No-Broadcasting Theorem for Quantum Asymmetry and Coherence and a Trade-off Relation for Approximate Broadcasting

Iman Marvian[1,*] and Robert W. Spekkens[2,†]
[1]*Departments of Physics and Electrical and Computer Engineering, Duke University, Durham, North Carolina 27708, USA*
[2]*Perimeter Institute for Theoretical Physics, 31 Caroline St. N, Waterloo, Ontario, Canada N2L 2Y5*

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Symmetries of both closed- and open-system dynamics imply many significant constraints. These generally have instantiations in both classical and quantum dynamics (Noether's theorem, for instance, applies to both sorts of dynamics). We here provide an example of such a constraint which has no counterpart for a classical system, that is, a uniquely quantum consequence of symmetric dynamics. Specifically, we demonstrate the impossibility of broadcasting asymmetry (symmetry breaking) relative to a continuous symmetry group, for bounded-size quantum systems. The no-go theorem states that if two initially uncorrelated systems interact by symmetric dynamics and asymmetry is created at one subsystem, then the asymmetry of the other subsystem must be reduced. We also find a quantitative relation describing the trade-off between the subsystems. These results cannot be understood in terms of additivity of asymmetry, because, as we show here, any faithful measure of asymmetry violates both subadditivity and superadditivity. Rather, they must be understood as a consequence of an (intrinsically quantum) information-disturbance principle. Our result also implies that if a bounded-size quantum reference frame for the symmetry group, or equivalently, a bounded-size reservoir of coherence (e.g., a clock with coherence between energy eigenstates in quantum thermodynamics) is used to implement any operation that is not symmetric, then the quantum state of the frame or reservoir is necessarily disturbed in an irreversible fashion, i.e., degraded.

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Introduction.—Finding the consequences of symmetries of a closed- or open-quantum dynamics is a problem that has a wide range of applications in physics, with Noether's theorem being perhaps the most prominent example. It is notable that the consequences that physicists have focused on, including the conservation of Noether charges and currents, generally hold in *both* quantum and classical contexts. A natural question, therefore, is whether there are consequences of symmetric dynamics that are unique to quantum theory.

Eugene P. Wigner pioneered the study of the consequences of symmetry in quantum theory and made various fundamental contributions to the topic. For instance, in 1952, he showed [1,2] that under the restriction of using only Hamiltonians which conserve an observable L that is additive across subsystems (e.g., the total angular momentum in a given direction), an exact measurement of another observable O becomes impossible unless O commutes with L. This fundamental no-go result, known as the Wigner-Araki-Yanase (WAY) theorem [3,4], can equivalently be phrased as a consequence of the restriction to Hamiltonians which are invariant under a continuous symmetry, namely the symmetry for which L is the generator.

In recent years, inspired by the success of entanglement theory [5], the problem of finding the consequences of symmetric dynamics has been further studied in the framework of quantum resource theories [6–9]. In the resource theory of asymmetry [10], any state which breaks the symmetry under consideration, i.e., any state which has some *asymmetry*, is treated as a resource (similar to entangled states in entanglement theory). A particular case of interest, which is relevant in the context of the WAY theorem for instance, is when the symmetry under consideration is the continuous set of translations generated by a fixed observable H, i.e., $\{e^{-iHt}: t \in \mathbb{R}\}$ (note that H need not be the Hamiltonian, nor t the time parameter, although the notation is meant to bring to mind this example). In this case, a state contains asymmetry iff it contains *coherence* (off-diagonal terms) with respect to the eigenspaces of H. It follows that the resource theory of asymmetry provides a natural framework to study this sort of coherence, which is known as *unspeakable* coherence [13,14], and which is the notion that is relevant for quantum metrology [15] and quantum thermodynamics [14,16,17] (as argued in Ref. [13]).

The resource-theoretic approach to the study of symmetric dynamics and asymmetry properties of quantum states has shed new light on earlier work. For instance, it was found that the *skew information*, a function introduced by Wigner and Yanase [18] as a replacement for the von Neumann entropy in the presence of symmetry, is, in fact, a measure of asymmetry [19–21]. Also, it was found in

Ref. [22] that the WAY theorem can be understood as a corollary of a deep result in quantum information theory, known as the *no-programming* theorem [23–25].

Another no-go theorem about continuous symmetries was uncovered in Ref. [12]: the *no-catalysis* theorem [26]. This result concerns state conversions using operations which are covariant (symmetric) with respect to a compact connected Lie group, and states that if the pure state conversion $\psi \to \phi$ is not achievable, then the catalyzed version of this same conversion, $\psi \otimes \eta \to \phi \otimes \eta$, is also not achievable for any choice of pure catalyst state, η, in a finite-dimensional Hilbert space [29] (see also Ref. [30] for related observations).

Taking the perspective of resource theories has also made evident that existing results on symmetric dynamics, including the no-catalysis and WAY theorems, are not uniquely quantum. This is because it has clarified that a key assumption in each of these no-go theorems is that the resource state being used is not perfectly asymmetric in the sense that the state and its translated versions (under the symmetry transformations) are not perfectly distinguishable. If one makes the analogous assumption classically—that the probability distribution over classical configurations constituting one's resource is not perfectly asymmetric in the same sense—then one obtains similar no-go results.

In this Letter, we find an example of a consequence of symmetric dynamics that *is* uniquely quantum, namely, a *no-broadcasting theorem* for asymmetry. It asserts that if, during a symmetric dynamics on an initially uncorrelated pair of systems, asymmetry is created at one subsystem, then the asymmetry of another subsystem should reduce. We also show that this result does not hold classically.

In fact, we prove a more general result, namely, that under symmetric dynamics, if one uses a bounded-size quantum system in an asymmetric state (a reference frame or coherence reservoir) as a resource to perform an asymmetric operation (i.e., a task which is impossible under symmetric dynamics), then one necessarily disturbs its state irreversibly—the frame or reservoir degrades. While it has been previously noted that quantum reference frames can degrade when used to implement certain asymmetric operations [31–34], this conclusion was established only for certain target operations and considered only for the case where the frame starts in a pure state (see, however, Ref. [35] for more recent work).

We also find a trade-off relation for approximate broadcasting, namely, a lower bound on the amount of disturbance caused by the broadcasting of asymmetry or coherence in the case of pure states [Eq. (6)]. This investigation also leads us to take note of a very general constraint on measures of asymmetry (Theorem 3).

Covariance condition.—We begin with some formalism. Consider an arbitrary physical process with input systems Q and S and output systems Q' and S', and let $\Lambda_{QS\to Q'S'}$ (or simply Λ) be the corresponding quantum operation (i.e., completely positive trace-preserving linear map) from the density operators of QS to the density operators of $Q'S'$. We are interested in the processes satisfying the *covariance* condition

$$\forall\, t\in\mathbb{R}: \Lambda\circ[\mathcal{U}_Q(t)\otimes\mathcal{U}_S(t)] = [\mathcal{U}_{Q'}(t)\otimes\mathcal{U}_{S'}(t)]\circ\Lambda. \quad (1)$$

Here, for each system $X\in\{Q,S,Q',S'\}$, we have defined $\mathcal{U}_X(t)[\cdot]\equiv e^{-iH_Xt}(\cdot)e^{iH_Xt}$, where H_X is a (Hermitian) observable defined on system X. Note that for each system X, the map $\mathbb{R}\ni t\to\mathcal{U}_X(t)$ can be interpreted as a representation of a group of translations. Equation (1) means that the description of the process $\Lambda_{QS\to Q'S'}$ is independent of which reference frame for translations one uses.

A particular case of interest is when the operator H_X is the Hamiltonian describing the closed-system dynamics of X, so that $\mathcal{U}_X(t)$ represents evolution for time $t\in\mathbb{R}$. In this case, the tensor product form of $\mathcal{U}_Q(t)\otimes\mathcal{U}_S(t)$ [and $\mathcal{U}_{Q'}(t)\otimes\mathcal{U}_{S'}(t)$] reflects the fact that systems Q and S (and systems Q' and S') are not interacting with one another before (and after) the process Λ, and, therefore, can be treated as separate noninteracting subsystems. Then, the covariance condition in Eq. (1) means that the effect of the process Λ on the inputs Q and S does not depend on the time at which the process acts on these systems. This property is satisfied, for instance, by any thermal machine that interacts a system with thermal baths and with work reservoirs (batteries).

Asymmetry as a resource.—A simple consequence of a process satisfying the covariance condition in Eq. (1) is that it cannot generate asymmetry. Suppose the input state ρ_{QS} is *symmetric* with respect to the symmetry represented by $t\to\mathcal{U}_Q(t)\otimes\mathcal{U}_S(t)$, i.e.

$$\forall\, t\in\mathbb{R}: \mathcal{U}_Q(t)\otimes\mathcal{U}_S(t)[\rho_{QS}] = \rho_{QS}. \quad (2)$$

Note that this holds iff ρ_{QS} is diagonal, or *incoherent* relative to the eigenspaces of $H_Q\otimes I_S + I_Q\otimes H_S$, where I_S and I_Q are the identity operators on S and Q. Then, it can be easily seen that the covariance of process Λ implies that incoherent states of the input systems are mapped to incoherent states of the output systems. In this sense, asymmetry, or coherence, is a *resource* that cannot be generated under covariant operations. Obviously, the physical interpretation of this resource depends on the nature of the symmetry. For instance, only for states that are asymmetric with respect to time translations is a system useful as a clock and only for states that are asymmetric with respect to rotations is a system useful as a gyroscope.

Under the restriction to processes which satisfy the covariance condition in Eq. (1), having access to a resource of asymmetry allows one to perform operations that would otherwise be impossible. For any fixed state ρ_Q of system Q, let $\mathcal{E}_{S\to S'}$ be the quantum operation from S to S' induced by the covariant operation $\Lambda_{QS\to Q'S'}$

$$\mathcal{E}_{S\to S'}(\cdot) \equiv \mathrm{Tr}_{Q'}[\Lambda_{QS\to Q'S'}(\rho_Q \otimes \cdot)]. \quad (3)$$

(Note that Q and S are assumed to be initially uncorrelated.) It can be easily seen that if ρ_Q is a symmetric state, then the map $\mathcal{E}_{S\to S'}$ is covariant, i.e., satisfies $\forall\, t\in\mathbb{R}: \mathcal{E}\circ\mathcal{U}_S(t) = \mathcal{U}_{S'}(t)\circ\mathcal{E}$. On the other hand, using a state ρ_Q which contains asymmetry, we can implement a noncovariant operation $\mathcal{E}_{S\to S'}$.

For instance, if the process Λ satisfies time-translation symmetry, then using an input system Q, whose state ρ_Q contains asymmetry with respect to time translations (or equivalently, contains coherence relative to the energy eigenspaces), one can implement on S operations which do not satisfy time-translation symmetry. Therefore, for an agent who seeks to implement an operation at a particular time relative to some time standard (i.e., reference clock) but who lacks access to it, such a system can constitute a token of the standard, a quantum clock that is synchronized with the reference clock.

Irreversibility and degradation.—Suppose that there is a covariant process under which $\rho_Q \to \sigma_{Q'}$. We say that the state conversion $\rho_Q \to \sigma_{Q'}$ is *reversible* in the resource theory if there exists a covariant process $\mathcal{R}_{Q'\to Q}$ which recovers ρ_Q from $\sigma_{Q'}$, i.e., $\mathcal{R}_{Q'\to Q}(\sigma_{Q'}) = \rho_Q$; otherwise, we say that the state conversion is irreversible and that the asymmetry of ρ_Q is *degraded* under the state conversion.

Asymmetry degradation theorem.—The following theorem shows that using a bounded-size system Q in an asymmetric state ρ_Q to implement a noncovariant operation on system S necessarily degrades the asymmetry of ρ_Q.

Theorem 1(asymmetry degradation).—Let Q be a system with a finite-dimensional Hilbert space, prepared in state ρ_Q. Suppose system S, initially uncorrelated with Q, interacts with system Q via a covariant process $\Lambda_{QS\to Q'S'}$. Let $\mathcal{E}_{S\to S'}$, defined in Eq. (3), be the effective map which determines how the reduced state of output S' depends on the state of S (for a fixed ρ_Q). If $\mathcal{E}_{S\to S'}$ is not covariant, then, for some states of S (including the completely mixed state) the conversion from ρ_Q to $\sigma_{Q'}$ is irreversible; i.e., state ρ_Q cannot be recovered from state $\sigma_{Q'}$ via a covariant process $\mathcal{R}_{Q'\to Q}$ (see Fig. 1).

The proof of this result will be given in the Supplemental Material [36]. It leverages our result on no-broadcasting of asymmetry (Proposition 2), which is a special case of Theorem 1, and whose proof will be presented in this Letter.

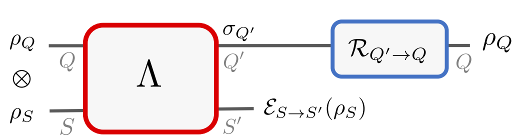

FIG. 1. If, using a covariant operation $\mathcal{R}_{Q'\to Q}$, state ρ_Q can be recovered from $\sigma_{Q'}$, then the effective operation $\mathcal{E}_{S\to S'}$ is covariant, and therefore can be implemented without having access to ρ_Q.

It is worth noting that, unlike the no-catalysis theorem of [29], here we do not assume that systems Q and S' are uncorrelated after the recovery process $\mathcal{R}_{Q'\to Q}$ is applied; rather, the result concerns the reduced state of Q itself. Such correlations become relevant, for instance, if we want to repeat this process to implement $\mathcal{E}_{S\to S'}$ multiple times, i.e., to implement $\mathcal{E}^{\otimes n}_{S\to S'}$ for arbitrary integer n. As we see in the following, if one requires such a notion of repeatability, which amounts to assuming lack of correlations, then the proof of degradation becomes much simpler and can be achieved by using arguments similar to the no-catalysis theorem of Ref. [29] or the arguments of Ref. [30]. However, interestingly, according to Theorem 1, even if we relax this requirement and ignore correlations, the degradation still holds, i.e., using state ρ_Q to implement a noncovariant process $\mathcal{E}_{S\to S'}$ will necessarily imply that Q undergoes a state conversion $\rho_Q \to \sigma_{Q'}$ that is irreversible.

No-broadcasting of asymmetry or coherence.—The special case of Theorem 1 that is the focus of this work concerns a map that incorporates both the process $\Lambda_{QS\to Q'S'}$ as well as any recovery operation $\mathcal{R}_{Q'\to Q}$ on it, and which is specialized to the case where S is trivial. We can conceptualize such a map as a *broadcast map* from Q to the pair of systems Q and S'. Unlike the usual discussions of broadcasting [37], where there is a set of possible states at the input and no restriction on the nature of the broadcast map, we are here interested in the case where there is a single state at the input, but the broadcast map is constrained to be covariant.

We will say that asymmetry or coherence can be broadcast if there is a covariant map that takes any input state ρ_Q to a state $\sigma_{QS'}$ with the property that (i) the input state ρ_Q is reproduced in the output Q, i.e., $\sigma_Q = \rho_Q$ where $\sigma_Q \equiv \mathrm{Tr}_{S'}(\sigma_{QS'})$, and (ii) the state of system S' has nontrivial asymmetry/coherence, i.e., $[\sigma_{S'}, H_{S'}] \neq 0$. We can now state precisely the no-go result advertised in the title.

Proposition 2.—(No-broadcasting of asymmetry or coherence) For bounded-size system Q, there does not exist a covariant broadcast map (defined by conditions (i) and (ii) above). In other words

$$\rho_Q \to \sigma_{QS'} \Rightarrow \mathrm{NOT}\ \ (\sigma_Q = \rho_Q\ \ \mathrm{AND}\ \ [\sigma_{S'}, H_{S'}] \neq 0). \quad (4)$$

We prove this proposition later by appealing to a lemma that concerns the standard notion of broadcasting (Lemma 4).

To see that this no-go result does not apply to classical asymmetry, it suffices to note that a map that clones any point distribution on a classical configuration space is covariant relative to any symmetry and consequently such a map achieves broadcasting of asymmetry when acted on any distribution that breaks the symmetry of interest.

Nonadditivity of asymmetry.—At first glance, it may appear that the impossibility of broadcasting asymmetry should follow from an intuitive idea, namely, that asymmetry might be a kind of extensive quantity, so that to create asymmetry in the system S' one needs to reduce the asymmetry of Q. This intuition can be formalized using the notion of measures of asymmetry (see, e.g., Refs. [11,20,44,45]): a function f from states to real numbers is called a measure of asymmetry if (i) it is nonincreasing under covariant operations, i.e., $\rho_A \to \sigma_B$ implies $f(\rho_A) \geq f(\sigma_B)$ and (ii) it vanishes on symmetric states. A measure of asymmetry is called *faithful* if it vanishes *only* on symmetric states. The Wigner-Yanase Skew information, $f(\rho_X) \equiv -\mathrm{Tr}([\sqrt{\rho_X}, H_X]^2)/2$, is an example of a faithful measure of asymmetry where H_X is the generator of the symmetry (e.g., H_X is the Hamiltonian if the symmetry is time translations).

A measure of asymmetry, f, is called *subadditive* if for any state σ_{AB} of a composite system AB, $f(\sigma_{AB}) \leq f(\sigma_A) + f(\sigma_B)$, where σ_A and σ_B are the reduced states of σ_{AB} on A and B, respectively. It is called *superadditive* if $f(\sigma_{AB}) \geq f(\sigma_A) + f(\sigma_B)$. A measure of asymmetry is called *additive* if it is both subadditive and superadditive.

Suppose that there was even a single faithful superadditive measure of asymmetry, f. In this case, $\rho_Q \to \sigma_{QS'}$ would imply that $f(\rho_Q) \geq f(\sigma_{QS'}) \geq f(\sigma_Q) + f(\sigma_{S'})$. Since f is assumed to be faithful, if $\sigma_{S'}$ is not symmetric, then $f(\sigma_{S'}) > 0$, and we would be able to infer that $f(\rho_Q) > f(\sigma_Q)$ and consequently that $\rho_Q \to \sigma_Q$ is irreversible, which would prove the impossibility of broadcasting asymmetry.

However, interestingly, as we show in the Supplemental Material [36], there is no such measure:

Theorem 3.—A faithful measure of asymmetry is neither superadditive, nor subadditive.

It follows that the argument articulated above—wherein one seeks to justify no-broadcasting of asymmetry from superadditivity of asymmetry—is not sound. Indeed, the fact that our no-broadcasting result holds in spite of Theorem 3 makes it more surprising. As we discuss in the Supplemental Material [36], the failure of superadditivity can be shown using the fact that it is possible to create arbitrarily many systems in symmetry-breaking states starting from a single system in such a state, e.g., using an approximate cloner (see also Ref. [46] for a related result on skew information).

It is worth noting that some faithful measures of asymmetry, such as skew information, are additive on *product states*. Therefore, the argument articulated above *does* yield a proof of our no-broadcasting theorem, Eq. (4), for the special case where $\sigma_{QS'} = \sigma_Q \otimes \sigma_{S'}$. However, to prove the theorem in the general case, we need more powerful tools from quantum information theory.

Approximate broadcasting.—Next, we derive a quantitative version of our no-broadcasting theorem. Specifically, we assume that there is a covariant process which transforms ρ_Q to $\sigma_{QS'}$, and we seek to find a quantitative limit on the degree of success in broadcasting in terms of the amount of asymmetry (unspeakable coherence) in the initial state ρ_Q. We express our trade-off relation in terms of (i) the degree of irreversibility of the state conversion $\rho_Q \to \sigma_Q$ [where $\sigma_Q \equiv \mathrm{Tr}_{S'}(\sigma_{QS'})$] and (ii) the amount of asymmetry (unspeakable coherence) left in state $\sigma_{S'}$ [where $\sigma_{S'} \equiv \mathrm{Tr}_Q(\sigma_{QS'})$].

To quantify the degree of irreversibility in a state conversion $\rho_Q \to \sigma_Q$, we consider the minimum achievable infidelity in recovering the initial state ρ_Q from the final state σ_Q

$$\mathrm{irrev}(\rho_Q, \sigma_Q) \equiv 1 - \max_{\mathcal{R}} \mathrm{Fid}^2[\rho_Q, \mathcal{R}(\sigma_Q)], \tag{5}$$

where the maximization is over all covariant quantum operations. Here, $\mathrm{Fid}(\tau_1, \tau_2) \equiv \|\sqrt{\tau_1}\sqrt{\tau_2}\|_1$ is the (Uhlmann) fidelity [47–49]. This definition implies that $\mathrm{irrev}(\rho_Q, \sigma_Q)$ is between 0 and 1, and the state conversion $\rho_Q \to \sigma_Q$ is reversible iff $\mathrm{irrev}(\rho_Q, \sigma_Q) = 0$.

To quantify the asymmetry left in state $\sigma_{S'}$, we consider a measure of asymmetry defined in terms of the fidelity. For any $t \in \mathbb{R}$, define $f_t(\rho) \equiv 1 - \mathrm{Fid}(\rho, e^{-iHt}\rho e^{iHt}) = 1 - \|\sqrt{\rho}e^{-iHt}\sqrt{\rho}\|_1$. As we show in the Supplemental Material [36], f_t is a measure of asymmetry for any $t \in \mathbb{R}$, and it takes values in [0, 1]. $f_t(\rho)$ quantifies how distinguishable ρ is from $e^{-iHt}\rho e^{iHt}$.

The trade-off relation we prove, unlike our no-broadcasting theorem, is limited to the case where the initial state is pure, a fact which we denote by writing $\rho_Q = \psi_Q$. Specifically, if $\psi_Q \to \sigma_{QS'}$, then

$$\forall\, t \in \mathbb{R}\colon\ f_t(\sigma_{S'}) \leq 4\frac{\sqrt{\mathrm{irrev}(\psi_Q, \sigma_Q)}}{1 - f_t(\psi_Q)}. \tag{6}$$

This trade-off relation states that for any $t \in \mathbb{R}$, the asymmetry of $\sigma_{S'}$, as quantified by f_t, is upper bounded by a multiple of the degree of irreversibility of the state conversion $\psi_Q \to \sigma_Q$, as quantified by $\sqrt{\mathrm{irrev}(\psi_Q, \sigma_Q)}$. Note that as $f_t(\psi_Q)$ increases, the derived lower bound on $\mathrm{irrev}(\psi_Q, \sigma_Q)$ decreases (see also Ref. [35] for related work).

The proof is given in the Supplemental Material [36]. There, we also demonstrate that this trade-off relation immediately implies our no-broadcasting theorem, Eq. (4), for the special case where the state ρ_Q is pure.

Finally, we present the proof of our no-broadcasting theorem for asymmetry in the general case, where ρ_Q may be mixed and $\sigma_{QS'}$ may have correlations between Q and S'.

Proof of no-broadcasting of asymmetry or coherence.—To prove Proposition 2, we make use of the following lemma concerning broadcasting of an unknown state.

Lemma 4.—(No-broadcasting of information encoded quantumly) Let $\{\rho_Q^{(x)}\}_x$ be an arbitrary set of states of system Q. Suppose that under a quantum operation $\mathcal{E}_{Q\to QS'}$ (the putative broadcasting map), the state $\rho_Q^{(x)}$ is converted to the state $\sigma_{QS'}^{(x)}$ of systems Q and S'. Assume that under this map, the state of system Q is preserved at the output, so that the reduced state on output Q, defined as $\sigma_Q^{(x)} \equiv \mathrm{Tr}_{S'}(\sigma_{QS'}^{(x)})$, satisfies $\sigma_Q^{(x)} = \rho_Q^{(x)}$ for all states in the set $\{\rho_Q^{(x)}\}_x$. In this case, the reduced state on the output S', defined as $\sigma_{S'}^{(x)} \equiv \mathrm{Tr}_Q(\sigma_{QS'}^{(x)})$, can be obtained from the input state $\rho_Q^{(x)}$ by performing a projective measurement with projectors $\{\Pi_Q^{(\mu)}\}_\mu$ that commute with all states in the set $\{\rho_Q^{(x)}\}_x$, followed by a state preparation which depends only on the outcome of this measurement. That is, the reduced state of S' is constrained to be of the form $\sigma_{S'}^{(x)} = \sum_\mu p_\mu^{(x)} \sigma_{S'}^{(\mu)}$, where $p_\mu^{(x)} = \mathrm{Tr}(\rho_Q^{(x)} \Pi_Q^{(\mu)})$ is the probability of obtaining the μ outcome, and where $\{\sigma_{S'}^{(\mu)}\}_\mu$ is an arbitrary set of states.

This is proven in the Supplemental Material [36] using a result of Koashi and Imoto [50]. Note that one can conceptualize this result not only as a no-go for broadcasting of quantumly encoded information, but also as a type of information-disturbance principle. Specifically, it asserts that if the channel is disturbance free on Q, then the only information about the identity of the unknown state of Q that can be obtained from S' is information that is encoded *classically* in Q. In the case where the set of states is noncommuting, this implies a nontrivial and intrinsically quantum constraint on information gain. (See the Supplemental Material [36] for further discussion).

Next, we leverage this lemma to prove Proposition 2. We assume that the asymmetry at the input Q is preserved in the output Q and show that this implies that the state of S' is symmetric. We assume, therefore, that there exists a covariant operation that achieves the conversion $\rho_Q \to \sigma_{QS'}$ such that $\sigma_Q = \rho_Q$. We now note that, by virtue of its covariance, this operation also achieves the conversion $\mathcal{U}_Q(t)[\rho_Q] \to \mathcal{U}_Q(t) \otimes \mathcal{U}_{S'}(t)[\sigma_{QS'}]$ for all $t \in \mathbb{R}$. Given the assumption that $\sigma_Q = \rho_Q$, the marginal on Q of $\mathcal{U}_Q(t) \otimes \mathcal{U}_{S'}(t)[\sigma_{QS'}]$ is $\mathcal{U}_Q(t)[\rho_Q]$, and so an operation that converts an unknown state drawn from the set $\{\mathcal{U}_Q(t)[\rho_Q] : t \in \mathbb{R}\}$ into the corresponding state in the set $\{\mathcal{U}_Q(t) \otimes \mathcal{U}_{S'}(t)[\sigma_{QS'}] : t \in \mathbb{R}\}$ is precisely a broadcasting map satisfying the assumption of Lemma 4, where t plays the role of x.

We then infer from Lemma 4 that under such a broadcasting map, the state of the output S' must be prepared based on the outcome μ of a projective measurement $\{\Pi_Q^{(\mu)}\}_\mu$ on the input Q, where $\{\Pi_Q^{(\mu)}\}_\mu$ is a complete set of orthogonal projectors that commute with all states in $\{\mathcal{U}_Q(t)[\rho_Q] : t \in \mathbb{R}\}$.

The next step of the argument is where the restriction of scope to continuous symmetries occurs. For continuous symmetries, we can consider the derivative with respect to the parameter t. Defining $\rho_Q^{(t)} \equiv \mathcal{U}_Q(t)[\rho_Q] = e^{-iH_Q t} \rho_Q e^{iH_Q t}$, we have

$$i \frac{d}{dt} \mathrm{Tr}(\rho_Q^{(t)} \Pi_Q^{(\mu)}) = \mathrm{Tr}([\Pi_Q^{(\mu)}, H_Q] \rho_Q^{(t)}) = \mathrm{Tr}(H_Q [\rho_Q^{(t)}, \Pi_Q^{(\mu)}]),$$

where the first equality is Ehrenfest's theorem, and the second equality follows from the cyclic property of the trace. Recalling that $[\rho_Q^{(t)}, \Pi_Q^{(\mu)}] = 0$ for all $t \in \mathbb{R}$ and for all μ, it follows that $\mathrm{Tr}(\Pi_Q^{(\mu)} \rho_Q^{(t)})$ is independent of t. Because the probability distribution over μ induced by $\rho_Q^{(t)}$ is independent of t, the state of S', which, as established above, can only depend on $\rho_Q^{(t)}$ via the mediary of μ, is also independent of t. This can be expressed as $e^{-iH_{S'} t} \sigma_{S'} e^{iH_{S'} t} = \sigma_{S'}$, or equivalently, as $[\sigma_{S'}, H_{S'}] = 0$, which concludes the proof.

Conclusion.—In this work, we have demonstrated a uniquely quantum constraint on the manipulation of asymmetry (equivalently, unspeakable coherence), namely, that it cannot be broadcast. Note that a similar result is found independently in Ref. [51], which is published concurrently with this letter. We have also found a trade-off relation which quantifies the amount of irreversibility in a covariant state conversion that achieves *approximate* broadcasting of asymmetry or coherence. Furthermore, we showed that for bounded-size systems, asymmetry necessarily degrades with use.

It is worth noting that the constraints we have described here are generic to symmetries described by connected Lie groups. This is because any symmetry transformation in such groups is an element of a one-parameter subgroup in the form of e^{-iLx} for a generator L, and $x \in \mathbb{R}$, and covariance with respect to the original group implies covariance with respect to this subgroup.

The results are also *specific* to continuous symmetries in that they generally do *not* hold for discrete symmetries. This parallels the situation with the celebrated WAY theorem [1–4] and the no-catalysis theorem of Ref. [29].

A broader question suggested by our work is: for which quantum resources theories is it impossible to broadcast a resourceful state using the free operations defined by that resource theory? Reference [52] can be seen as providing another example in addition to the one described here.

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*iman.marvian@duke.edu
†rspekkens@perimeterinstitute.ca

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