, S. Huang, K. P. Chen, Y. Chong, and M. C. Rechtsman, Nat. Photonics (in press).
[70] K. Luo, J. Feng, Y. X. Zhao, and R. Yu, arXiv:1810.09231.
[71] C. H. Lee, G. Li, Y. Liu, T. Tai, R. Thomale, and X. Zhang, arXiv:1812.02011.
[72] E. J. Bergholtz and J. C. Budich, arXiv:1903.12187.
[73] V. V. Konotop, J. Yang, and D. A. Zezyulin, Rev. Mod. Phys. **88**, 035002 (2016); L. Feng, R. El-Ganainy, and L. Ge, Nat. Photonics **11**, 752 (2017); R. El-Ganainy, K. G. Makris, M. Khajavikhan, Z. H. Musslimani, S. Rotter, and D. N. Christodoulides, Nat. Phys. **14**, 11 (2018); M.-A. Miri and A. Alù, Science **363**, eaar7709 (2019).
[74] T. Kato, *Perturbation Theory for Linear Operators* (Springer, New York, 1966).
[75] M. V. Berry, Czech. J. Phys. **54**, 1039 (2004).
[76] W. D. Heiss, J. Phys. A **45**, 444016 (2012).
[77] In Ref. [74], an exceptional point is defined as a generic degeneracy point at which the matrix may be diagonalizable. On the other hand, Ref. [74] defines a degeneracy point of a nondiagonalizable matrix as a defective point.
[78] Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, and D. N. Christodoulides, Phys. Rev. Lett. **106**, 213901 (2011).
[79] A. Regensburger, C. Bersch, M.-A. Miri, G. Onishchukov, D. N. Christodoulides, and U. Peschel, Nature (London) **488**, 167 (2012).
[80] L. Feng, Y.-L. Xu, W. S. Fegadolli, M.-H. Lu, J. E. B. Oliveira, V. R. Almeida, Y.-F. Chen, and A. Scherer, Nat. Mater. **12**, 108 (2013).
[81] B. Peng, Ş. K. Özdemir, F. Lei, F. Monifi, M. Gianfreda, G. L. Long, S. Fan, F. Nori, C. M. Bender, and L. Yang, Nat. Phys. **10**, 394 (2014).
[82] C. Dembowski, H.-D. Gräf, H. L. Harney, A. Heine, W. D. Heiss, H. Rehfeld, and A. Richter, Phys. Rev. Lett. **86**, 787 (2001).
[83] T. Gao, E. Estrecho, K. Y. Bliokh, T. C. H. Liew, M. D. Fraser, S. Brodbeck, M. Kemp, C. Schneider, S. Höfling, Y. Yamamoto, F. Nori, Y. S. Kivshar, A. G. Truscott, R. G. Dall, and E. A. Ostrovskaya, Nature (London) **526**, 554 (2015).

[84] J. Doppler, A. A. Mailybaev, J. Böhm, U. Kuhl, A. Girschikm, F. Libisch, T. J. Milburn, P. Rabl, N. Moiseyev, and S. Rotter, Nature (London) **537**, 76 (2016).
[85] H. Xu, D. Mason, L. Jiang, and J. G. E. Harris, Nature (London) **537**, 80 (2016).
[86] J. W. Yoon, Y. Choi, C. Hahn, G. Kim, S. H. Song, K.-Y. Yang, J. Y. Lee, Y. Kim, C. S. Lee, J. K. Shin, H.-S. Lee, and P. Berini, Nature (London) **562**, 86 (2018).
[87] J. Wiersig, Phys. Rev. Lett. **112**, 203901 (2014).
[88] Z.-P. Liu, J. Zhang, S. K. Özdemir, B. Peng, H. Jing, X.-Y. Lü, C.-W. Li, L. Yang, F. Nori, and Y.-x. Liu, Phys. Rev. Lett. **117**, 110802 (2016).
[89] H. Hodaei, A. U. Hassan, S. Wittek, H. Garcia-Gracia, R. El-Ganainy, D. N. Christodoulides, and M. Khajavikhan, Nature (London) **548**, 187 (2017).
[90] W. Chen, Ş. K. Özdemir, G. Zhao, J. Wiersig, and L. Yang, Nature (London) **548**, 192 (2017).
[91] H.-K. Lau and A. A. Clerk, Nat. Commun. **9**, 4320 (2018).
[92] M. Zhang, W. Sweeney, C. W. Hsu, L. Yang, A. D. Stone, and L. Jiang, arXiv:1805.12001.
[93] T. E. Lee and C.-K. Chan, Phys. Rev. X **4**, 041001 (2014).
[94] K. Kawabata, Y. Ashida, and M. Ueda, Phys. Rev. Lett. **119**, 190401 (2017).
[95] M. Nakagawa, N. Kawakami, and M. Ueda, Phys. Rev. Lett. **121**, 203001 (2018).
[96] J. Li, A. K. Harter, J. Liu, L. de Melo, Y. N. Joglekar, and L. Luo, Nat. Commun. **10**, 855 (2019).
[97] B. Dóra, M. Heyl, and R. Moessner, Nat. Commun. **10**, 2254 (2019).
[98] Y. Wu, W. Liu, J. Geng, X. Song, X. Ye, C.-K. Duan, X. Rong, and J. Du, Science **364**, 878 (2019).
[99] L. Xiao, K. Wang, X. Zhan, Z. Bian, K. Kawabata, M. Ueda, W. Yi, and P. Xue, arXiv:1812.01213.
[100] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.123.066405 for detailed discussions on the 38-fold symmetry, topological classification, and illustrative examples.
[101] N. Hatano and D. R. Nelson, Phys. Rev. Lett. **77**, 570 (1996); Phys. Rev. B **56**, 8651 (1997); **58**, 8384 (1998).
[102] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998); C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002); C. M. Bender, Rep. Prog. Phys. **70**, 947 (2007).
[103] Particle-hole symmetry in non-Hermitian superconductors and superfluids should be defined in terms not of complex conjugation but of transposition [45].
[104] A. Mostafazadeh, J. Math. Phys. (N.Y.) **43**, 205 (2002); **43**, 2814 (2002); **43**, 3944 (2002).
[105] D. Bernard and A. LeClair, in *Statistical Field Theories*, edited by A. Cappelli and G. Mussardo (Springer, Dordrecht, 2002), pp. 207–214.
[106] Reference [45] points out that the 43-fold Bernard-LeClair symmetry overcounted some non-Hermitian symmetry classes and overlooked others. Consequently, the non-Hermitian generalization of the Altland-Zirnbauer symmetry actually consists of 38 classes [45]. Furthermore, the physical meaning of each symmetry is identified in Ref. [45].
[107] Our discussion is applicable to the generalized Bloch wave functions [31,44,45,51] as long as the corresponding symmetry is respected with complex wave numbers.
[108] M. Karoubi, *K-Theory: An Introduction* (Springer, Berlin, 2008).
[109] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B **78**, 195125 (2008); S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. Ludwig, New J. Phys. **12**, 065010 (2010).
[110] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B **81**, 134508 (2010).
[111] T. Kawakami, T. Okamura, S. Kobayashi, and M. Sato, Phys. Rev. X **8**, 041026 (2018).