One-Dimensional Quasicrystals with Power-Law Hopping

X. Deng,[1] S. Ray,[2] S. Sinha,[2] G. V. Shlyapnikov,[3,4,5,6,7] and L. Santos[1]

[1]*Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstr. 2, 30167 Hannover, Germany*

[2]*Indian Institute of Science Education and Research, Kolkata, Mohanpur, Nadia 741246, India*

[3]*LPTMS, CNRS, Universite Paris Sud, Universite Paris-Saclay, Orsay 91405, France*

[4]*SPEC, CEA, CNRS, Universite Paris-Saclay, CEA Saclay, Gif sur Yvette 91191, France*

[5]*Russian Quantum Center, Skolkovo, Moscow 143025, Russia*

[6]*Van der Waals-Zeeman Institute, University of Amsterdam, Science Park 904, 1098 XH Amsterdam, Netherlands*

[7]*Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, 430071 Wuhan, China*

(Received 23 August 2018; revised manuscript received 4 January 2019; published 10 July 2019)

One-dimensional quasiperiodic systems with power-law hopping, $1/r^a$, differ from both the standard Aubry-André (AA) model and from power-law systems with uncorrelated disorder. Whereas in the AA model all single-particle states undergo a transition from ergodic to localized at a critical quasidisorder strength, short-range power-law hops with $a > 1$ can result in mobility edges. We find that there is no localization for long-range hops with $a \leq 1$, in contrast to the case of uncorrelated disorder. Systems with long-range hops rather present ergodic-to-multifractal edges and a phase transition from ergodic to multifractal (extended but nonergodic) states. Both mobility and ergodic-to-multifractal edges may be clearly revealed in experiments on expansion dynamics.

DOI: 10.1103/PhysRevLett.123.025301

Quasicrystals constitute an intriguing intermediate case between disordered and periodic systems. In the former case arbitrarily small disorder results in localization for all single-particle states (SPS) in both one- and two-dimensional (1D and 2D) systems, whereas in three dimensions a mobility edge separates extended and localized SPSs [1,2]. The situation is very different in quasiperiodic systems formed by two incommensurate lattices, which for 1D in the tight-binding regime (with nearest-neighbor hopping) are well described by the Aubry-André (AA) model [3–5]. This model has been realized in experiments with ultracold atoms in bichromatic optical lattices, in which single-particle localization, Bose glasses, and many-body localization have been observed [6–9]. Due to the self-duality of the AA model [5,10,11], above a critical quasidisorder strength all SPS change from ergodic to localized.

In disordered systems extended states were commonly believed to be ergodic, except at the mobility edge, where the states are multifractal, i.e., neither localized nor ergodic [12–17]. However, recent studies of the artificial Bethe lattice [18,19], random matrix models [20], and dipolar excitations in 3D random systems [21] have revealed finite-width bands of extended nonergodic states next to the ergodic bands, raising fundamental questions concerning ergodic-to-nonergodic transitions [22].

Beyond nearest-neighbor hopping breaks the self-duality of the AA model, and energy-dependent mobility edges appear [5,23–27]. This is the case in shallow lattices, where intermediate regimes with both extended and localized SPS have been predicted [24–27] and recently observed [28], or in zig-zag lattices with next-to-nearest neighbor hopping [29]. Self-duality is also lost when the hopping amplitude decays with the interparticle distance r as $1/r^a$ [26,30]. This is particularly interesting since power-law interactions occur in many systems. Dipole-dipole interactions ($\propto 1/r^3$) play a crucial role for magnetic atoms [31], polar molecules [32], Rydberg atoms [33], nitrogen-vacancy centers [34], and nuclear spins in solid-state systems [35]. Moreover, tunable power-law interactions are achievable for laser-driven ions ($0 \leq a \leq 3$) [36,37] and for atoms in photonic crystal waveguides [38]. These interactions induce power-law exchange, e.g., between rotational states in polar molecules [32] or hyperfine states in trapped ions [36,37], resulting in power-law hopping of excitations.

In this Letter, we study the SPS of generalized AA (GAA) models with power-law hops. Short-range hops ($a > 1$) are characterized by a hierarchy of regimes with mobility edges (Fig. 1). Remarkably, for long-range hops, $a \leq 1$, all SPS are extended, in stark contrast to power-law models with uncorrelated disorder [39–41]. However, there are finite-width bands of both ergodic and nonergodic (multifractal) states. We classify these states and show that there is a phase transition at the ergodic-to-multifractal edge, characterized by an abrupt change of fractal dimensions. Moreover, we show that the expansion dynamics of excitations can reveal the presence of mobility and ergodic-to-multifractal edges.

Generalized AA model.—We consider pinned particles (with unit filling) at the sites of a deep 1D lattice. The particles have two internal states $\{\uparrow, \downarrow\}$. Interactions result

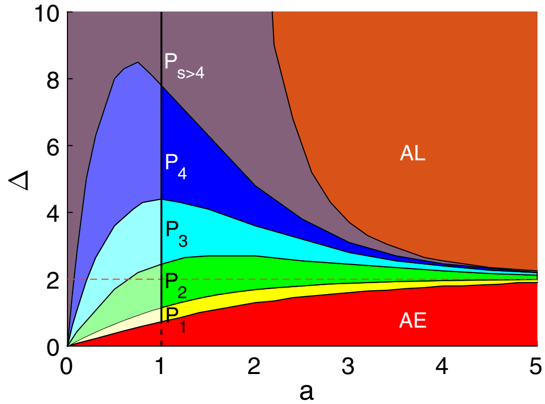

FIG. 1. Regimes of 1D quasicrystals with power-law hopping, for $\beta = (\sqrt{5}-1)/2$. For small quasidisorder strength Δ all SPS are ergodic (AE) and for large Δ (for hopping power $a > 1$) all are localized (AL). The P_s regimes are characterized by a fraction β^s of ergodic SPS, whereas the rest are localized ($a > 1$) or multifractal ($a \leq 1$). The different behavior for $a > 1$ and $a < 1$ is indicated in the figure with a slightly different color. The results were obtained for 987 sites, with periodic boundary conditions. Calculations for larger systems do not modify the results [42].

in power-law exchange between particles. A second lattice, incommensurate with the primary one, induces a quasidisordered variation of the energy difference between $\uparrow$ and $\downarrow$ [47]. The transport of an $\uparrow$ excitation in a sample of $\downarrow$ particles is described by a GAA model:

$$\hat{H} = -J\sum_{i,j\neq i}\frac{1}{|i-j|^a}|i\rangle\langle j| + \Delta\sum_j \cos[\beta(2\pi j+\phi)]|j\rangle\langle j|, \quad (1)$$

where $|j\rangle$ denotes the state in which the excitation is localized at the site j, and $J/|i-j|^a$ is the hopping rate between the sites i and j. We set $J = 1$ for simplicity. The quasidisorder potential is characterized by its strength Δ, the incommesurabilty β (the ratio of the period of the primary lattice to the one of the second lattice), and the displacement ϕ. For $a \gg 1$, the GAA model approaches the AA model [42]. For the latter, all SPS are ergodic for $\Delta < 2$, all localized for $\Delta > 2$, and all multifractal (extended but nonergodic) at $\Delta = 2$ [5,10,11]. Note that model (1) breaks the self-duality of the original AA model. A self-dual generalization of the AA model with power-law hops have been previously discussed [48]. However, that self-dual model demands power-law interactions between the sites, which lead to a radically different physics than that discussed in our paper [49].

Determination of the localization properties.—A way of discerning between localized, multifractal, and ergodic SPS, which is especially useful for spectra with edges, is given by the analysis of the eigenenergies E_n (indexed in growing energy order), and in particular by the even-odd (odd-even) spacings $s_n^{e-o} = E_{2n} - E_{2n-1}$ ($s_n^{o-e} = E_{2n+1} - E_{2n}$). Ergodic SPS present a doubly degenerate spectrum ($s_n^{o-e} \simeq 0$) [5], and hence a gap between s_n^{e-o} and s_n^{o-e}. In contrast, for localized SPS both subsets are of the same form, and the gap vanishes. This is illustrated for the AA model in Fig. 2(a1-a3). For the multifractal case ($\Delta = 2$) the distribution of both s_n^{e-o} and s_n^{o-e} is strongly scattered [Fig. 2(a2)]. We also characterize the SPS $|\psi_n\rangle = \sum_j \psi_n(j)|j\rangle$ by the moments $I_q(n) = \sum_j |\psi_n(j)|^{2q} \propto N^{-D_q(q-1)}$, where D_q are the fractal dimensions. Localized states are characterized by $D_q = 0$, ergodic extended states by $D_q = 1$, while multifractal states have nontrivial $0 < D_q < 1$ [12–17,19,21]. As shown below, the study of D_2 (obtained from the inverse participation ratio, I_2) is particularly useful to characterize transitions at the mobility and ergodic-to-multifractal edges. The study of the multifractal spectrum of the SPS and $D_{q>2}$ [19] confirms the classification provided by the level spacing and D_2 analyses [42].

Mobility edge.—Figure 1 summarizes our results for $\beta = (\sqrt{5}-1)/2$, but similar physics is found for other values of β. We assume periodic boundary conditions in our exact-diagonalization calculations, choosing the number of sites L within the Fibonacci series (up to $L = 75025$). For $a \gg 1$ we recover the AA model, and hence all SPS are ergodic (AE regime) for $\Delta < 2$, or localized (AL regime [50]) for $\Delta > 2$. For finite $a > 1$ there is a critical value $\Delta_0(a)$ at which a mobility edge splits ergodic and localized SPS [51]. For $\beta = (\sqrt{5}-1)/2$ we numerically find that states with energies $E_{\beta L\leq n\leq L}$ become localized [Fig. 2(b1)], whereas those with $E_{n<\beta L}$ remain ergodic. This regime, which we call P_1, exists up to a critical $\Delta_1(a)$, at which states with energies $E_{\beta^2 L<n<\beta L}$ also become localized [Fig. 2(b2)]. The localization transition is observable from the behavior of D_2, which in our calculations springs from 1 to a value that within our numerical accuracy is compatible with $D_2 = 0$ [Fig. 2(b3)]. A sequence of P_s regimes is present for higher Δ values (Fig. 1) [52]. In the P_s regime the lowest $\beta^s L$ states are ergodic and the rest are localized. The blocklike nature of the transitions may be well understood from the analysis of the dispersion of the subbands [42]. We note that the above mentioned particular blocks of states that localize or become multifractal for $\beta = (\sqrt{5}-1)/2$ result from the form of the corresponding bands. Although the overall form of the diagram of Fig. 1 is maintained for other β values, the sizes of the eigenstate blocks, as well as the specific boundaries, $\Delta_s(a)$, of the P_s regimes, depend on the value of β.

Ergodic-to-multifractal edge.—Interestingly, the SPS properties radically change for long-range hops ($a \leq 1$). The AE regime extends all the way down to $a = 0$, where it vanishes. The sequence of P_s regimes is maintained, but localization is absent, in stark contrast to the case of power-law hopping in the presence of uncorrelated disorder [41]. In contrast, the spectrum presents an edge between ergodic and multifractal (extended but nonergodic) SPS. Within the P_s regime, the lowest $\beta^s L$ states are ergodic, whereas the

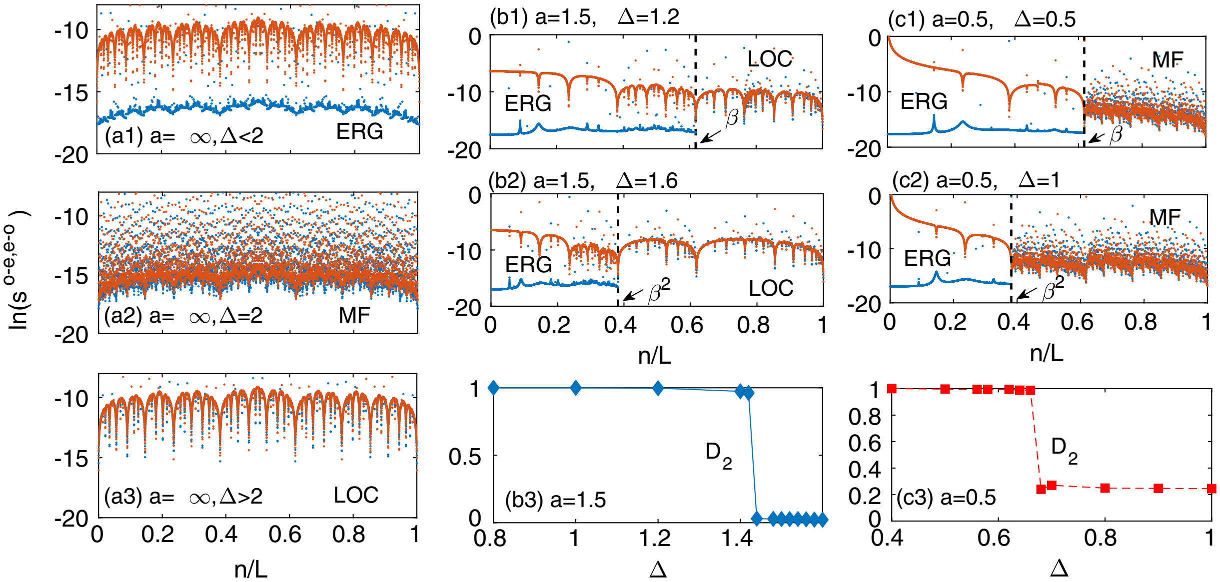

FIG. 2. Results for $\beta = (\sqrt{5}-1)/2$. Level spacing s^{e-o} (red) and s^{o-e} (blue) for the AA model (a1-a3), $a = 1.5$ (b1,b2), and $a = 0.5$ (c1,c2) for different Δ. In the AA model all SPS are either localized (LOC), multifractal (MF), or ergodic (ERG). In the GAA model, P_s regimes appear, in which the lowest β^s fraction of SPS is ergodic, whereas the rest is localized ($a > 1$) or multifractal ($a < 1$). These graphs were obtained from calculations for $L = 28657$ sites with periodic boundary conditions. (b3) and (c3) show D_2 for the SPS between $\beta^2 L$ and βL at $a = 1.5$ and 0.5, respectively. For $a = 1.5$ (0.5) a blocklike localization (ergodic-to-multifractal) transition occurs when crossing from P_1 to P_2. The results of panels (b3) and (c3) were obtained from calculations with up to $L = 75025$ sites with periodic boundary conditions and then extrapolated to infinite systems. See Ref. [42] for more details about the energy and Δ dependence of D_2.

rest are multifractal. This behavior is illustrated for $a = 0.5$ in Figs. 2(c1) and 2(c2). When crossing the P_{s-1} to P_s boundary, D_2 jumps from 1 to $0 < D_2 < 1$ for the states with energies $E_{\beta^s L<n<\beta^{s-1}L}$ [53]. This confirms the ergodic-to-multifractal character of the transition [Fig. 2(c3)] [42].

Excitation dynamics.—The nature of the SPS results in a peculiar excitation dynamics. We consider all particles ↓, except an initially localized ↑ excitation, which for simplicity is placed at the center of a lattice with open boundary conditions. We define the survival probability, $F(R)$, as the probability of finding the excitation after a given time in a site within the region $(-R/2, R/2)$. As recently shown for random matrix models [54–57], $F(R)$ provides crucial information about the localization properties. Figure 3 shows $F(R = L/2)$ as a function of Δ for $a = 3$ and $a = 0.5$ for open boundary conditions and $L = 987$ sites for long times t ($Jt = 10^4$), although similar results are found for smaller lattices and shorter times. In the AE regime, $F(R)$ vanishes for infinitely large L and long times. For finite L the probability of finding the excitation at a given lattice site is the same for all sites and is equal to $1/L$. In contrast, the P_s regimes present localized and extended

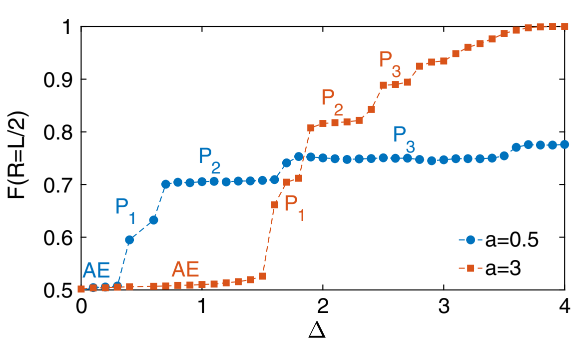

FIG. 3. Survival probability $F(R = L/2)$ in the long-time limit ($Jt = 10^4$) for $a = 0.5$ (blue circles) and $a = 3$ (red squares), for open boundary conditions with $L = 987$ sites and $\beta = (\sqrt{5}-1)/2$. The intermediate P_s regimes lead to a step-wise dependence on Δ.

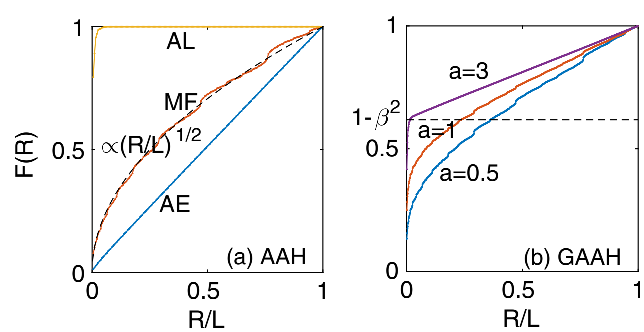

FIG. 4. (a) Long-time survival probability $F(R)$ for the AA model with $L = 100$, for the AE, AL, and MF cases, assuming an initially localized excitation at $x = 0$ in $L = 100$ sites. (b) $F(R)$ for the GAA model with open boundary conditions and $L = 987$ sites for $a = 0.5$, 1, and 3, within the P_2 regime.

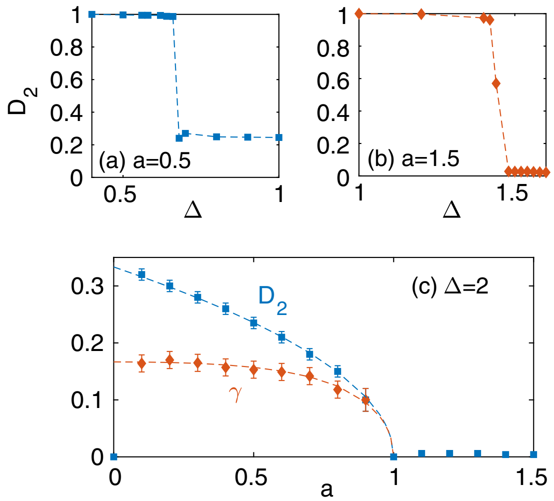

FIG. 5. Fractal dimension D_2 for $a = 0.5$ (a) and 1.5 (b) as a function of Δ. Figure (c) shows D_2 and γ [with $l(t) \propto t^{-\gamma}$] as a function of a for $\Delta = 2$. Dashed lines depict the fitted relations $D_2(a) \approx \frac{1}{3}(1-a)^{1/2}$ and $\gamma \simeq D_2/(2-a)$. See Ref. [42] for a detailed discussion on our calculations of D_2 and γ, as well as on the error bars.

SPS, and hence the excitation wave packet presents a bimodal distribution, partially escaping, and partially remaining localized close to the initial position. As a result $F(R)$ presents a steplike growth when entering the P_s regimes (see Fig. 3).

The dynamics in the presence of multifractal SPS differs from that of localized and ergodic ones. This is best illustrated in the AA model [see Fig. 4(a)]. For sufficiently large R/L, for long times, $F(R) \simeq 1$ for $\Delta > 2$, $F(R) \simeq R/L$ for $\Delta < 2$, and $F(R) \simeq (R/L)^{1/2}$ for $\Delta = 2$. The latter reflects the nonergodic character of the multifractal expansion. For the GAA model with $\beta = (\sqrt{5}-1)/2$, in the P_s regime the lowest $\beta^s L$ states remain extended. If the rest of the SPS are localized ($a > 1$), with a localization length smaller than $R/4L$, then $F(R)$ approaches $F_0(R) = (1-\beta^s) + \beta^s R/L$ [$a = 3$ in Fig. 4(b)]. However, for $a \leq 1$ there are ergodic and multifractal SPS, and the latter also contribute to the escape probability. Hence, for $L \to \infty$ the function $F(R)$ should vanish for all P_s regimes. For finite systems, $F(R)$ remains finite, but $F(R) < F_0(R)$ and presents a nonlinear dependence [Fig. 4(b)] [58].

The time dependence of $F(R)$ constitutes as well a clear indicator of the presence of multifractal SPS [54–57]. Figure 5 shows our results for $l(t) = F(R = 0, t) = |\langle\psi(t)|\psi(0)\rangle|^2$ [i.e., the Loschmidt echo amplitude, where $\psi(0)$ and $\psi(t)$ are the initial state and its evolved state, respectively]. For all cases $l(t) \sim t^{-\gamma}$. Fitting our numerical data to this dependence we find that ergodic (localized) SPS result in $\gamma \simeq 1$ (0), whereas our numerics reveals that the multifractal SPS appearing for $a \leq 1$ result in $\gamma \simeq D_2/(2-a)$ [42]. The analysis of the excitation dynamics, which can be monitored using spin-resolved quantum microscopes [59], can hence reveal not only the structure of intermediate regimes, but also the multifractal nature of the SPS for long-range hops.

Outlook.—Quasicrystals with power-law hops, $1/r^a$, present nontrivial localization properties. They are characterized by mobility edges for $a > 1$, by ergodic-to-multifractal edges for $a \leq 1$, and by the existence of a ladder of intermediate regimes in which SPS blocks become localized or multifractal. These properties may be readily tested using expansion experiments. Mobility edges and step-wise dynamics may be experimentally probed for polar molecules pinned in deep bichromatic optical lattices. Powers $0 \leq a \leq 3$ may be directly realized in ions [36,37]. Hence ion experiments are particularly interesting for the comparative study of mobility versus ergodic-to-multifractal edges.

Ising-like interactions, which for the case of spin excitations in polar molecules may be induced by an external polarizing electric field, are expected to lead to an intriguing physics including the possibility of a many-body mobility or ergodic-to-non-ergodic edge, due to the interaction-induced coupling between ergodic and localized or multifractal SPS [60,61], and the possible localization instability [62]. The presence of nonergodic SPS bands opens fascinating possibilities for the realization of a bad metal phase [63,64] and for the observation of ergodic to nonergodic phase transition.

We acknowledge fruitful discussions with V. E. Kravtsov and O. M. Yevtushenko. X. D. and L. S. are thankful for the support of the DFG (SFB 1227 DQ-mat and FOR2247). G. V. S. acknowledges funding from the European Research Council under European Community's Seventh Framework Programme (FP7/2007-2013 Grant Agreement No. 341197). S. S. would like to thank the Institute of Theoretical Physics of the Leibniz University of Hannover for its hospitality.

[1] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[2] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).
[3] P. G. Harper, Proc. Phys. Soc. London Sect. A **68**, 874 (1955).
[4] M. Ya. Azbel, Sov. Phys. JETP **17**, 665 (1963); **19**, 634 (1964); Phys. Rev. Lett. **43**, 1954 (1979).
[5] S. Aubry and G. André, Ann. Israel Phys. Soc. **3**, 133 (1980).
[6] G. Roati, C. D'Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, Nature (London) **453**, 895 (2008).
[7] B. Deissler, M. Zaccanti, G. Roati, C. D'Errico, M. Fattori, M. Modugno, G. Modugno, and M. Inguscio, Nat. Phys. **6**, 354 (2010).
[8] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, Science **349**, 842 (2015).

[9] H. P. Lüschen, P. Bordia, S. Scherg, F. Alet, E. Altman, U. Schneider, and I. Bloch, Phys. Rev. Lett. **119**, 260401 (2017).
[10] I. M. Suslov, Sov. Phys. JETP **56**, 612 (1982).
[11] M. Wilkinson, Proc. R. Soc. A **391**, 305 (1984).
[12] F. Wegner, Z. Phys. B **36**, 209 (1980).
[13] H. Aoki, J. Phys. C **16**, L205 (1983); Phys. Rev. B **33**, 7310 (1986).
[14] B. L. Altshuler, V. E. Kravtsov, and I. V. Lerner, JETP Lett. **43**, 441 (1986).
[15] A. D. Mirlin, Y. V. Fyodorov, A. Mildenberger, and F. Evers, Phys. Rev. Lett. **97**, 046803 (2006).
[16] M. Janssen, Phys. Rep. **295**, 1 (1998).
[17] F. Evers and A. D. Mirlin, Rev. Mod. Phys. **80**, 1355 (2008).
[18] G. Biroli, A. Ribeiro-Teixeira, and M. Tarzia, arXiv:1211.7334.
[19] A. De Luca, B. L. Altshuler, V. E. Kravtsov, and A. Scardicchio, Phys. Rev. Lett. **113**, 046806 (2014).
[20] V. E. Kravtsov, J. M. Khaimovich, E. Cuevas, and M. Amini, New J. Phys. **17**, 122002 (2015).
[21] X. Deng, B. L. Altshuler, G. V. Shlyapnikov, and L. Santos, Phys. Rev. Lett. **117**, 020401 (2016).
[22] B. L. Altshuler, E. Cuevas, L. B. Ioffe, and V. E. Kravtsov, Phys. Rev. Lett. **117**, 156601 (2016).
[23] R. Riklund, Y. Liu, G. Wahlström, and Z. Zhao-bo, J. Phys. C **19**, L705 (1986).
[24] D. J. Boers, B. Goedeke, D. Hinrichs, and M. Holthaus, Phys. Rev. A **75**, 063404 (2007).
[25] J. Biddle and S. Das Sarma, Phys. Rev. Lett. **104**, 070601 (2010).
[26] J. Biddle, D. J. Priour, Jr., B. Wang, and S. Das Sarma, Phys. Rev. B **83**, 075105 (2011).
[27] X. Li, X. Li, and S. Das Sarma, Phys. Rev. B **96**, 085119 (2017).
[28] H. P. Lüschen, S. Scherg, T. Kohlert, M. Schreiber, P. Bordia, X. Li, S. Das Sarma, and I. Bloch, Phys. Rev. Lett. **120**, 160404 (2018).
[29] F. A. An, E. J. Meier, and B. Gadway, Phys. Rev. X **8**, 031045 (2018).
[30] For other functional dependences of the hopping amplitude see Refs. [25,26].
[31] A. de Paz, A. Sharma, A. Chotia, E. Maréchal, J. H. Huckans, P. Pedri, L. Santos, O. Gorceix, L. Vernac, and B. Laburthe-Tolra, Phys. Rev. Lett. **111**, 185305 (2013).
[32] B. Yan, S. A. Moses, B. Gadway, J. P. Covey, K. R. A. Hazzard, A. M. Rey, D. S. Jin, and J. Ye, Nature (London) **501**, 521 (2013).
[33] M. Saffman, T. G. Walker, and K. Mølmer, Rev. Mod. Phys. **82**, 2313 (2010).
[34] G. Waldherr, Y. Wang, S. Zaiser, M. Jamali, T. Schulte-Herbrueggen, H. Abe, T. Ohshima, J. Isoya, P. Neumann, and J. Wrachtrup, Nature (London) **506**, 204 (2014).
[35] G. A. Álvarez, D. Suter, and R. Kaiser, Science **349**, 846 (2015).
[36] P. Richerme, Z.-X. Gong, A. Lee, C. Senko, J. Smith, M. Foss-Feig, S. Michalakis, A. V. Gorshkov, and C. Monroe, Nature (London) **511**, 198 (2014).
[37] P. Jurcevic, B. P. Lanyon, P. Hauke, C. Hempel, P. Zoller, R. Blatt, and C. F. Roos, Nature (London) **511**, 202 (2014).
[38] C.-L. Hung, A. Gonzalez-Tudela, J. I. Cirac, and H. J. Kimble, Proc. Natl. Acad. Sci. U.S.A. **113**, E4946 (2016).
[39] A. L. Burin and L. A. Maksimov, JETP Lett. **50**, 338 (1989).
[40] G. L. Celardo, R. Kaiser, and F. Borgonovi, Phys. Rev. B **94**, 144206 (2016).
[41] X. Deng, V. E. Kravtsov, G. V. Shlyapnikov, and L. Santos, Phys. Rev. Lett. **120**, 110602 (2018).
[42] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.123.025301, which includes Refs. [43–46], for more details on an effective band-dependent tight-binding model that allows for a simple understanding of the blocklike transitions between P_s regimes, on the analysis of multifractality of single-particle states, on the special case of $a = 1$, on the relation between inverse participation ratio and survival probability, on the convergence of the GAA model towards the AA model when $a \to \infty$, and on details of localization and ergodic-to-multifractal transitions.
[43] A. Szabó and U. Schneider, Phys. Rev. B **98**, 134201 (2018).
[44] R. Ketzmerick, G. Petschel, and T. Geisel, Phys. Rev. Lett. **69**, 695 (1992).
[45] J. X. Zhong and R. Mosseri, J. Phys. Condens. Matter **7**, 8383 (1995).
[46] I. García-Mata, O. Giraud, B. Georgeot, J. Martin, R. Dubertrand, and G. Lemarié, Phys. Rev. Lett. **118**, 166801 (2017).
[47] Differential polarizability results in the quasidisordered variation of energy between both states. Differential polarizability is irrelevant for the primary lattice, as long as both spin components can be considered as pinned.
[48] S. Gopalakrishnan, Phys. Rev. B **96**, 054202 (2017).
[49] The model discussed in Ref. [48] presents in addition to the power-law hopping a power-law intersite interaction potential, very different than the second term at the right-hand side of Eq. (1). This peculiar potential is introduced in order to preserve self-duality. As a result, the physical properties of the system are radically different than those we discuss in our work. In particular, multifractal states, mobility edges, and ergodic-to-nonergodic transitions are not found.
[50] For a discussion on the existence of the AL regime in the GAA model see A. M. García-García and E. Cuevas, Phys. Rev. B **79**, 073104 (2009); C. Monthus, Fractals **27**, 1950007 (2019).
[51] A finer analysis, as well as the study of the subband dispersion discussed in the Supplemental Material [42], shows that the transition between P_s regimes actually occurs within a very narrow but finite window of Δ values, in which subbands become localized. These subband transitions may be as well split into even narrower sub-sub-bands transitions. However, for all practical purposes this self-similar set of closely lying transitions may be considered as a single abrupt jump between P_s regimes.
[52] For larger Δ, we observe in our numerics a series of P_s regimes, due to the self-similar nature form of the band spectrum. However, for large s, the P_s regime is characterized for $\beta = (\sqrt{5} - 1)/2$ by a tiny fraction $\beta^s L$ of ergodic states. Hence, for finite systems eventually the P_s ladder

ends into an AL phase. We note, however, that in our numerics for $a < 2$ we have always found, even for very large Δ values a finite region of ergodic states when considering larger and larger systems. This explains the form of the diagram in Fig. 1. For the $P_{s\gg 1}$ regime the number of ergodic states is tiny and becomes irrelevant for the expansion dynamics.
[53] The value of D_2 for the multifractal states does not variate significantly either as a function of the eigenenergy or of the quasidisorder strength Δ [42].
[54] V. E. Kravtsov, O. M. Yevtushenko, P. Snajberk, and E. Cuevas, Phys. Rev. E **86**, 021136 (2012).
[55] E. J. Torres-Herrera, A. M. García-García, and Lea F. Santos, Phys. Rev. B **97**, 060303(R) (2018).
[56] L. F. Santos and E. J. Torres-Herrera, in *Thermodynamics in the Quantum Regime. Fundamental Theories of Physics*, edited by F. Binder, L. Correa, C. Gogolin, J. Anders, and G. Adesso (Springer, Cham, 2018), Vol. 195.
[57] G. De Tomasi, M. Amini, S. Bera, I. M. Khaymovich, and V. E. Kravtsov, SciPost Phys. **6**, 014 (2019).
[58] The form of $F(R)$ becomes independent of the size L for the case of an ergodic-to-multifractal edge, showing that the states are actually multifractal and not only localized with a large localization length.
[59] J. P. Covey, L. De Marco, O. L. Acevedo, A. M. Rey, and J. Ye, New J. Phys. **20**, 043031 (2018).
[60] X. Li, S. Ganeshan, J. H. Pixley, and S. Das Sarma, Phys. Rev. Lett. **115**, 186601 (2015).
[61] R. Modak and S. Mukerjee, Phys. Rev. Lett. **115**, 230401 (2015).
[62] M. Žnidarič and M. Ljubotina, Proc. Natl. Acad. Sci. U.S.A. **115**, 4595 (2018).
[63] M. Pino, L. B. Ioffe, and B. L. Altshuler, Proc. Natl. Acad. Sci. U.S.A. **113**, 536 (2016).
[64] M. Pino, V. E. Kravtsov, B. L. Altshuler, and L. B. Ioffe, Phys. Rev. B **96**, 214205 (2017).